Fall 2007

Course: **ADVANCED STRENGTH OF MATERIALS (ASE/EM 339)**

Unique No. ASE 13615 EM 14115

Room: ~~WRW 312~~ RLM 5.116

Time: MWF 2:00 - 3:00 pm

Instructor: Stelios Kyriakides (WRW 108)

Office Hours: MWF 3:00 - 4:00
4-5pm

Teaching Assistant: Rong Jiao (WRW 214) Office Time Th 1:30-2:30 pm

Recommended Text: "Advanced Mechanics of Materials," 2nd Ed. 1999 Cook, R. D. and Young, W. C., Prentice Hall,

Evaluation:

Homework	10%
Two Quizzes	50%
Final Exam	40%

Homework:

Homework will be assigned weekly; it will be due EXACTLY one week from the day of assignment. 10% of credit will be deducted for every day it is late. Homework turned in 5 or more days after the due date will not receive any credit.

C. HOVELL

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ADVANCED STRENGTH OF MATERIALS (ASE/EM 339)

QUIZ #1

Wednesday, October 17, 2007; 2:00 – 3:00 p.m., RLM 5.116

Closed Book Exam

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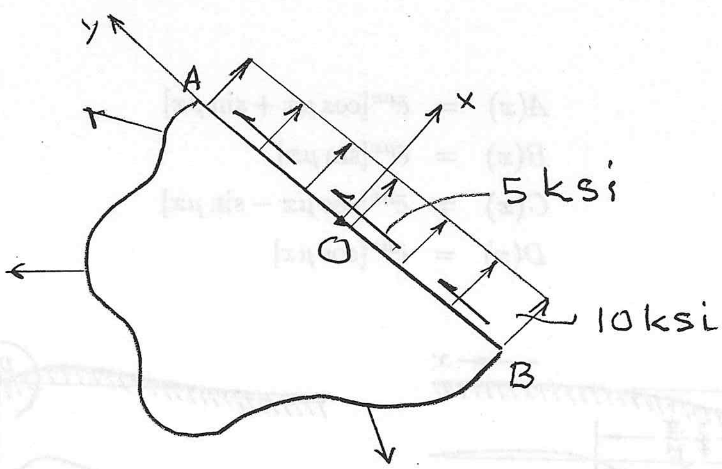
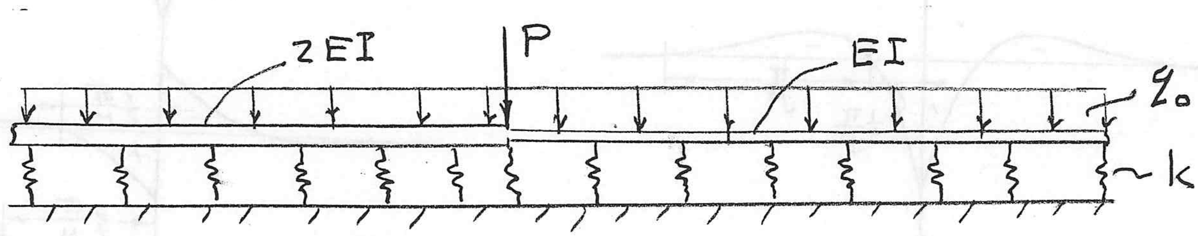
1. Side AB of a plate is loaded by a tensile stress of 10 ksi and by a shear stress of 5 ksi. The strain along AB at point O is measured to be 0.0002.

   a. Calculate the stresses and strains at point O relative the frame $(x-y)$O given that E = 10,000 ksi and $\nu = 0.3$ (assume also that a state of plane exists).

   b. If all the loads applied to the plate are increased proportionately such that the stresses are $\alpha(\sigma_x, \sigma_y, \sigma_{xy})$ where α = const., find the maximum allowable value of α if the material yields at an axial stress of 70 ksi.
   Hint : Use the von Mises yield criterion.

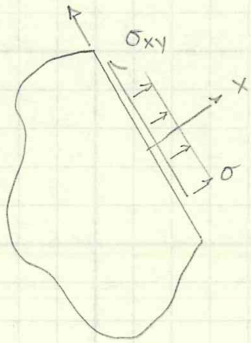


2. A long beam is made of two halves of different bending rigidity, with the left half having rigidity $2EI$ and the right half EI (see figure). The beam rests on an elastic foundation with stiffness k per unit length, carries a uniform load q_o per unit length and a point force P at the point of discontinuity. Sketch the deflected shape of the beam and calculate expressions for the deflections in the two halves. For a bonus of 2 points sketch also the deformed shape if the load P is not applied.



QUIZ #1

1.



$\sigma_{xy} = 5\,\text{ksi}$, $\sigma_x = 10\,\text{ksi}$

$E = 10000\,\text{ksi}$

$\nu = 0.3$

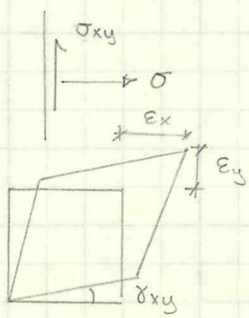
$\varepsilon_y = 0.0002$

$$\varepsilon_y = \frac{\sigma_y}{E} - \left[\nu \frac{\sigma_x}{E}\right]$$

$\sigma_z = 0$ – plane strain

$$\varepsilon_x = \frac{\sigma_x}{E} - \left[\nu \frac{\sigma_y}{E}\right]$$

$$\gamma_{xy} = \frac{\sigma_{xy}}{G}, \quad G = \frac{E}{2(1+\nu)}$$



$$\sigma_y = \varepsilon_y E + \nu \sigma_x$$
$$= (0.0002)(10{,}000\,\text{ksi}) + 0.3(10\,\text{ksi})$$
$$= 5\,\text{ksi}$$

$$\varepsilon_x = \frac{10\,\text{ksi}}{10000\,\text{ksi}} - 0.3\,\frac{(5\,\text{ksi})}{10000\,\text{ksi}} = 8.5\times10^{-4}$$

$$\gamma_{xy} = \frac{\sigma_{xy}}{E}\cdot 2(1+\nu) = \frac{5\,\text{ksi}}{10000\,\text{ksi}}\cdot 2(1+0.3)$$
$$= 0.0013$$ ✓

a. $\sigma_x = 10\,\text{ksi}$ ✓, $\varepsilon_x = 0.00085$ ✓
$\sigma_y = 5\,\text{ksi}$ ✓, $\varepsilon_y = 0.0002$ ✓
$\sigma_{xy} = 5\,\text{ksi}$ ✓, $\gamma_{xy} = 0.0013$ ✓

von Mises: $\sigma_o^2 = \sigma_x'^2 + \sigma_y'^2 - \sigma_x'\sigma_y' + 3\sigma_{xy}'^2$

$\sigma_o = 70\,\text{ksi}$

$\sigma_x' = \alpha\sigma_x$
$\sigma_y' = \alpha\sigma_y$
$\sigma_{xy}' = \alpha\sigma_{xy}$

$$(70\,\text{ksi})^2 = \alpha^2\left[(10\,\text{ksi})^2 + (5\,\text{ksi})^2 - (10\,\text{ksi})(5\,\text{ksi}) + 3(5\,\text{ksi})^2\right]$$

$$4900\,\text{ksi}^2 = \alpha^2\,(150\,\text{ksi}^2)$$

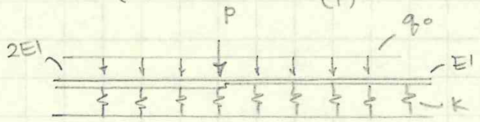
$$\alpha = 5.72$$

b. $\alpha = 5.72$ ✓

20/20

QUIZ #1

2.



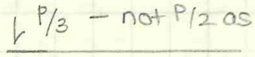
$v_p = \frac{q_0}{K}$, add to v at end ✓

$\mu_1 = \left(\frac{K}{4EI}\right)^{1/4} \quad \mu_2 = \left(\frac{K}{8EI}\right)^{1/4}$ ✓ 2

$v_h = e^{-\mu x}(c_3 \sin \mu x + c_4 \cos \mu x)$ ✓

$c_1 = c_2 = 0$ as $v_h'(\infty) = v_h(\infty) = 0$ 5

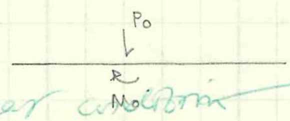
$v_{h1} = e^{-\mu_1 x}(c_3 \sin \mu_1 x + c_4 \cos \mu_1 x)$ ✓



– not P/2 as beam is less stiff, takes less load

o.k. but how can you decide 1/3 P, 2/3 P from the beginning

X -4



shear condition is not correct.

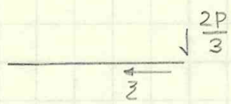
$V(0) = \frac{-P}{3} = -\frac{P_0}{2} - \frac{M_0\mu_1}{2}$, $P_0 = \frac{2P}{3} - \frac{M_0\mu_1}{2}$

$v(0) = \Delta = \frac{P_0\mu_1}{2K} + \frac{M_0\mu_1^2}{K}$ ✓, $\Delta = \frac{P\mu_1}{3K} - \frac{M_0\mu_1^2}{4K} + \frac{M_0\mu_1^2}{K}$

1.5

$\Delta = \frac{P\mu_1}{3K} + \frac{3}{4}\frac{M_0\mu_1^2}{K}$

similarly,



$V(0) = \frac{-2P}{3} = -\frac{P_1}{2} - \frac{M_1\mu_2}{2}$

$v(0) = \Delta = \frac{P_1\mu_2}{2K} + \frac{M_1\mu_2^2}{K}$ ✓, $P_1 = \frac{4P}{3} - \frac{M_1\mu_2}{2}$

$\Delta = \frac{2P}{3K}\mu_2 - \frac{M_1\mu_2^2}{4K} + \frac{M_1\mu_2^2}{K}$

$\Delta = \frac{2P}{3K}\mu_2 + \frac{3}{4}\frac{M_1\mu_2^2}{K}$

$$\frac{P\mu_1}{3K} + \frac{3}{4}\frac{M_0\mu_1^2}{K} = \frac{2P\mu_2}{3K} + \frac{3}{4}\frac{M_1\mu_2^2}{K}$$

$$P\left(\frac{\mu_1}{3K} - \frac{2\mu_2}{3K}\right) = \frac{3}{4K}\left(M_1\mu_2^2 - M_0\mu_1^2\right)$$

$$\frac{1}{\mu_2^2}\left[\frac{4K}{3}\frac{P}{3K}(\mu_1 - 2\mu_2) + M_0\mu_1^2\right] = M_1, \quad M_0 = \frac{-1}{\mu_1^2}\left[\frac{4P}{9}(\mu_1 - 2\mu_2) - M_1\mu_2^2\right]$$

$$v(\xi) = \frac{P_1\mu_2}{2K}A(\xi) + \frac{M_1\mu_2^2}{K}B(\xi)$$

$$v(x) = \frac{P_0\mu_1}{2K}A(x) + \frac{M_0\mu_1^2}{K}B(x)$$

Also know $V_1(0) + V_2(0) = -P$ ✓

$$\frac{-P_0}{2} - \frac{M_0\mu_1}{2} - \frac{P_1}{2} - \frac{M_1\mu_2}{2} = -P \qquad \frac{M_0\mu_1}{2} = \frac{-M_1\mu_2}{2}, \text{ or } M_0\mu_1 = -M_1\mu_2$$

no slope condition -1.5
no moment condition -1.5

QUIZ #1

2. solve for M_1, M_0, P_1, P_0

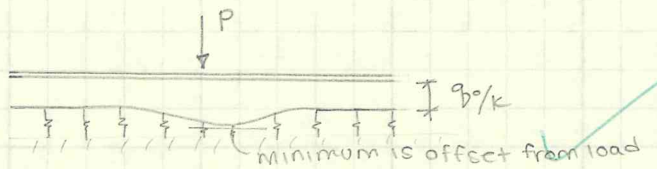
Keys: – $v_{right} = v_{left}$ – two different μ's

– $v'(0) \neq 0$
not necessarily

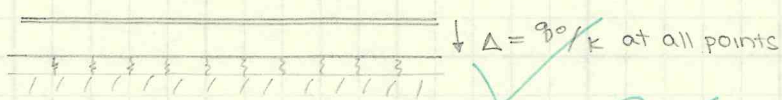
– shear is not split to either side evenly
$V_{left} = 2/3 P, \; V_{right} = P/3$

– At end, add constant $v_p = q_0/k$ to answer

Deflected shape:



Without load P:



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More math...

$$\frac{1}{\mu_2^2}\left[\frac{4P}{9}(\mu_1-2\mu_2) - M_1\mu_2\mu_1\right] = M_1$$

$$\left[\frac{4P}{9}\mu_1 - \frac{8P}{9}\mu_2\right]\frac{1}{\mu_2^2} = M_1 + M_1\frac{\mu_1}{\mu_2}$$

$$M_1 = \frac{4P/9\,(\mu_1-2\mu_2)}{\mu_2^2\,(1+\mu_1/\mu_2)}$$

$$P_1 = \frac{4P}{3} - \frac{4P/9\,(\mu_1-2\mu_2)}{2\mu_2+2\mu_1}$$

$$v_h(\xi) = \frac{P_1\mu_2}{2k} + \frac{M_1\mu_2^2}{k}$$, $\xi > 0$ defined to the left

$$M_0 = \frac{-4P/9\,(\mu_1-2\mu_2)}{\mu_1\mu_2+\mu_1^2}$$

$$P_0 = \frac{2P}{3} + \frac{4P/9\,(\mu_1-2\mu_2)}{2(\mu_2+\mu_1)}$$

$$v_h(x) = \frac{P_0\mu_1}{2k} + \frac{M_0\mu_1^2}{k}$$, $x > 0$ defined to the right

check $v_h(\xi=0) = v_h(x=0)$

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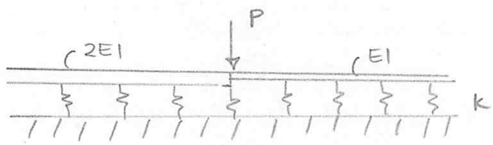
EM339: MATERIALS

2.



$v_1(\xi) = e^{-\mu_1 \xi}(C_3 \sin\mu_1\xi + C_4\cos\mu_1\xi)$ $\qquad$ $v_2(x) = e^{-\mu_2 x}(D_3\sin\mu_2 x + D_4\cos\mu_2 x)$

Boundaries:

(1) $v_1(0) = v_2(0)$ $\qquad$ (3) $M_1(0) = M_2(0)$

(2) $v_1'(0) = -v_2'(0)$ $\qquad$ (4) $V_1(0) = P_0$; $V_2(0) = P - P_0$

(1) $\underline{C_4 = D_4}$

(2) $\mu_1 e^{-\mu_1\xi}(-C_3\sin\mu_1\xi - C_4\cos\mu_1\xi + C_3\cos\mu_1\xi - C_4\sin\mu_1\xi) = \frac{dv_1}{dx}$

$$\frac{dv_1(0)}{d\xi} = \mu_1(-C_4 + C_3)$$

Similarly, $\frac{dv_2(0)}{d\xi} = \mu_2(D_3 - D_4)$

$$-\mu_1(C_3 - C_4) = \mu_2(D_3 - D_4) \qquad \text{define } \alpha = \frac{\mu_1}{\mu_2}$$

$$D_3 = -\alpha(C_3 - C_4) + D_4 = -\alpha C_3 + \alpha C_4 + C_4$$

$$\underline{D_3 = -\alpha C_3 + C_4(1+\alpha)}$$

(3) $M_1 = -EI(2)\frac{d^2 v_1}{d\xi^2}$

$\frac{d^2 v_1}{d\xi^2} = \mu_1^2 e^{-\mu_1\xi}\left[-2C_3\cos\mu_1\xi + 2C_4\sin\mu_1\xi\right]$; canceling $-EI, 2$:

$$2\mu_1^2 e^{-\mu_1\xi}\left[-C_3\cos\mu_1\xi + C_4\sin\mu_1\xi\right] = \mu_2^2 e^{-\mu_2 x}\left[-D_3\cos\mu_2 x + D_4\sin\mu_2 x\right]$$

at $x = \xi = 0$,

$$2\mu_1^2(-C_3) = \mu_2^2(-D_3)$$

$$\underline{D_3 = 2\alpha^2 C_3}$$

EM 339: MATERIALS

2.

$$C_4 = D_4 \,, \quad D_3 = 2\alpha^2 C_3 \,, \quad D_3 = -\alpha C_3 + C_4(1+\alpha)$$

$$2\alpha^2 C_3 = -\alpha C_3 + C_4(1+\alpha)$$

$$C_3(2\alpha^2+\alpha) = C_4(1+\alpha)$$

$$C_4 = \frac{2\alpha^2+\alpha}{1+\alpha} C_3 = D_4$$

(4) $$\frac{d^2 v_1}{d\xi^2} = 2\mu_1^2 e^{-\mu_1 \xi}(-C_3 \cos\mu_1\xi + C_4 \sin\mu_1\xi)$$

$$\frac{d^3 v_1}{d\xi^3} = -2\mu_1^3 e^{-\mu_1\xi}(-C_3\cos\mu_1\xi + C_4 \sin\mu_1\xi)$$
$$+ 2\mu_1^3 e^{-\mu_1\xi}(C_3 \sin\mu_1\xi + C_4\cos\mu_1\xi)$$
$$= 2\mu_1^3 e^{-\mu_1\xi}\left[C_3(\cos\mu_1\xi + \sin\mu_1\xi) + C_4(\cos\mu_1\xi - \sin\mu_1\xi)\right]$$

$$-2EI\frac{d^3 v_1(0)}{d\xi^3} = -P_1 \,, \quad -EI\frac{d^3 v_2(0)}{dx^3} = -P + P_1 = -P + 2EI\frac{d^3 v_1(0)}{d\xi^3}$$

$$\frac{d^3 v_1(0)}{d\xi^3} = 2\mu_1^3\left[C_3 + C_4\right] \,, \quad \frac{d^3 v_2(0)}{dx^3} = 2\mu_2^3(D_3 + D_4)$$

$$-EI\left[2\mu_2^3(D_3+D_4)\right] = -P + 2EI\left[2\mu_1^3(C_3+C_4)\right]$$

$$\mu_2^3(D_3+D_4) = \frac{P}{2EI} - 2\mu_1^3(C_3+C_4)$$

$$D_3 + D_4 = \frac{P}{2EI\mu_2^3} - \frac{2\mu_1^3}{\mu_2^3}(C_3+C_4) \qquad \mu_2^4 = \frac{K}{4EI}$$

$$\uparrow \qquad \uparrow$$

$$2\alpha^2 C_3 + D_4 = \frac{2P\mu_2}{k} - \frac{1}{\alpha}(C_3 + D_4)$$

$$D_4\left(1+\frac{1}{\alpha}\right) = \frac{2P\mu_2}{K} + C_3(-2\alpha^2 - 1/\alpha)$$

$$\frac{\alpha(2\alpha+1)(\alpha+1)}{\alpha(\alpha+1)} C_3 + \frac{2\alpha^3+1}{\alpha} \cdot C_3 = \frac{2P\mu_2}{K}$$

simplifies to: $$\left[\frac{2\alpha^3+2\alpha^2+\alpha+1}{\alpha}\right] C_3 = \frac{2P\mu_2}{K}$$

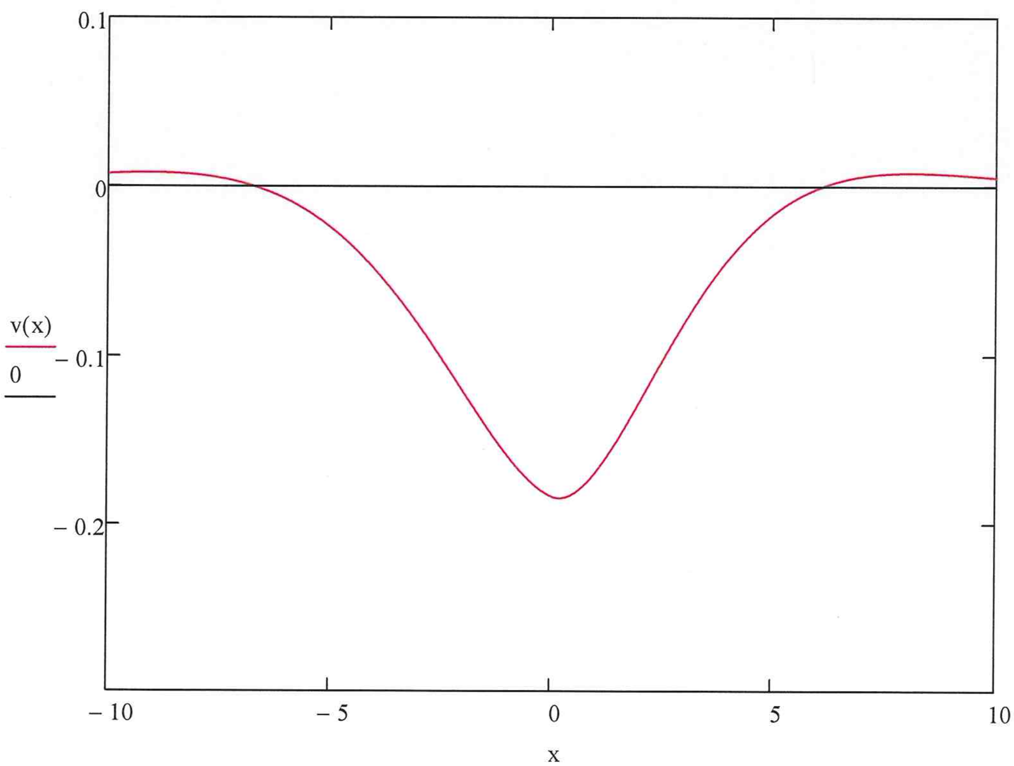
$P := 1 \qquad EI := 10$

$k := 1 \qquad \mu_1 := \left(\frac{k}{8\cdot EI}\right)^{0.25} \qquad \mu_2 := \left(\frac{k}{4\cdot EI}\right)^{0.25} \qquad \alpha := \frac{\mu_1}{\mu_2}$

$$C_3 := \frac{2\cdot P\cdot \mu_2}{k}\cdot\frac{\alpha}{2\alpha^3 + 2\cdot\alpha^2 + \alpha + 1} \qquad D_3 := 2\cdot\alpha^2\cdot C_3$$

$$C_4 := \frac{2\cdot\alpha^2 + \alpha}{1+\alpha}\cdot C_3 \qquad D_4 := C_4$$

$$v(x) := \begin{cases} -e^{-\mu_2\cdot x}\cdot\left(D_3\cdot\sin(\mu_2\cdot x) + D_4\cdot\cos(\mu_2\cdot x)\right) & \text{if } x \geq 0 \\ -e^{\mu_1\cdot x}\cdot\left(C_3\cdot\sin(-\mu_1\cdot x) + C_4\cdot\cos(-\mu_1\cdot x)\right) & \text{if } x < 0 \end{cases}$$



QUIZ #2

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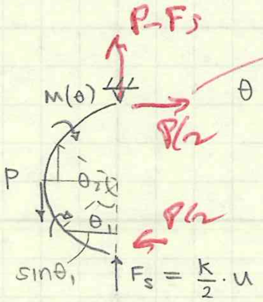
1. use symmetry to only consider one side



Moment.

$\theta \le \pi/2$ (e.g. θ_1)

$$M(\theta) = F_s \cdot \sin\theta \, R$$ ✓

$\theta \ge \pi/2$ (e.g. θ_2) ✓

$$M(\theta) = F_s R \underbrace{\cos(\theta - \pi/2)}_{=\sin\theta} + PR(1 - \underbrace{\cos(\theta - \pi/2)}_{=\sin\theta})$$

$$M(\theta) = F_s R \sin\theta + PR(1 - \sin\theta)$$

– Because the load F_s exists at B, no dummy force is needed
– Assuming linear elastic behavior,

$$U = U^c$$

$$\Delta = \frac{\partial U^c}{\partial F_s} = \frac{\partial U}{\partial F_s}$$

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$$\Delta_B = \int \frac{M(\theta)}{EI} \frac{\partial M}{\partial F_s} R\,d\theta$$

$$EI\,\Delta_B = \int_0^{\pi/2} F_s R \sin\theta \cdot R\sin\theta \cdot R\,d\theta + \int_{\pi/2}^{\pi} (F_s R\sin\theta + PR(1 - \sin\theta))R^2 \sin\theta\,d\theta$$

$$= \int_0^{\pi/2} F_s R^3 \sin^2\theta\,d\theta + \int_{\pi/2}^{\pi} \left[F_s R^3 \sin^2\theta + PR^3 \sin\theta - PR^3 \sin^2\theta\right] d\theta$$

$$\int \sin^2 = \int 1 - \cos^2 = \theta - \frac{\theta}{2} - \frac{1}{4}\sin 2\theta$$
$$= \frac{\theta}{2} - \frac{1}{4}\sin 2\theta$$

$$\frac{EI}{R^3}\Delta_B = F_s\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} + (F_s - P)\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_{\pi/2}^{\pi} - P\cos\theta\Big|_{\pi/2}^{\pi}$$

$$= F_s(\pi/4) + (F_s - P)(\pi/2 - \pi/4) + P$$

$$= \frac{3\pi}{4}F_s + (\pi/4 + 1)P \quad \text{, but } F_s = -\frac{k}{2}\Delta_B$$

$$\frac{EI}{R^3}\Delta_B = \frac{3\pi}{4} \cdot \frac{k}{2}\Delta_B + (\pi/4 + 1)P$$

$$\Delta_B = \frac{(\pi/4 + 1)P}{EI/R^3 + \frac{3\pi}{8}k}$$

↑F_s
↓F_s

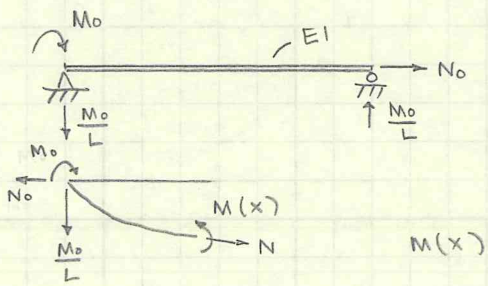
check units ✓
using reasonable values, Δ is reasonable ✓

(–) Means the spring increases Δ_B!

QUIZ #2

$M(x) = -EIv''$!!

2.



$$M(x) = M_o - N_o v(x) - \frac{M_o}{L}x$$

$$= M_o(1 - x/L) - N_o v(x)$$

$$-EI\frac{d^2v}{dx^2} + N_o v(x) = M_o(1 - x/L)$$

$$\frac{d^2v}{dx^2} \; - \; + \frac{N_o}{EI}v(x) = \frac{M_o}{EI}(1 - x/L)$$

$$-\frac{M_o}{N_o}\left(1 - \frac{x}{L}\right)$$

assume $k^2 = \frac{N_o}{EI}$

h h

$$v(x) = A\sin kx + B\cos kx + F(x) \leftarrow (?) \text{ No.}$$

$$\frac{d^2F(x)}{dx^2} = \frac{-M_o}{EI} + \frac{M_o}{EI}\cdot\frac{x}{L}$$

$$\frac{dF}{dx} = \frac{-M_o}{EI}x + \frac{M_o}{2EI}\frac{x^2}{L} + C$$

$$F(x) = \frac{-M_o x^2}{2EI} + \frac{M_o x^3}{6EIL} + Cx + D$$

$$v(x) = A\sin kx + B\cos kx - \frac{M_o}{2EI}\left(x^2 - \frac{x^3}{3L}\right) + Cx + D$$

$$v'(x) = kA\cos kx - Bk\sin kx - \frac{M_o}{EI}\left(x - \frac{x^2}{2L}\right) + C$$

$$v''(x) = -k^2A\sin kx - Bk^2\cos kx - \frac{M_o}{EI}(1 - x/L)$$

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$$v(x) = A\sin kx + B\cos kx, \quad \frac{d^2v}{dx^2} + k^2v(x) = 0, \text{ not } \frac{M_o}{EI}(1 - x/L)$$

I can't remember how to solve the differential equation that is not equal to zero.

0

Part b: $\Delta_B = \frac{-N_o L}{EA} + \frac{1}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx$ defined to the left.

with $\frac{dv}{dx}$, integrate for expression.

QUIZ #2

2. (cont'd)

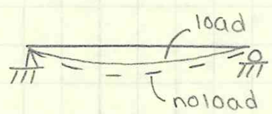
$$\frac{d^2v}{dx^2} + k^2 v(x) = \frac{M_o}{EI}(1 - x/L), \quad k^2 = \frac{N_o}{EI}$$

with general solution, solve for constants considering:

- $v(0) = 0$
- $v''(0) = 0$
- $v''(L) = 0$
- $v(L) = 0$

can also use shears $(\pm \frac{M_o}{L})$
cannot use $v'(L/2) = 0$, as shape will not be symmetric.

— Response should yield deflection less than that of a beam without axial load.



— max deflection should be left of center

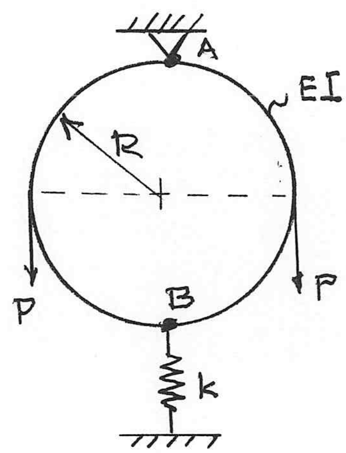
ADVANCED STRENGTH OF MATERIALS (ASE/EM 339)

QUIZ #2

Monday, November 19, 2007; 2:00 – 3:00 pm, RLM 5.116

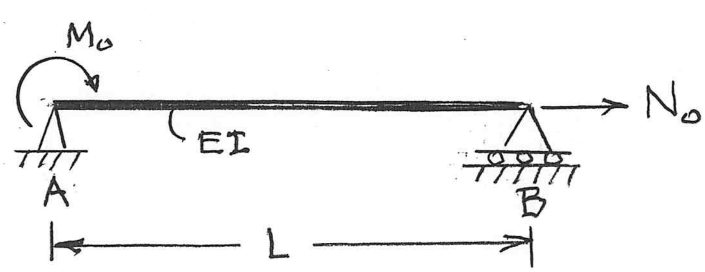
Closed Book Exam

1. A thin-walled ring structure of bending rigidity EI consists of two half rings pinned together as shown in the figure. The top hinge is fixed in space and the bottom is connected to a spring of stiffness k. Use Castigliano's theorems to calculate the deflection of point B due to the two point loads P hanging from the ring.



2. A simply supported beam carries an axial tensile load N_o and a point moment M_o applied at one end (A).

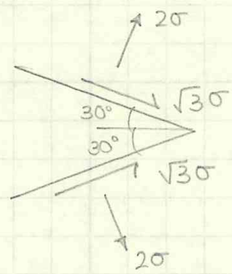
   (a) Calculate an expression for the deflected shape of the beam.
   (b) Write down an expression for the horizontal displacement of point B.



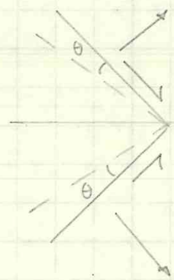
54.5/60

HOMEWORK #1

1.



recalculate stresses if planes are at 90°, not 60° — move each by 15° ?



$\sigma_y = 2\sigma\cos\theta - \sqrt{3}\,\sigma\sin\theta$

$\sigma_{xy} = -2\sigma\sin\theta + \sqrt{3}\,\sigma\cos\theta$

$\sigma_{xy} = -2\sigma\sin\theta + \sqrt{3}\,\sigma\cos\theta$

] same ✓

$\sigma_x = 2\sigma\cos\theta + \sqrt{3}\,\sigma\sin\theta$

$$\tan 2\theta_p = \left[\frac{2\sigma_{xy}}{\sigma_x - \sigma_y}\right] = \left[\frac{\cancel{2}(-2\cancel{\sigma}\sin\theta + \sqrt{3}\cancel{\sigma}\cos\theta)}{\cancel{2}\sqrt{3}\cancel{\sigma}\sin\theta}\right]$$

$$\tan 2\theta_p = \left[-\frac{2}{\sqrt{3}} + \frac{1}{\tan\theta}\right] = 2.58$$ ✓

$$2\theta_p = 68.8°, \quad \underline{\theta_p = 34.4°}$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \sigma_{xy}^2}$$

$$= 2\sigma\cos\theta \pm \left[(\sqrt{3}\sigma\sin\theta)^2 + \sigma^2(\sqrt{3}\cos\theta - 2\sin\theta)^2\right]^{\frac{1}{2}}$$ ✓

5

Not correct! $= \sigma\left[1.93 \pm 1.24\right]$

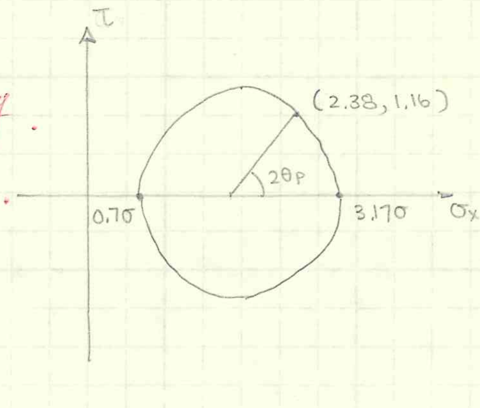
Using $\overset{\hat{n}}{\underset{\sim}{T}} = \underset{\sim}{\sigma}\,\underset{\sim}{\hat{n}}$.

$\sigma_1 = 3.17\sigma$

$\underline{\sigma_2 = 0.693\sigma}$

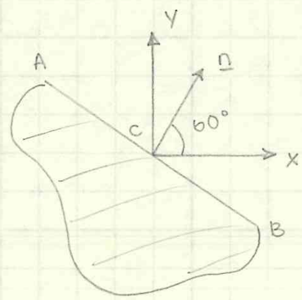
Find $\overset{\hat{n}}{T}$ for two surfaces with different $\hat{n}$, then solve σ_{xx}, σ_{yy}, σ_{xy}.

See the solution post or your TA.



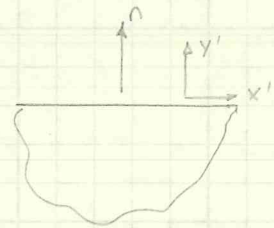
HOMEWORK #1

2.



$\tau_{max_c} = 9$ ksi

consider:

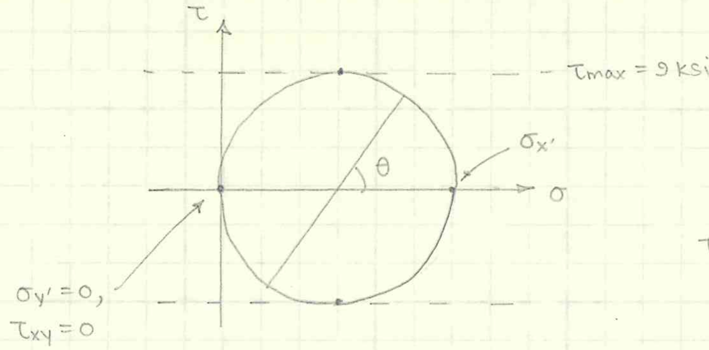


$$T = \begin{bmatrix} \sigma_{x'} & \sigma_{xy'} \\ \sigma_{xy'} & \sigma_{y'} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

but, free surface means $T = 0$,
so $\sigma_{xy'} = \sigma_{y'} = 0$

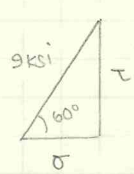
$\sigma_{x'} = ?$

Draw Mohr's circle:



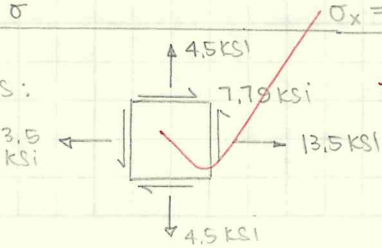
To capture $\tau_{max} = 9$ ksi and the point (0,0), $\sigma_{x'} = 18$ ksi

Now, x, y axis are 30° ↻ from x', y'
calculate stresses at $\theta = 60°$ (2×30°)



$\tau = (9\text{ksi})\sin 60° = 7.79$ ksi $= \frac{9}{2}\sqrt{3}$ ksi
$\sigma = (9\text{ksi})\cos 60 = 4.5$ ksi
$\sigma_x = 9\text{ksi} + \sigma$, $\sigma_y = 9\text{ksi} - \sigma$

State of Stress:



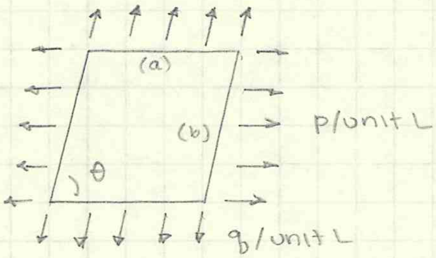
Another possible answer.

-0.5

9.5

HOMEWORK #1

3.



$\theta = 60°$

$\overset{n}{\underline{T}} = \sigma_n \underline{n}$

known — known — solve

$\underline{n}_a = \langle 0, 1, 0 \rangle$

$\underline{n}_b = \langle \sin\theta, -\cos\theta, 0 \rangle$

$\underline{T}_a = \langle q\cos\theta, q\sin\theta, 0 \rangle$

$\underline{T}_b = \langle p, 0, 0 \rangle$

Surface A:

$$T_{ax} = q\cos\theta = \sigma_x(0) + \sigma_{xy}(1) + \sigma_{xz}(0)$$
$$T_{ay} = q\sin\theta = \sigma_{yx}(0) + \sigma_y(1) + \sigma_{yz}(0)$$
$$T_{az} = 0 = \sigma_{yz}(0) + \sigma_{zy}(1) + \sigma_z(0)$$

$$q\cos\theta = \sigma_{xy}$$
$$q\sin\theta = \sigma_y$$

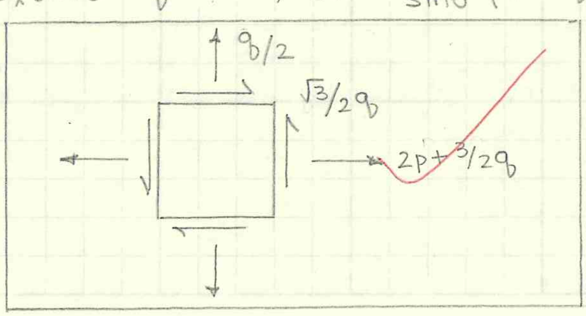
Surface B:

$$T_{bx} = p = \sigma_x \sin\theta + \sigma_{xy}(-\cos\theta)$$
$$T_{by} = 0 = \sigma_{yx}\sin\theta + \sigma_y(-\cos\theta)$$
$$\sigma_y \cos\theta = \sigma_{xy}\sin\theta$$

Combining:

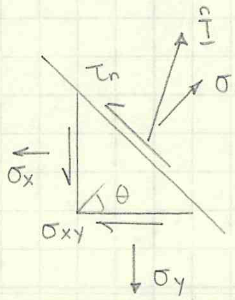
$$\sigma_y \cos\theta = q\cos\theta \cdot \sin\theta$$

$$p = \sigma_x \sin\theta - q\cos^2\theta, \quad \sigma_x = \frac{1}{\sin\theta}\left(p + q\cos^2\theta\right), \quad \theta \neq 0$$



HOMEWORK #1

4.



$$\tau_n^2 = \overset{n}{\underline{T}} \cdot \overset{n}{\underline{T}} - \sigma_n^2$$

$$\tau_n = -\sigma_x \cos\theta \sin\theta + \sigma_y \cos\theta \sin\theta + \sigma_{xy}\underbrace{(\cos^2\theta - \sin^2\theta)}_{\sigma_{xy}\cos 2\theta}$$

$$\overset{n}{\underline{T}} = \begin{bmatrix} \sigma_x n_1 + \sigma_{xy} n_2 \\ \sigma_{xy} n_1 + \sigma_y n_2 \end{bmatrix}, \quad n_1 = \cos\theta, \; n_2 = \sin\theta$$

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \sigma_{xy} \sin 2\theta$$

no angle (θ) means $\cos\theta, \sin\theta$

$$\underline{T} \cdot \underline{T} = \sigma_x^2 \cos^2 + 2\sigma_x \sigma_{xy} \cos \cdot \sin + \underbrace{\sigma_{xy}^2 \sin^2 + \sigma_{xy}^2 \cos^2}_{\sigma_{xy}^2(\sin^2 + \cos^2) = \sigma_{xy}^2} + 2\sigma_{xy}\sigma_y \sin \cdot \cos + \sigma_y^2 \sin^2$$

$$= \sigma_x^2\left(\frac{\cos 2\theta + 1}{2}\right) + \sigma_y^2\left(\frac{1 - \cos 2\theta}{2}\right) + \sigma_{xy}^2 + 2\sigma_{xy}\sin 2\theta(\sigma_x + \sigma_y)$$

$$= \cancel{\sigma_x^2}\cos^2 + \cancel{\sigma_y^2}\sin^2 + 2\sigma_{xy}\sigma_x \sin 2\theta + 2\sigma_{xy}\sigma_y \sin 2\theta + \cancel{\sigma_{xy}^2}$$

$$\sigma_n^2 = \cancel{\sigma_x^2}\cos^4 + 2\sigma_x\sigma_y\sin^2\cos^2 + 4\sigma_x\sigma_{xy}\cos^2\sin 2\theta + \cancel{\sigma_y^2}\sin^4 + 4\sigma_y\sigma_{xy}\sin^2\sin 2\theta + \sigma_{xy}^2\cancel{\sin^2 2\theta}$$

$$\tau_n^2 = \underline{\sigma_x^2\cos^2\sin^2} - 2\sigma_x\sigma_y\cos^2\sin^2 - 2\sigma_x\sigma_{xy}\sin 2\theta\cos 2\theta + \underline{\sigma_y^2\cos^2\sin^2}$$
$$+ 2\sigma_y\sigma_{xy}\sin 2\theta\cos 2\theta + \underline{\sigma_{xy}^2\cos^2 2\theta}$$

$\sigma_y^2\sin^2 - \sigma_y^2\sin^4$

$\sigma_{xy}^2(1 - \sin^2 2\theta)$

$\sigma_x^2\cos^2 - \sigma_x^2\cos^4$

goal: $\underline{T} \cdot \underline{T} - \sigma_n^2 = \tau_n^2$

HOMEWORK #1

4. (cont'd)

Removing cancelled terms,

$$\cancel{2}\cancel{\sigma_{xy}}\cancel{\sin 2\theta}\,(\sigma_x + \sigma_y - 2\sigma_x\cos^2 - 2\sigma_y\sin^2) = \cancel{2}\cancel{\sigma_{xy}}\cancel{\sin 2\theta}\cos 2\theta\,(-\sigma_x + \sigma_y)$$

$$\sigma_x(1 - 2\cos^2) + \sigma_y(1 - 2\sin^2) = \cos 2\theta\,(-\sigma_x + \sigma_y)$$

To be equal,

$$1 - 2\cos^2 = -\cos 2\theta \quad ✓$$

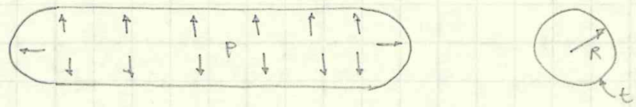
$$1 - 2\sin^2 = \cos 2\theta \quad ✓$$

Thus,

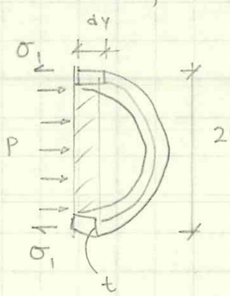
$$\underline{\overset{n}{T}} \cdot \underline{\overset{n}{T}} - \sigma_n^2 = \tau_n^2 \text{ as defined}$$

HOMEWORK #1

5.



combined hoop, axial stresses

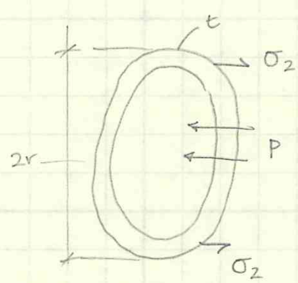


$\Sigma F = 0$

$$p\,dy(2r) - 2\sigma_1\,dy\,t = 0$$

$$2pr = 2\sigma_1 t$$

$$\sigma_1 = \frac{pr}{t}$$ ← hoop stress



$\Sigma F = 0$

$$p(\pi r^2) - \sigma_2(2\pi r)t = 0$$

$$pr = \sigma_2(2t)$$

$$\sigma_2 = \frac{pr}{2t}$$ ← axial stress

Strains: direct strain

$$\varepsilon_x = \frac{\sigma_x}{E}$$

but other directions reduce effect:

$$-\frac{\nu\sigma_y}{E}$$

$$\varepsilon_{hoop} = \frac{pr}{tE} - \frac{\nu pr}{2tE} = \frac{pr}{tE}(1 - \nu/2)$$

$$\varepsilon_{axial} = \frac{pr}{2tE} - \frac{\nu pr}{tE} = \frac{pr}{tE}(1/2 - \nu)$$

$$\varepsilon_{rad} = -\frac{\nu}{E}\left(\frac{pr}{t} + \frac{pr}{2t}\right) = -\frac{\nu pr}{Et}\frac{3}{2}$$

$$\boxed{\begin{aligned}\varepsilon_{hoop} &= \frac{pr}{tE}(1-\nu/2)\\ \varepsilon_{axial} &= \frac{pr}{tE}(1/2-\nu)\\ \varepsilon_{radial} &= -\frac{3}{2}\frac{\nu pr}{Et}\end{aligned}}$$

EM 339: MATERIALS | | C. HOVELL 14 Sept 07 | 7/8

HOMEWORK #1

5. (cont'd)

yield at τ_o in direct shear

$$\tau_y = k_o = \sigma_o / \sqrt{3} = \tau_o$$

$$\sigma_o = \tau_o \sqrt{3}$$

von Mises said

$$\sigma_o^2 = \sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 + 3\cancel{\tau_{xy}^2}$$ → 0 for this problem

$$3\tau_o^2 = \sigma^2 - \sigma \cdot \frac{\sigma}{2} + \frac{\sigma^2}{4} = \frac{3}{4}\sigma^2$$

$$\sigma = \frac{pr}{t}$$

$$\tau_o^2 = \frac{\sigma^2}{4}, \quad \sigma = 2\tau_o$$

$$\frac{pr}{t} = 2\tau_o$$

$$\boxed{p = \frac{2\tau_o t}{r} \text{ to cause yield}}$$

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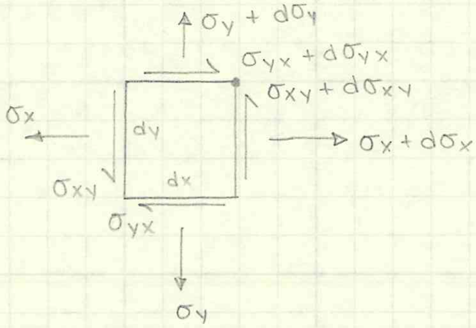
HOMEWORK #1

6.



Show that $\sigma_{xy} = \sigma_{yx}$

sum moments around • ⟲+

$\Sigma M = 0$

$$-\cancel{\sigma_y}dx\frac{dx}{2} + \sigma_{yx}dxdy + \cancel{\sigma_x}dy\frac{dy}{2} - \sigma_{xy}dydx$$
$$+(\cancel{\sigma_y} + d\sigma_y)dx\frac{dx}{2} - (\cancel{\sigma_x} + d\sigma_x)dy\frac{dy}{2} = 0$$

$$\sigma_{yx}dxdy + d\sigma_y\frac{dx^2}{2} = \sigma_{xy}dxdy + d\sigma_x\frac{dy^2}{2}$$

$dx^2, dy^2 \to 0$, as dx and dy are small, squaring makes them smaller...

$$\sigma_{yx}dx\,dy = \sigma_{xy}\,dx\,dy$$

$$\therefore\ \sigma_{yx} = \sigma_{xy}$$

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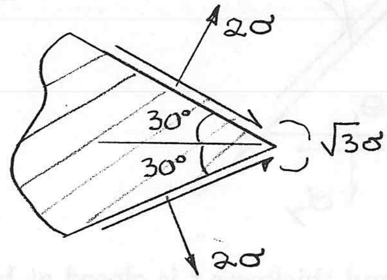
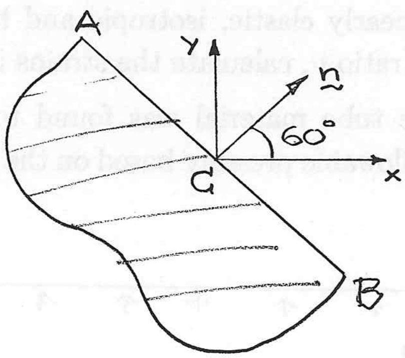
ADVANCED STRENGTH OF MATERIALS (ASE/EM 339)

Homework #1 due 14 Sept 07

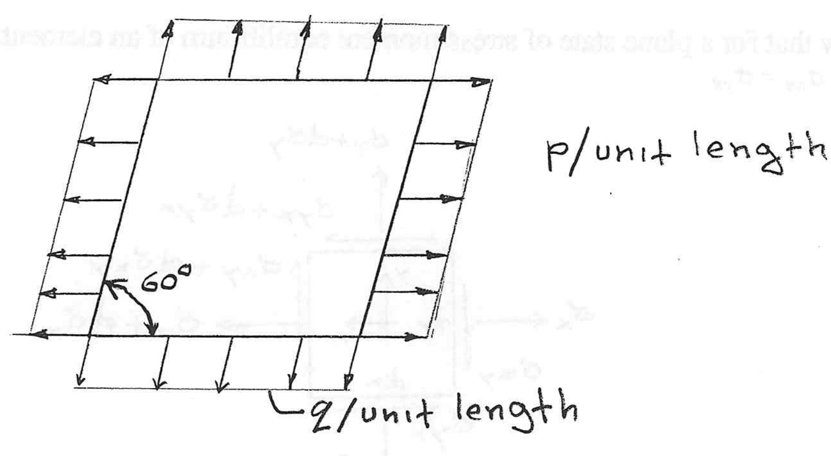
1. Given the stresses on the two intersecting surfaces shown, calculate the principal stresses at this point:



2. The edge AB of a plate is unloaded. The maximum shear stress at point C on this surface is 9 ksi. Find the state of stress at point C with respect to the system of axes given.



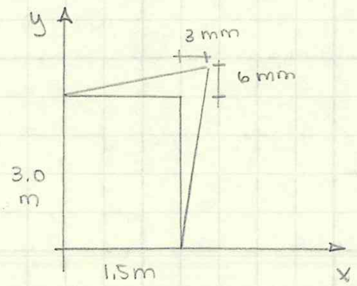
3. A flat plate of unit thickness is loaded with uniformly distributed forces p and q per unit length as shown in the figure. Calculate the stresses in the plate.
Hint: First calculate the traction vectors on the boundaries. Use these to calculate the stresses.



HOMEWORK #2

54/6

1.



$$u(x,y) = \frac{x \cdot y\ (3\,mm)}{(1.5m)(3.0m)}$$

$$v(x,y) = \frac{x \cdot y\ (6\,mm)}{(1.5m)(3.0m)}$$

u, v = displacement of a point at (x, y) during deformation

considering small strains,

$$\varepsilon_x = \frac{du}{dx}$$

$$\varepsilon_y = \frac{dv}{dy}$$

$$\gamma_{xy} = \frac{dv}{dx} + \frac{du}{dy}$$

$$\frac{\partial u}{\partial x} = \frac{0.003y}{4.5} = 0.6 \times 10^{-3}\, y \qquad \frac{\partial u}{\partial y} = 0.6 \times 10^{-3}\, x$$

$$\frac{\partial v}{\partial y} = \frac{0.006x}{4.5} = 1.2 \times 10^{-3}\, x \qquad \frac{\partial v}{\partial x} = 1.2 \times 10^{-3}\, y$$

$\varepsilon_x(B) = (0.6 \times 10^{-3})(3.0) = 1.8 \times 10^{-3}$

$\varepsilon_y(B) = (1.2 \times 10^{-3})(1.5) = 1.8 \times 10^{-3}$

$\gamma_{xy}(B) = (0.6 \times 10^{-3})(1.5) + (1.2 \times 10^{-3})(3.0) = 4.5 \times 10^{-3}$

$$\boxed{\begin{array}{c} u(x,y) = (0.6 \times 10^{-3}/m)\,xy \\ v(x,y) = (1.2 \times 10^{-3}/m)\,xy \\ \varepsilon_x = 1.8 \times 10^{-3} \\ \varepsilon_y = 1.8 \times 10^{-3} \\ \gamma_{xy} = 4.5 \times 10^{-3} \end{array}}$$

6.5

please see the solution post.
If need help. see your TA.

HOMEWORK #2

2. verify that: $\varepsilon_x = Ky^2 + Cxy$
$\varepsilon_y = K(x^2+y^2) + Cy$
$\gamma_{xy} = Cxy$

represents a possible state of strain if $C=4K$

subbing 4K in for C:

$$\varepsilon_x = Ky^2 + 4Kxy$$
$$\varepsilon_y = K(x^2+y^2) + 4Ky$$
$$\gamma_{xy} = 4Kxy$$

compatibility must exist:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\frac{\partial \varepsilon_x}{\partial y} = 2Ky + 4Kx \quad , \quad \frac{\partial^2 \varepsilon_x}{\partial y^2} = 2K$$

$$\frac{\partial \varepsilon_y}{\partial x} = 2Kx \quad , \quad \frac{\partial^2 \varepsilon_y}{\partial x^2} = 2K$$

$$\frac{\partial \gamma_{xy}}{\partial x} = 4Ky \quad , \quad \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 4K$$

6

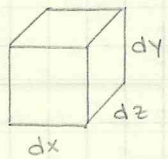
$2K + 2K = 4K$ ✓

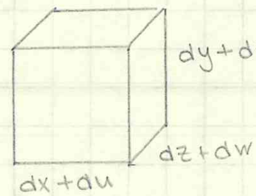
Thus, strains are compatible and state of strain is possible.

HOMEWORK #2

3. By considering the deformation of a cube, show that the change in volume for small strains is given by:

$$\Delta V = (\varepsilon_x + \varepsilon_y + \varepsilon_z)V$$





$V = dx\,dy\,dz$ $\qquad$ $V' = (dx+du)(dy+dv)(dz+dw)$

For small strains, higher order values are assumed to equal 0
Thus,

$$\varepsilon_x = \frac{du}{dx}, \quad \varepsilon_y = \frac{dv}{dy}, \quad \varepsilon_z = \frac{dw}{dz}$$

considering the original equation,

$$\Delta V = V' - V$$

$$\Delta V = V\left(V'/V - 1\right) \quad \text{vs.} \quad \Delta V = (\varepsilon_x + \varepsilon_y + \varepsilon_z)V$$

$$\frac{V'}{V} - 1 = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\frac{V'}{V} = \frac{(dx\,dy\,dz)(1 + du/dx)(1 + dv/dy)(1 + dw/dz)}{dx\,dy\,dz}$$

$$= (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)$$

$$= (1 + \varepsilon_y + \varepsilon_x + \varepsilon_x\varepsilon_y)(1+\varepsilon_z)$$

$$= 1 + \varepsilon_y + \varepsilon_x + \cancel{\varepsilon_x\varepsilon_y} + \varepsilon_z + \cancel{\varepsilon_y\varepsilon_z} + \cancel{\varepsilon_x\varepsilon_z} + \cancel{\varepsilon_x\varepsilon_y\varepsilon_z} \to 0$$

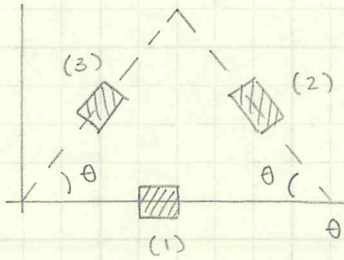
assumption stated above

$$\frac{V'}{V} = 1 + \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\frac{V'}{V} - 1 = \varepsilon_x + \varepsilon_y + \varepsilon_z \text{, as necessary} \checkmark$$

HOMEWORK #2

4.



$\theta = 60°$

$\varepsilon_1 = 5 \times 10^{-3}$
$\varepsilon_2 = -3 \times 10^{-3}$
$\varepsilon_3 = -2 \times 10^{-3}$

$$\varepsilon_x = \frac{du}{dx}, \quad \varepsilon_y = \frac{dv}{dy}, \quad \gamma_{xy} = \frac{du}{dy} + \frac{dv}{dx}$$

$$\frac{\partial^2 \varepsilon_x}{dy} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\begin{bmatrix} \varepsilon_x & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{xy}/2 & \varepsilon_y & \gamma_{yz}/2 \\ \gamma_{xz}/2 & \gamma_{yz}/2 & \varepsilon_z \end{bmatrix}$$

same as stress tensor;
same transformations
$\sigma_{xy} \to \frac{1}{2}\gamma_{xy}$

$$\varepsilon_n = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy} \sin\theta \cos\theta$$

$$\varepsilon_1 = \varepsilon_x = 5 \times 10^{-3}$$
$$\varepsilon_2 = \varepsilon_x \cos^2(120°) + \varepsilon_y \sin^2(120°) + \gamma_{xy} \sin(120°)\cos(120°) = -3 \times 10^{-3}$$
$$\varepsilon_3 = \varepsilon_x \cos^2(60°) + \varepsilon_y \sin^2(60°) + \gamma_{xy} \sin(60°)\cos(60°) = -2 \times 10^{-3}$$

3 equations, 3 unknowns:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.25 & 0.75 & -\sqrt{3}/4 \\ 0.25 & 0.75 & \sqrt{3}/4 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 5 \times 10^{-3} \\ -3 \times 10^{-3} \\ -2 \times 10^{-3} \end{bmatrix}$$

$$\boxed{\begin{aligned} \text{a. } \varepsilon_x &= 0.005 \\ \varepsilon_y &= -0.005 \\ \gamma_{xy} &= 0.00115 \end{aligned}}$$

HOMEWORK #2

4. (cont'd)

$\varepsilon_x = 0.005$
$\varepsilon_y = -0.005$
$\gamma_{xy} = 0.00115$

Calculate principal strains using transformation equations

$$\tan 2\theta_p = \left(\frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}\right) = \left[\frac{0.00115}{0.005 + 0.005}\right] = 0.115$$

$$\theta_p = \frac{1}{2}\tan^{-1}(0.115) = \underline{3.29°}$$ or 93.28

$$\varepsilon_{\substack{max \\ min}} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{1}{2}\gamma_{xy}\right)^2}$$

$$= 0 \pm \left[(0.005)^2 + \frac{1}{4}(0.00115)^2\right]^{\frac{1}{2}} = \pm 0.00503$$

$$\boxed{\begin{array}{l} \text{b.}\ \varepsilon_{max} = 5.03 \times 10^{-3} \\ \varepsilon_{min} = -5.03 \times 10^{-3} \\ \text{at } 3.29° \text{ rotation} \end{array}}$$

Maximum shear strain:

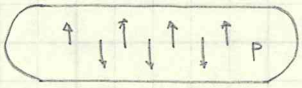
$$\gamma_{xy} = 2\sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \frac{1}{4}\gamma_{xy}^2} = 2\,\varepsilon_{max} = 0.0101$$

$$\boxed{\text{c. } \gamma_{max} = 10.1 \times 10^{-3}}$$

6

HOMEWORK #2

5.



$R = 2.0$ in $E = 30{,}000$ ksi

$t = 0.030$ in $\sigma_0 = 50$ ksi

$L = 30$ in $\nu = 0.3$

$$\varepsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \nu(\cancel{\sigma_r} + \sigma_z)\right]$$

$$\varepsilon_r = \frac{1}{E}\left[\cancel{\sigma_r} - \nu(\sigma_\theta + \sigma_z)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\cancel{\sigma_r} + \sigma_\theta)\right]$$

$\sigma_r = 0$

$\sigma_\theta = \frac{pR}{t}$

$\sigma_z = \frac{pR}{2t}$

$$\varepsilon_\theta = \frac{1}{E}\frac{pR}{t}\left(1 - \nu/2\right)$$

$$\varepsilon_r = \frac{-1}{E}\frac{\nu pR}{t}\left(1 + 1/2\right)$$

$$\varepsilon_z = \frac{1}{E}\frac{pR}{t}\left(1/2 - \nu\right)$$

Change in volume:

$$V = \pi R^2 L + 4/3\,\pi R^3$$

$\Delta L = \varepsilon_z L$ $L' = L(1 + \varepsilon_z)$

$\Delta R = R\varepsilon_\theta$ $R' = R(1 + \varepsilon_\theta)$

$$\Delta V = V' - V$$

$$\pi\left[R^2(1+\varepsilon_\theta)^2 L(1+\varepsilon_z) + 4/3 R^3(1+\varepsilon_\theta)^3 - R^2L - 4/3 R^3\right]$$

multiplying out, ignoring second-order terms,

$$\Delta V = \pi R^2\left[L\varepsilon_z + (2L + 4R)\varepsilon_\theta\right]$$

please see the solution post.

HOMEWORK #2

5. (cont'd)

von MISES: $\sigma_0^2 = \sigma_\theta^2 - \sigma_\theta \sigma_z + \sigma_z^2$

$= \sigma^2 - \frac{1}{2}\sigma^2 + \frac{1}{4}\sigma^2$

$= \frac{3}{4}\sigma$ $\quad \sigma = \frac{2}{\sqrt{3}}\sigma_0 = \frac{pR}{t}$ — divide by 2 for safety factor

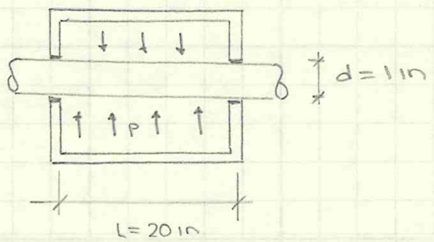
$p = \frac{2}{\sqrt{3}}\sigma_0 t/R$

sub in and solve

7.5

HOMEWORK #2

6.



$f_y = 50\text{ ksi}$
$E = 30 \times 10^6\text{ psi}$
$\nu = 0.3$

$\sigma_r = -p$

$\sigma_z = 0$

$$\sigma_\theta = \text{unknown} = \frac{E}{1+\nu}\left[\cancel{\varepsilon_\theta} + \frac{\nu}{1-2\nu}\left(\varepsilon_r + \cancel{\varepsilon_\theta} + \varepsilon_z\right)\right]$$

how do you know $\varepsilon_\theta = 0$?

$\varepsilon_\theta \neq 0$. But $\sigma_1 = \sigma_2 = -p$, $\sigma_z = 0$. it is good for you to think about this problem.

$\boxed{\varepsilon_\theta = 0} \rightarrow$

$$\varepsilon_r = \text{unknown} = \frac{1}{E}\left[\sigma_r - \nu\left(\sigma_\theta + \cancel{\sigma_z}\right)\right]$$

$$\varepsilon_z = \text{unknown} = \frac{1}{E}\left[\cancel{\sigma_z} - \nu\left(\sigma_r + \sigma_\theta\right)\right]$$

combining,

$$\sigma_\theta = \frac{E}{1+\nu}\cdot\frac{\nu}{1-2\nu}\cdot\frac{1}{E}\left[\left(\sigma_r - \nu\sigma_\theta\right) + (-\nu)\left(\sigma_r + \sigma_\theta\right)\right]$$

$$= \frac{\nu}{(1+\nu)(1-2\nu)}\left[\sigma_r(1-\nu) - 2\sigma_\theta\nu\right]$$

$$= \frac{\nu(1-\nu)}{(1+\nu)(1-2\nu)}\sigma_r - \frac{2\nu^2}{(1+\nu)(1-2\nu)}\sigma_\theta$$

$$\sigma_\theta\left[1 + \frac{2\nu^2}{(1+\nu)(1-2\nu)}\right] = \frac{\nu(1-\nu)}{(1+\nu)(1-2\nu)}\sigma_r$$

solve for σ_θ substitute into ε_z (measure of axial elongation)

$$\varepsilon_z = \frac{-\nu}{E}\left[\sigma_r + \frac{\dfrac{\nu(1-\nu)}{(1+\nu)(1-2\nu)}\sigma_r}{1 + \dfrac{2\nu^2}{(1+\nu)(1-2\nu)}}\right]$$

$$= \frac{-\nu}{E}\left[\sigma_r + \frac{\nu(1-\nu)\sigma_r}{(1+\nu)(1-2\nu) + 2\nu^2}\right] = \frac{-\nu\sigma_r}{E}(1+\nu)$$

$$\underline{\underline{\varepsilon_z = \frac{\nu p}{E}(1+\nu)}}$$

HOMEWORK #2

6. (cont'd)

Yield occurs when $\sigma_o^2 = \sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\sigma_{xy}^2$

$$\sigma_o^2 = \sigma_\theta^2 - \sigma_\theta\sigma_r + \sigma_r^2 + 0$$

$$\sigma_\theta = \nu\sigma_r$$

$$\sigma_r = -p$$

$$\sigma_o^2 = (\nu p)^2 - (\nu p)(p) + (-p)^2$$

$$= \nu^2 p^2 - \nu p^2 + p^2 = p^2(\nu^2 - \nu + 1)$$

$$p = \sqrt{\frac{(50{,}000\text{psi})^2}{0.3^2 - 0.3 + 1}} = 56.25\text{ K}$$

$$\boxed{p = 56.3\text{ kip}}$$

$$\Delta L = \varepsilon_z \cdot L_o = \frac{\nu p}{E}(1+\nu)L$$

$$= \frac{(0.3)(56.3\text{ K})}{30{,}000\text{ ksi}}(1+0.3)(20\text{ in})$$

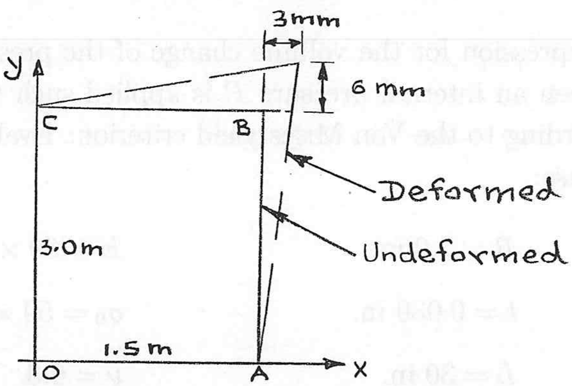
$$\boxed{\Delta L = 0.015\text{ in}}$$

10

ADVANCED STRENGTH OF MATERIALS (ASE/EM 339)

Homework #2

1. The plate in the figure is loaded in its plane and deformed as shown. Calculate the displacements $u(x, y)$ and $v(x, y)$ and the strains at point B.



2. Verify that
$$\epsilon_x = Ky^2 + Cxy$$
$$\epsilon_y = K(x^2 + y^2) + Cy$$
$$\gamma_{xy} = Cxy$$
represent a possible state of strain if $C = 4K$.

3. By considering the deformation of a cube, show that the change of volume for <u>small strains</u> is given by

$$\Delta V = (\epsilon_x + \epsilon_y + \epsilon_z)V$$

(V = volume of the cube)

4. A strain gage rosette is used for measuring the strains at a point on a plate. Given the arrangement of the rosette (see figure) and the measured strains, calculate:

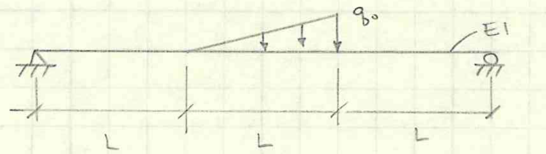
(a) ϵ_x , ϵ_y and γ_{xy} ,

(b) the principal strains and the principal directions,

HOMEWORK #3

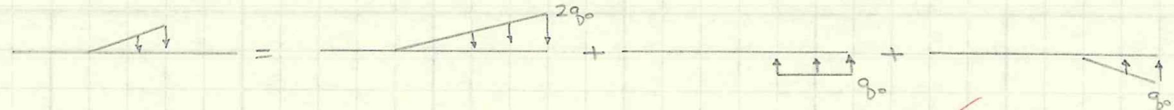
47.5/50

1. (a)



$R_1 = \frac{2}{9} q_0 L$

$R_2 = 5/18\, q_0 L$



$$M(x) = R_1 x - \frac{1}{2}(2q_0)\frac{\langle x-L\rangle^3}{2L}\cdot\frac{1}{3} + q_0\langle x-2L\rangle^2\cdot\frac{1}{2} + \frac{1}{2}q_0\langle x-2L\rangle^3\frac{1}{L}\cdot\frac{1}{3}$$

$$-EIv''(x) = \frac{2}{9}q_0 L x - \frac{q_0}{6L}\langle x-L\rangle^3 + \frac{q_0}{2}\langle x-2L\rangle^2 + \frac{q_0}{6L}\langle x-2L\rangle^3$$

$$-EIv'(x) = \frac{1}{9}q_0 L x^2 - \frac{q_0}{24L}\langle x-L\rangle^4 + \frac{q_0}{6}\langle x-2L\rangle^3 + \frac{q_0}{24L}\langle x-2L\rangle^4 + A$$

$$-EIv(x) = \frac{1}{27}q_0 L x^3 - \frac{q_0}{120L}\langle x-L\rangle^5 + \frac{q_0}{24}\langle x-2L\rangle^4 + \frac{q_0}{120L}\langle x-2L\rangle^5 + Ax + \cancel{B}$$

$v(0)=0,\quad B=0$

$v(3L)=0,$

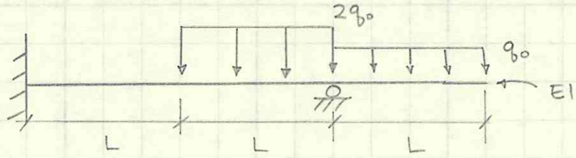
$$A = \frac{q_0}{3L}\left[\frac{L^4 3^3}{27} - \frac{(2L)^5}{120L} + \frac{L^4}{24} + \frac{L^5}{120L}\right] = -\frac{47}{180}q_0 L^3$$

$$\boxed{v(x) = \frac{-1}{EI}\left[\frac{q_0 L x^3}{27} - \frac{q_0\langle x-L\rangle^5}{120L} + \frac{q_0\langle x-2L\rangle^4}{24} + \frac{q_0\langle x-2L\rangle^5}{120L} \cancel{+} - \frac{47 q_0 L^3}{180}x\right]}$$

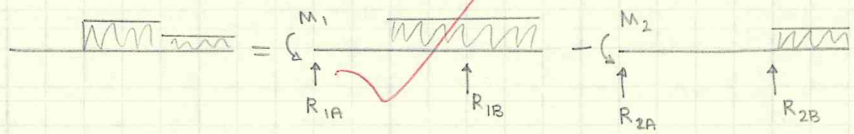
9.5/10

HOMEWORK #3

1. (b)



(1) (2)



$$R_{1A} + R_{1B} = 2q_0(2L)$$

$$M_1 = (2q_0)(2L)(2L) - R_{1B}(2L)$$

$$= 8q_0L^2 - 2LR_{1B}$$

$$R_{2A} + R_{2B} = q_0L$$

$$M_2 = q_0L\cdot\frac{5}{2}L - R_{2B}(2L)$$

$$= \frac{5}{2}q_0L^2 - 2LR_{2B}$$

(1): $M(x) = R_{1A}x - M_1 - q_0\langle x-L\rangle^2 + R_{1B}\langle x-2L\rangle$ $+\frac{q_0}{2}\langle x-2L\rangle^2$

$$-EIv' = R_{1A}\frac{x^2}{2} - M_1x - \frac{q_0}{3}\langle x-L\rangle^3 + \frac{R_{1B}}{2}\langle x-2L\rangle^2 + A_1$$

$$-EIv = R_{1A}\frac{x^3}{6} - \frac{M_1}{2}x^2 - \frac{q_0}{9}\langle x-L\rangle^4 + \frac{R_{1B}}{6}\langle x-2L\rangle^3 + A_1x + B_1$$

$v(0) = 0,\ B_1 = 0$

$v'(0) = 0,\ A_1 = 0$

$v(2L) = 0,$

$$R_{1A}\frac{4}{3}L^3 - 2M_1L^2 - \frac{q_0L^4}{9} = 0$$

$$R_{1A} = \frac{3}{4L^3}\left[\frac{q_0L^4}{9} + 2M_1L^2\right]$$

$$= \frac{q_0L}{12} + \frac{2}{3L}\left[8q_0L^2 - 2L(4q_0L - R_{1A})\right]$$

$$= \frac{q_0L}{12} + \frac{4}{3}R_{1A}$$

$$\left[\begin{array}{cc} R_{1A} = \frac{q_0L}{4} & R_{1B} = \frac{15}{4}q_0L \\ & M_1 = \frac{1}{2}q_0L^2 \end{array}\right]$$

$$-EIv_1 = \frac{q_0Lx^3}{8} - \frac{q_0L^2x^2}{4} - \frac{q_0}{9}\langle x-L\rangle^4 + \frac{5}{8}q_0L\langle x-2L\rangle^3$$

8.5

HOMEWORK #3

1. (b) (cont'd)

(2): $M(x) = R_{2A}x - M_2 - \frac{q_0}{2}\langle x-2L\rangle^2 + R_{2B}\langle x-2L\rangle$

$$-EIv' = R_{2A}\frac{x^2}{2} - M_2 x - \frac{q_0}{6}\langle x-2L\rangle^3 + \frac{R_{2B}}{2}\langle x-2L\rangle^2 + A_2$$

$$-EIv_2 = R_{2A}\frac{x^3}{6} - \frac{M_2}{2}x^2 - \frac{q_0}{24}\langle x-2L\rangle^4 + \frac{R_{2B}}{6}\langle x-2L\rangle^3 + A_2 x + B_2$$

$v_2(0) = 0, \quad B_2 = 0$

$v_2'(0) = 0, \quad A_2 = 0$

$v(2L) = 0,$

$R_{2A}\frac{4}{3}L^3 - 2M_2L^2 = 0, \quad \cancel{2}M_2\cancel{L^2} = \frac{\cancel{4}^{\,2}}{3}R_{2A}L^{\cancel{3}}$

$\frac{5}{2}q_0L^{\cancel{2}} - 2\cancel{L}(q_0L - R_{2A}) = \frac{2}{3}R_{2A}\cancel{L}$

$\frac{1}{2}q_0L = -\frac{4}{3}R_{2A}$

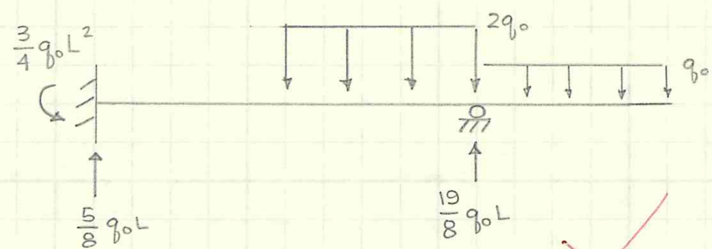
$$\boxed{R_{2A} = \frac{-3}{8}q_0L \qquad R_{2B} = \frac{11}{8}q_0L \qquad M_2 = \frac{-q_0L^2}{4}}$$

$$-EIv_2 = \frac{-q_0L}{16}x^3 + \frac{q_0L^2}{8}x^2 - \frac{q_0}{24}\langle x-2L\rangle^4 + \frac{11}{48}q_0L\langle x-2L\rangle^3$$

combine: $v = v_1 - v_2$

$$-EIv(x) = \frac{3}{16}q_0Lx^3 - \frac{3}{8}q_0L^2x^2 - \frac{q_0}{9}\langle x-L\rangle^4 + \frac{q_0}{24}\langle x-2L\rangle^4 + \frac{q_0L}{48}\langle x-2L\rangle^3$$

Reactions add, too

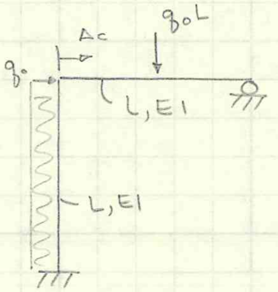


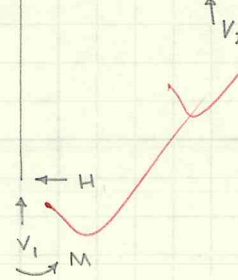
check: $R_A + R_B = 3q_0L$ ✓
$\Sigma M = 0$ ✓

$$v(3L) = \frac{q_0L^4}{36EI} \text{ down}$$

HOMEWORK #3

1. (c)

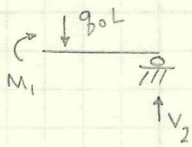


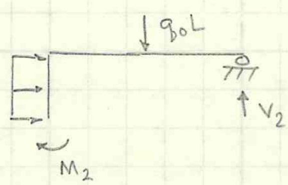


$\Sigma F_x: \; H = q_0 L$

$\Sigma F_y: \; V_1 + V_2 = q_0 L$

$\Sigma M: \; q_0 \frac{L^2}{2} + q_0 \frac{L^2}{2} - V_1 L = M$





$$M_1(x) = -EIv_1''(x) = V_2 x - q_0 L \langle x - L/2 \rangle$$

$$-EIv_1'(x) = V_2 \frac{x^2}{2} - \frac{q_0 L}{2} \langle x - L/2 \rangle^2 + A_1$$

$$-EIv_1(x) = V_2 \frac{x^3}{6} - \frac{q_0 L}{6} \langle x - L/2 \rangle^3 + A_1 x + \cancel{B_1}$$

$$v_1(0) = 0, \; B_1 = 0$$

$$M_2(y) = -EIv_2''(y) = V_2 L - q_0 \frac{L^2}{2} - q_0 \frac{y^2}{2}$$

$$-EIv_2'(y) = V_2 L y - q_0 \frac{L^2}{2} y - q_0 \frac{y^3}{6} + A_2$$

$$-EIv_2(y) = V_2 L \frac{y^2}{2} - q_0 \frac{L^2}{4} y^2 - q_0 \frac{y^4}{24} + A_2 y + B_2$$

$$v_2(0) = \Delta_c, \; B_2 = \Delta_c$$

$$v_2'(L) = 0,$$

$$0 = V_2 L^2 - q_0 \frac{L^3}{2} - q_0 \frac{L^3}{6} + A_2$$

$$\underline{A_2 = \frac{2}{3} q_0 L^3 - V_2 L^2}$$

$$v_2(L) = 0$$

$$0 = V_2 \frac{L^3}{2} - q_0 \frac{L^4}{4} - q_0 \frac{L^4}{24} + \frac{2}{3} q_0 L^4 - V_2 L^3 + \Delta_c$$

$$\underline{\Delta_c = \frac{1}{2} V_2 L^3 - \frac{3}{8} q_0 L^4}$$

$$v_2'(0) = \theta$$

$$\underline{A_2 = \theta = \frac{2}{3} q_0 L^3 - V_2 L^2}$$

HOMEWORK #3

1. (c) cont'd

$v_1(L) = 0$,

$$0 = V_2 \frac{L^3}{6} - \frac{q_o L}{6}(L/2)^3 + A_1 L$$

$$A_1 = \frac{1}{L}\left[\frac{L^4 q_o}{48} - V_2 \frac{L^3}{6}\right]$$

$$\underline{A_1 = q_o \frac{L^3}{48} - V_2 \frac{L^2}{6}}$$

$v_1'(L) = \theta$,

$$\theta = V_2 \frac{L^2}{2} - \frac{q_o L}{2}(L/2)^2 + q_o \frac{L^3}{48} - V_2 \frac{L^2}{6}$$

$$\underline{\theta = \frac{1}{3} V_2 L^2 - \frac{5}{48} q_o L^3}$$

Establishing compatibility at the joint,

$$\frac{1}{3} V_2 L^2 - \frac{5}{48} q_o L^3 = \frac{2}{3} q_o L^3 - V_2 L^2$$

$$\frac{4}{3} V_2 \cancel{L^2} = \frac{37}{48} q_o L^{\cancel{3}}$$

$$\underline{V_2 = \frac{37}{64} q_o L}$$

check using virtual work; matches that solution ✓

Solving for displacement,

$$\Delta_c = \frac{1}{2}\left(\frac{37}{64} q_o L\right) L^3 - \frac{3}{8} q_o L^4$$

$$\Delta_c = -\frac{11}{128} q_o L^4$$

divide by EI, wasn't included earlier in θ calculation

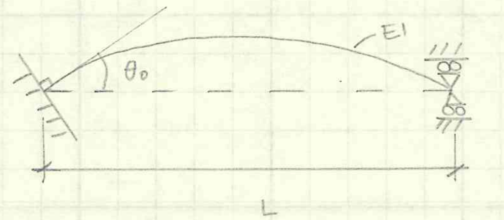
$$\boxed{\Delta_c = \frac{-11}{128} \frac{q_o L^4}{EI}}$$

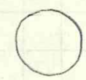
check using stiffness method; matches solution ✓

great! 10

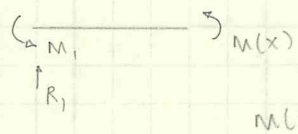
HOMEWORK #3

1. (d)





$R, t, R \gg t$



$$M(x) = R_1 x - M_1$$

$$-EIv' = \frac{R_1}{2}x^2 - M_1 x + A$$

$$-EIv = \frac{R_1}{6}x^3 - \frac{M_1}{2}x^2 + Ax + B$$

Boundaries:

$v(0) = 0$
$v'(0) = -\theta_0$
$v(L) = 0$
$v''(L) = 0$

$v(0) = 0, \; B = 0$
$v'(0) = \theta_0, \; A = EI\theta_0$
$v''(L) = 0,$

$$R_1 L = M_1$$

$v(L) = 0,$

$$\frac{R_1}{6}L^3 - \frac{R_1 L}{2}L^2 + EI\theta_0 L = 0$$

$$R_1 = \frac{3EI\theta_0}{L^2}, \quad M_1 = \frac{3EI\theta_0}{L}$$

$$-EIv(x) = EI\left[\frac{1}{2}\frac{\theta_0 x^3}{L^2} - \frac{3}{2}\frac{\theta_0 x^2}{L}\right] + EI\theta_0 x$$

$$M(x) = -EIv''(x)$$

$$v' = \frac{3}{2}\frac{\theta_0 x^2}{L^2} - 3\frac{\theta_0 x}{L} + EI\theta_0$$

$$v'' = 3\frac{\theta_0}{L^2}x - 3\frac{\theta_0}{L}, \quad M(x) = -EI\left[\frac{3\theta_0 x}{L^2} - \frac{3\theta_0}{L}\right]$$

EM 339 : MATERIALS | C. HOVELL 28 Sept 07 | 7/8

HOMEWORK #3

1. (d) cont'd

$$M(x) = -3EI\theta_0 \left(x/L^2 - 1/L \right)$$

σ_{max} at M_{max}, occurs at wall $(x=0)$

$$M_{max} = \frac{3EI\theta_0}{L}$$

$$\sigma_{max} = \frac{M_{max}\, c}{I} \quad , \quad \cancel{c = R/2} \quad \boxed{c = R}$$

$I = \pi r^3 t$, given that $r \gg t$

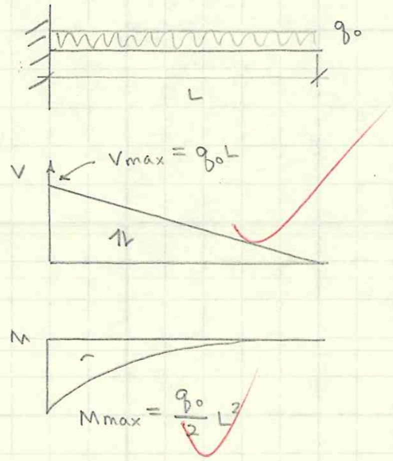
$$\sigma_{max} = \frac{3EI\theta_0 \cdot R/2}{\pi R^3 t} = \frac{3}{2} E\theta \frac{R}{L}$$

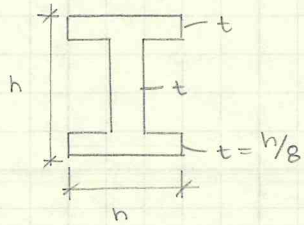
$$\boxed{\sigma_{max} = \frac{3E\theta_0 R}{\cancel{2}L} \text{ at wall}}$$

8.5

HOMEWORK #3

2.





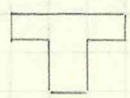
$$\sigma = \frac{Mc}{I}, \quad \tau = \frac{VQ}{It} \qquad \text{Need: } I, Q$$

$$I = \tfrac{1}{12} t h^3 + 2\left[\tfrac{1}{12}(h-t)t^3 + (h-t)t\left(\tfrac{h}{2} - \tfrac{t}{2}\right)^2\right]$$

$t = h/8$

$$= \frac{323}{6144} h^4 \quad \text{(from mathCAD)}$$

$Q = A'\bar{y}'$ of top half



$$Q = \frac{h}{2} t \left(\frac{h}{2} \cdot \frac{1}{2}\right) + (h-t)(t)(h/2 - t/2)$$

$$= \frac{h}{2} \cdot \frac{h}{8} \cdot \frac{h}{4} + \frac{7}{8} h \cdot \frac{h}{8} \cdot \frac{7}{16} h$$

$$= 0.0635 h^3 \quad (65/1024)$$

$$\sigma = \frac{Mc}{I} = \frac{q_0 L^2}{2} \cdot \frac{h/2}{\frac{323}{6144} h^3} = 4.76 \frac{q_0 L^2}{h^3}$$

$$\tau = \frac{VQ}{It} = \frac{q_0 L \cdot 0.0635 h^3}{323/6144\, h^3\, h/8} = 9.66 \frac{q_0 L}{h^2}$$

$$\boxed{\begin{array}{l} a.\ \sigma = 4.76 \dfrac{q_0 L^2}{h^3} \\ b.\ \tau = 9.66 \dfrac{q_0 L}{h^2} \end{array}}$$

Shear stress has a larger coefficient but is less dependent on geometry. Flexural stress can be very large or very small dependant on L/h

$$\sigma = \frac{32}{65} \frac{L}{h} \tau$$

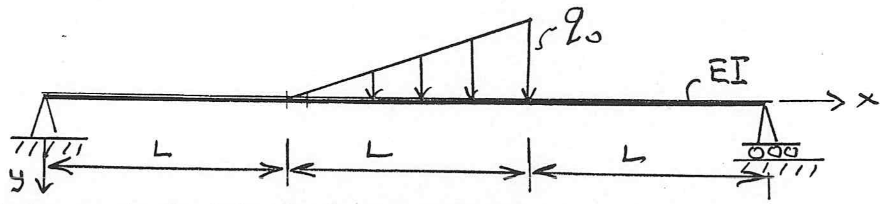
h

ADVANCED STRENGTH OF MATERIALS (ASE/EM 339)

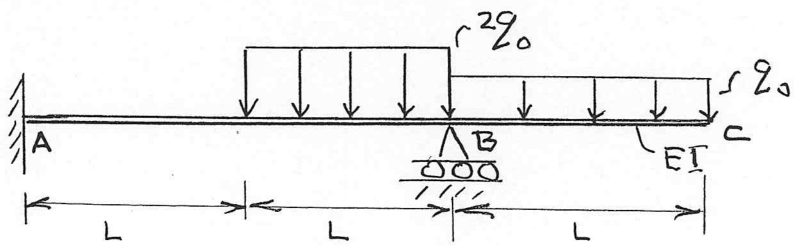
Homework #3

1. Solve the following beam problems:

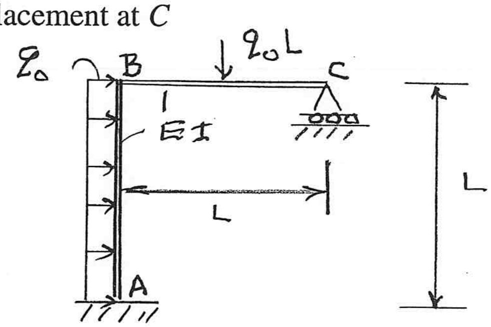
(a) Find an expression for the displacement $v(x)$



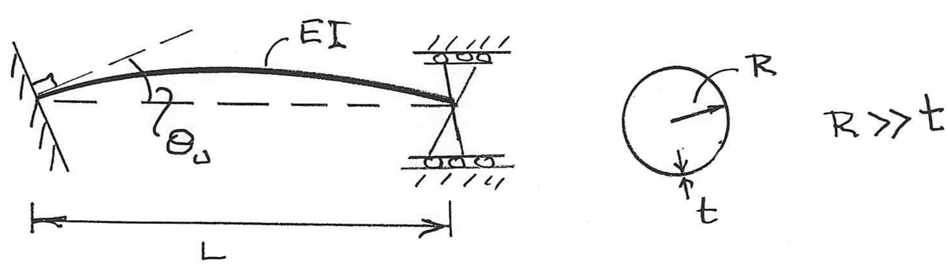
(b) Find the reactions and the displacement of *C*



(c) Find the displacement at *C*



(d) Find the maximum stress



HOMEWORK #4

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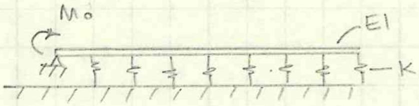
1. (a)



$$v_H = e^{\mu x}\left[c_1 \sin\mu x + c_2 \cos\mu x\right] + e^{-\mu x}\left[c_3 \sin\mu x + c_4 \cos\mu x\right]$$

considering $v(\infty)=0,\ v'(\infty)=0$

$$c_1 = c_2 = 0$$

$$v(x) = e^{-\mu x}\left[c_3 \sin\mu x + c_4 \cos\mu x\right]$$

$v(0)=0$, so $c_4 = 0$

$$v(x) = e^{-\mu x} c_3 \sin\mu x$$

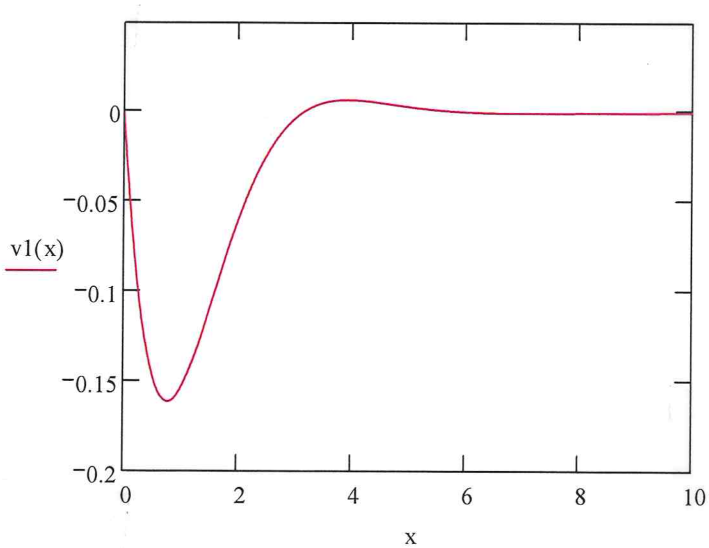
$$v'(x) = -\mu e^{-\mu x} c_3 \sin\mu x + \mu e^{-\mu x} c_3 \cos\mu x$$

$$v''(x) = -2\mu e^{-\mu x} c_3 \cos\mu x$$

$$v''(0) = \frac{M_o}{EI}, \quad c_3 = \frac{-M_o}{2\mu^2} \cdot \frac{1}{EI}$$

$$\boxed{v(x) = \frac{-M_o e^{-\mu x}}{2\mu^2 EI} \sin\mu x}$$

or $v(x) = -\frac{M_o}{2\mu^2} B(x) \cdot \frac{1}{EI}$



For all plots, constants $(k, \mu, a, \ldots)$ equal 1.0

Good!

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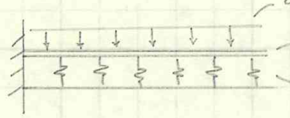
HOMEWORK #4

1. (b)



v_H as before, $v_p = \frac{q_0}{k}$ down

$$v_H(\infty) = 0, \quad v'_H(\infty) = 0 \longrightarrow c_1 = c_2 = 0$$

$$v(x) = e^{-\mu x}\left[c_3 \sin \mu x + c_4 \cos \mu x\right] + \frac{q_0}{k}$$

$$v(0) = 0, \quad c_4 = \frac{-q_0}{k}$$

$$v'(x) = \mu e^{-\mu x}\left[c_3(\cos \mu x - \sin \mu x) - c_4(\sin \mu x + \cos \mu x)\right]$$

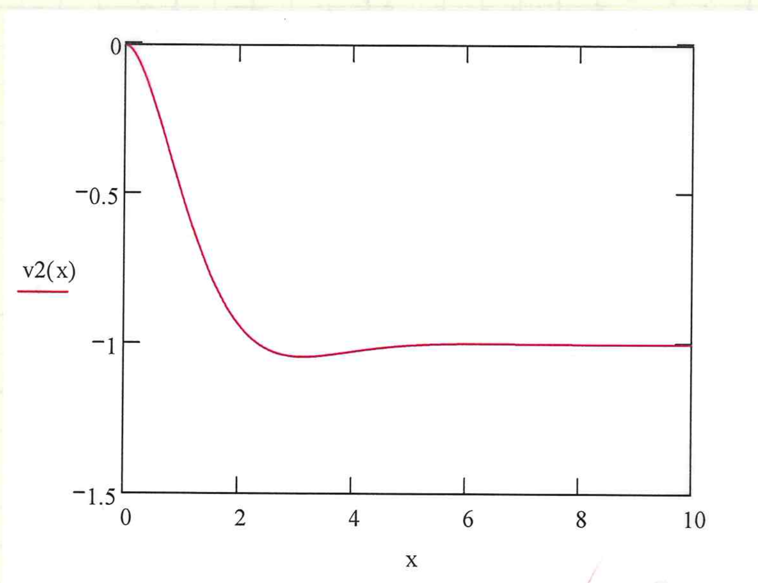
$$v'(0) = 0 = \mu c_3 - \mu c_4$$

$$c_3 = c_4 = \frac{-q_0}{k}$$

$$\boxed{v(x) = \frac{-q_0}{k} e^{-\mu x}\left[\sin \mu x + \cos \mu x\right] + \frac{q_0}{k}}$$

positive defined downward

$$\text{or } v(x) = \frac{-q_0}{k} A(x) + \frac{q_0}{k}$$



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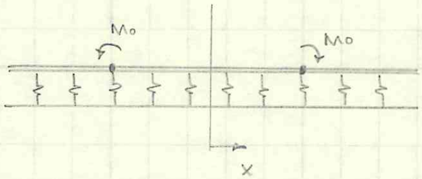
HOMEWORK #4

2.



$$v(x) = e^{-\mu x}\left[C_3 \sin \mu x + C_4 \cos \mu x\right]$$

$$v'(x) = \mu e^{-\mu x}\left[C_3(\cos \mu x - \sin \mu x) - C_4(\sin \mu x + \cos \mu x)\right]$$

$$v'(0) = 0, \quad C_3 = C_4$$

$$v''(x) = \mu^2 e^{-\mu x} C\left[2\sin \mu x - 2\cos \mu x\right]$$

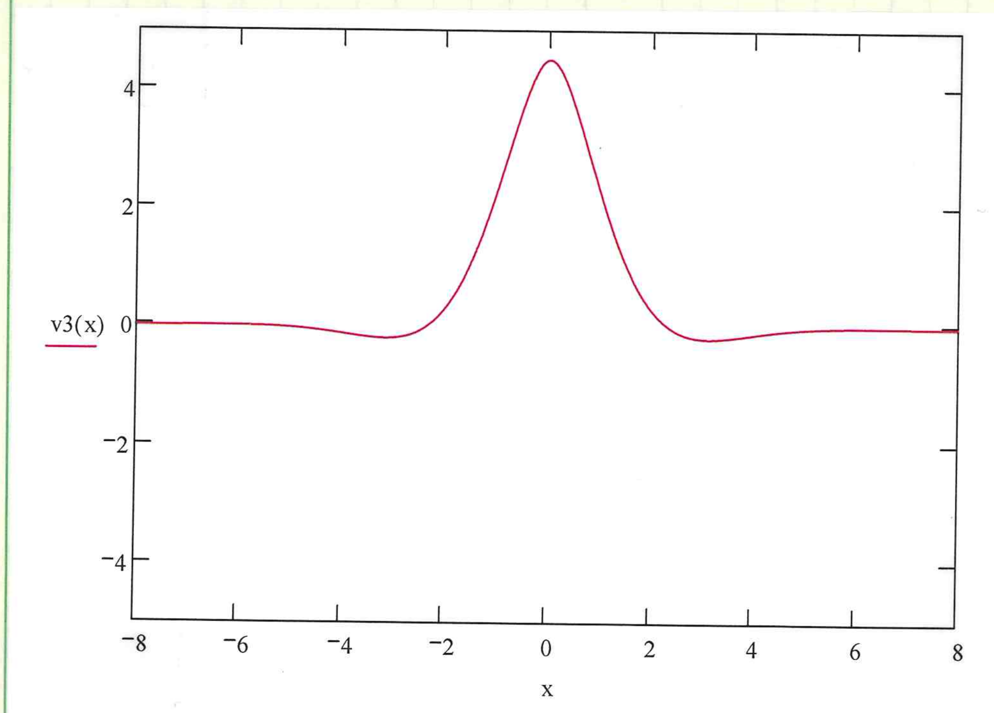
No, $-EIv''(a) = M$ there. M is the moment at that section, maybe is not the applied moment Mo.

$$v''(a) = \frac{M_o}{EI}, \quad C = \frac{M_o}{2EIe^{-\mu a}\mu^2(\sin \mu a - \cos \mu a)}$$

$$\boxed{v(x) = \frac{M_o e^{-\mu x}(\sin \mu x + \cos \mu x)}{2EI\mu^2 e^{-\mu a}(\sin \mu a - \cos \mu a)}}$$

for $x \geq 0$

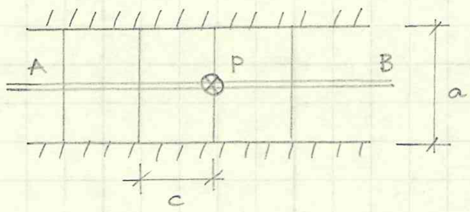
for $x < 0$, use $f(-x)$



8.5

HOMEWORK #4

3.



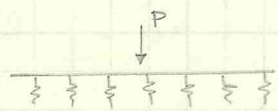
$c = 12$ in
$b = 64$ ft
$a = 16$ ft
$EI = 2E\hat{I}$

$$k = \frac{48E\hat{I}}{a^3 c}$$

As an infinite elastic foundation,

$$v(x) = e^{-\mu x}\left[c_1 \sin\mu x + c_2 \cos\mu x\right]$$

using boundaries, same as standard



$$\text{a.}\quad v(x) = \frac{P\mu}{2k} e^{-\mu x}\left(\sin\mu x + \cos\mu x\right)$$

max deflection at $x = 0$

$$v(x) = \frac{P\mu}{2k}, \quad \mu = \left[\frac{k}{4EI}\right]^{1/4}$$

$$k = \frac{48E\hat{I}}{a^3 c}$$

S6×12.5: $I_x = 22.1\ \text{in}^4$
$d = 6$ in

$$\sigma_y = \frac{Mc}{I}, \quad M = \frac{\sigma_y I}{c} = \frac{(20\text{ksi})(22.1\text{in}^4)}{3\text{in}} = 147.3\ \text{k}\cdot\text{in}$$

simply supported beam with point load,

$$M_{max} = \frac{PL}{4}, \quad P = \frac{4M_{max}}{a}$$

load doesn't all go to central beam:

P on beam → total P

$$\delta = \frac{Pa^3}{48E\hat{I}} = v(0) = \frac{P\mu}{2k}$$

$$\frac{4M_{max}a^2}{48E\hat{I}} = \frac{P\mu}{2}\cdot\frac{a^3 c}{48E\hat{I}}$$

$$8M_{max} = P\mu a \cdot c$$

$$P = \frac{8M_{max}}{\mu a c} = \frac{8(147.3\ \text{k}\cdot\text{in})}{(192\text{in})(12\text{in})}\cdot\left[\frac{48E\hat{I}/a^3c}{4EI}\right]^{-1/4}$$

$$4EI \to 2E\hat{I}, \quad \frac{6}{a^3 c}$$

b. $P_{max} = 31.4$ k

HOMEWORK #4

3. (cont'd)

$$v(0) = \frac{P\mu}{2k} = \frac{(31.4\,k)}{2} \cdot \frac{a^3 c}{48E\hat{I}} \cdot \left[\frac{48E\hat{I}/a^3c}{4(2E\hat{I})}\right]^{1/4}$$

$$= 15.7\,k\,\frac{(192\,in)^3(12\,in)}{48(29000\,ksi)(22.1\,in^4)}\left[\frac{6}{(192\,in)^3(12\,in)}\right]^{1/4}$$

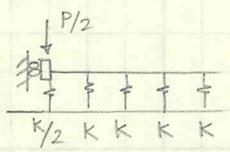
c. $v(0) = 0.71\,in$

$$v(32\,ft = 384\,in) = 0.71\,in\; e^{-\mu(384\,in)}\left[\sin\mu 384\,in + \cos\mu 384\,in\right]$$

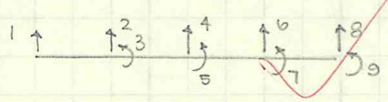
$$\mu = \left[6/(192\,in)^3(12\,in)\right]^{1/4} = 0.0163$$

c. $v(384\,in) = 1.32\times10^{-3}\,in$

Exact solution



$$K = \frac{48E\hat{I}}{a^3}$$



form k matrix:

$$k = \begin{bmatrix} 12EI/L^3 + k/2 & -12EI/L^3 & 6EI/L^2 & 0 & 0 & \cdots \\ & 24EI/L^3 + k & 0 & -12EI/L^3 & 6EI/L^2 & \cdots \\ & & \text{etc.} & & & \end{bmatrix}$$

L = 8 ft, spacing between beams

$$P = \begin{bmatrix} -P/2 \\ 0 \\ 0 \\ \cdots \end{bmatrix} \qquad u = k^{-1} \cdot P$$

on next page

EM 339: MATERIALS

HOMEWORK #4

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$$
k_{beam}(L_o, EI, k) := \begin{pmatrix}
\frac{12\cdot EI}{L_o^{3}}+\frac{k}{2} & \frac{-12\cdot EI}{L_o^{3}} & \frac{6\cdot EI}{L_o^{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-12\cdot EI}{L_o^{3}} & \frac{24\cdot EI}{L_o^{3}}+k & 0 & \frac{-12\cdot EI}{L_o^{3}} & \frac{6\cdot EI}{L_o^{2}} & 0 & 0 & 0 & 0 \\
\frac{6\cdot EI}{L_o^{2}} & 0 & \frac{8\cdot EI}{L_o} & \frac{-6\cdot EI}{L_o^{2}} & \frac{2\cdot EI}{L_o} & 0 & 0 & 0 & 0 \\
0 & \frac{-12\cdot EI}{L_o^{3}} & \frac{-6\cdot EI}{L_o^{2}} & \frac{24\cdot EI}{L_o^{3}}+k & 0 & \frac{-12\cdot EI}{L_o^{3}} & \frac{6\cdot EI}{L_o^{2}} & 0 & 0 \\
0 & \frac{6\cdot EI}{L_o^{2}} & \frac{2\cdot EI}{L_o} & 0 & \frac{8\cdot EI}{L_o} & \frac{-6\cdot EI}{L_o^{2}} & \frac{2\cdot EI}{L_o} & 0 & 0 \\
0 & 0 & 0 & \frac{-12\cdot EI}{L_o^{3}} & \frac{-6\cdot EI}{L_o^{2}} & \frac{24\cdot EI}{L_o^{3}}+k & 0 & \frac{-12\cdot EI}{L_o^{3}} & \frac{6\cdot EI}{L_o^{2}} \\
0 & 0 & 0 & \frac{6\cdot EI}{L_o^{2}} & \frac{2\cdot EI}{L_o} & 0 & \frac{8\cdot EI}{L_o} & \frac{-6\cdot EI}{L_o^{2}} & \frac{2\cdot EI}{L_o} \\
0 & 0 & 0 & 0 & 0 & \frac{-12\cdot EI}{L_o^{3}} & \frac{-6\cdot EI}{L_o^{2}} & \frac{12EI}{L_o^{3}}+k & \frac{-6\cdot EI}{L_o^{2}} \\
0 & 0 & 0 & 0 & 0 & \frac{6\cdot EI}{L_o^{2}} & \frac{2\cdot EI}{L_o} & \frac{-6\cdot EI}{L_o^{2}} & \frac{4\cdot EI}{L_o}
\end{pmatrix}
\qquad
F_b(P) := \begin{pmatrix} \frac{-P}{2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$EI_{prime} := 29000\cdot 22.1$ $\quad a := 192$ $\qquad EI_{prime} = 6.409\times 10^{5}$

$EI := 2\cdot EI_{prime}$ $\quad L_o := 96$ $\qquad EI = 1.282\times 10^{6}$

$k := \frac{48\cdot EI_{prime}}{a^{3}}$ $\quad P := 31.37$ $\qquad k = 4.346$

$$u(L_o, EI, k, P) := k_{beam}(L_o, EI, k)^{-1}\cdot F_b(P)$$

$$u(L_o, EI, k, P)\ \text{float}, 3 \rightarrow \begin{pmatrix} -3.33 \\ -1.85 \\ 0.0208 \\ -0.354 \\ 0.00985 \\ 0.126 \\ 0.00152 \\ 0.136 \\ -0.000604 \end{pmatrix}$$

obviously incorrect. I can't find where the error exists. Classmates using a similar method (same k matrix, etc) got $v = 0.83$ in, closer to approximation.

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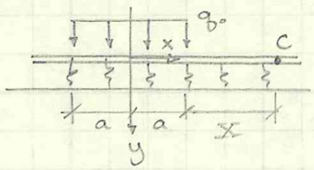
HOMEWORK #4

4.



$$dv(x) = \frac{q_0 d\xi \mu}{2k}\left[\cos(\mu(x-\xi)) + \sin(\mu(x-\xi))\right]e^{-\mu(x-\xi)}$$

$$v(x) = \frac{q_0\mu}{k}\int_0^a e^{-\mu(x-\xi)}\left[\cos\mu(x-\xi) + \sin\mu(x-\xi)\right]d\xi$$

why you didn't consider ξ from $\boxed{-a \text{ to } a}$

$$v(x) = \frac{q_0\mu}{k}\left[\frac{1}{\mu}e^{-\mu(x-\xi)}\cos\mu(x-\xi)\right]\Big|_0^a$$

$$v(x) = \frac{q_0}{k}e^{-\mu(x-a)}\cos\mu(x-a) - \frac{q_0}{k}e^{-\mu x}\cos\mu x$$

check $v(\infty)=0$ $\quad \frac{q_0}{k}(1) - \frac{q_0}{k} = 0$

Now, $x = X + a$

$$v(X+a) = \frac{q_0}{k}\left[e^{-\mu X}\cos\mu X - e^{-\mu(X+a)}\cos\mu(X+a)\right]$$

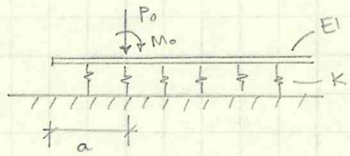
8

check: if $X \to \infty$, $v \to 0$

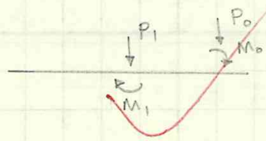
$\frac{q_0}{k}(1-1)=0$ ✓

HOMEWORK #4

5.



consider infinite beam:



$$V(0) = 0 = \frac{-P_1}{2} - \frac{M_1}{2}\mu + \frac{P_o}{2}D(a) - \frac{M_o}{2}\mu A(a)$$

$$M(0) = 0 = \frac{P_1}{4\mu} + \frac{M_1}{2} - \frac{P_o}{4\mu}C(a) - \frac{M_o}{2}D(a)$$

two unknowns:

$$\begin{bmatrix} -1/2 & -\mu/2 \\ 1/4\mu & 1/2 \end{bmatrix} \begin{bmatrix} P_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} \frac{M_o}{2}\mu A(a) - \frac{P_o}{2}D(a) \\ \frac{M_o}{2}D(a) + \frac{P_o}{4\mu}C(a) \end{bmatrix}$$

$$P_1 = P_o\left[2D(a) - C(a)\right] - 2\mu M_o\left[A(a) + D(a)\right]$$

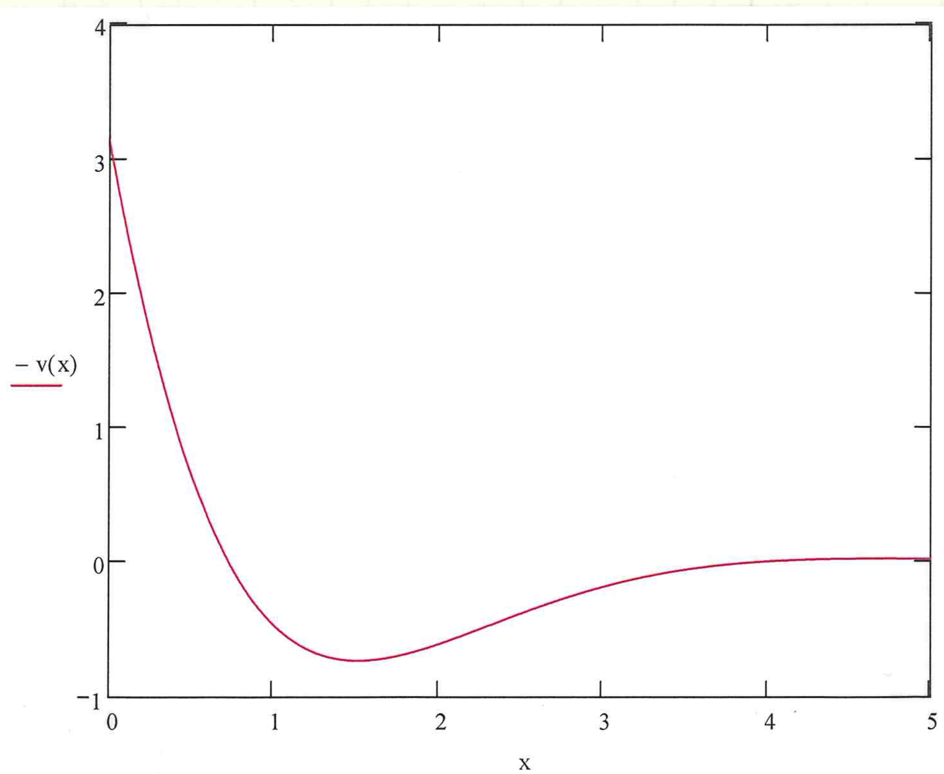
$$M_1 = \frac{P_o}{\mu}\left[C(a) - D(a)\right] + M_o\left[2D(a) + A(a)\right]$$

$$v(x) = \frac{P_1\mu}{2K}A(x) + \frac{M_1\mu^2}{K}B(x) + \frac{P_o\mu}{2K}A(x-a) + \frac{M_o\mu^2}{K}B(x-a)$$

if $x > a$

if $x < a$

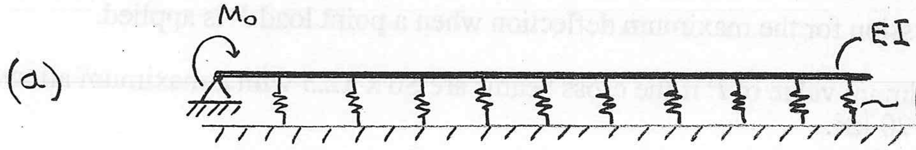
9.5

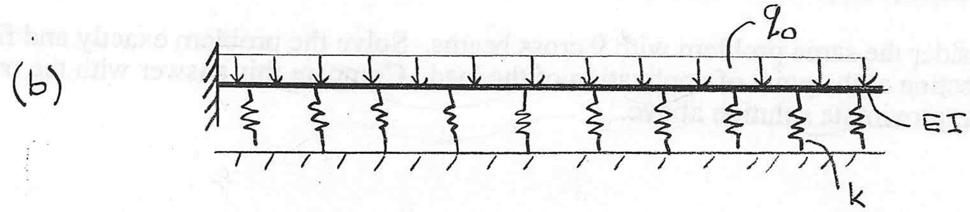


ADVANCED STRENGTH OF MATERIALS (ASE/EM 339)

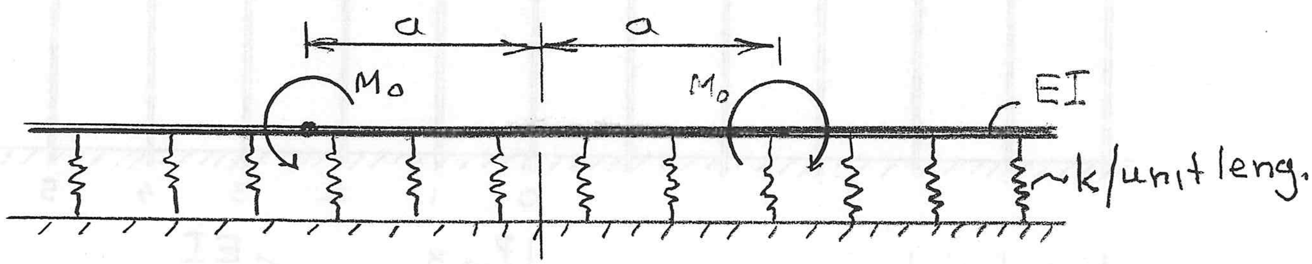
Homework #4 due 12 October 07

1. Find the equations for the deflections of the semi infinite beams shown below.

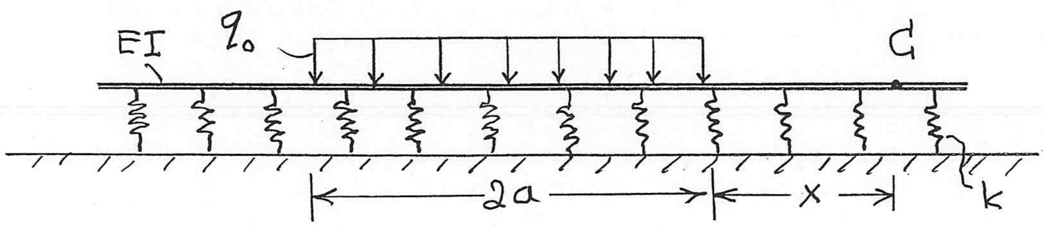




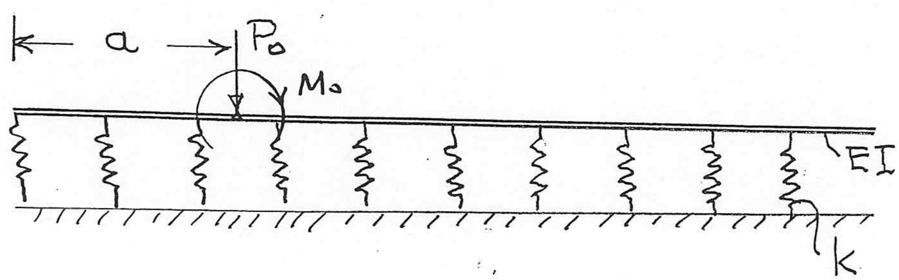
2. A long beam of bending rigidity EI is resting on an elastic foundation of stiffness k/per unit length. Calculate expressions for the deflected shape of the beam when two concentrated moments M_o are applied as shown in the figure.



4. A long beam rests on an elastic foundation and is loaded over part of its length with a uniformly distributed load q_o/unit length. Find the deflection at point C shown in the figure.



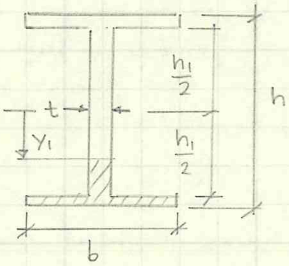
5. Find expressions for the deflection of the semi-infinite beam resting on an elastic foundation shown below.



~~27.5~~ / 30
28!

HOMEWORK #5

1.



$$Q = b\left(\frac{h}{2}-\frac{h_1}{2}\right)\left(\frac{h}{2}-\frac{1}{2}\left(\frac{h}{2}-\frac{h_1}{2}\right)\right)+t\left(\frac{h_1}{2}-y_1\right)\left(\frac{h_1}{2}-\frac{1}{2}\left(\frac{h_1}{2}-y_1\right)\right)$$

$$= b\left(\frac{h}{2}-\frac{h_1}{2}\right)\left(\frac{h}{4}+\frac{h_1}{4}\right)+t\left(\frac{h_1}{2}-y_1\right)\left(\frac{h_1}{4}+\frac{y_1}{2}\right)$$

$$= b\left(\frac{h^2}{8}+\frac{hh_1}{8}-\frac{h_1h}{8}-\frac{h_1^2}{8}\right)+t\left(\frac{h_1^2}{8}-\frac{h_1y_1}{4}+\frac{h_1y_1}{8}-\frac{y_1^2}{2}\right)$$

$$= \frac{b}{2}\left(\frac{h^2}{4}-\frac{h_1^2}{4}\right)+\frac{t}{2}\left(\frac{h_1^2}{4}-y_1^2\right)$$

$$\tau = \frac{QV}{It}$$

$$\tau = \frac{V}{It}\left[\frac{b}{2}\left(\frac{h^2}{4}-\frac{h_1^2}{4}\right)+\frac{t}{2}\left(\frac{h_1^2}{4}-y_1^2\right)\right]$$

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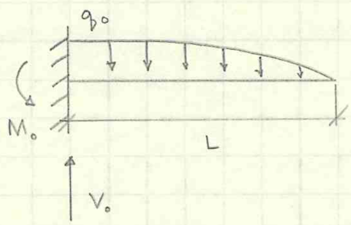
$\tau_{max}\cdot A\cdot\alpha = V$, τ_{max} at $y_1 = 0$

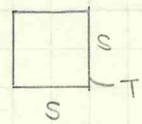
$$\frac{V}{It}\left[\frac{b}{2}\left(\frac{h^2}{4}-\frac{h_1^2}{4}\right)+\frac{t}{2}\frac{h_1^2}{4}\right]A\alpha = V$$

$$\alpha = \frac{It}{A}\left[\frac{bh^2}{8}-\frac{h_1^2}{8}(b-t)\right]^{-1}$$

HOMEWORK #5

2.





T = 0.050 in
S = 1 in

$E = 30 \times 10^{6}$ psi
$G = 1.5 \times 10^{6}$ psi

L = 20 in

$$q(x) = q_0\left[1 - (x/L)^2\right]$$

$q_0 = 10$ lbf/in

$$V_0 = 2/3\, q_0 L$$

$$M_0 = 2/3\, q_0 L \cdot 3/8\, L = 1/4\, q_0 L^2$$

$$-EI\frac{d^4 v}{dx^4} = q(x) = q_0\left[1 - (x/L)^2\right]$$

considering: $v(0) = 0$, $v'(0) = 0$

$$-EI\, v''(0) = \frac{1}{4} q_0 L^2$$

$$-EI\, v'''(0) = 2/3\, q_0 L$$

$$v_b(x) = \frac{-q_0}{EI}\left[\frac{x^4}{24} - \frac{x^6}{360L^2} - \frac{Lx^3}{9} - \frac{L^2x^2}{8}\right]$$

$$\frac{dV(x)}{dx} = \alpha GA\frac{d^2 v_s}{dx^2} = -q(x)$$

$$\alpha GA\frac{d^2 v_s}{dx^2} = q_0\left[\left(\frac{x}{L}\right)^2 - 1\right]$$

$$\alpha GA\frac{dv_s}{dx} = q_0\left[\frac{x^3}{3L^2} - x\right] + c$$

$$\alpha GA\, v_s = q_0\frac{x^4}{12L^2} - \frac{x^2}{2}q_0 + Cx + D$$

$v_s(0) = 0$, $D = 0$

$v_s'(0) = 0$, $C = 0$

OR $v_s'(0) = \gamma$, $C = \gamma$

$$v = v_b + v_s$$

$$v = \frac{-q_0}{EI}\left[\frac{x^4}{24} - \frac{x^6}{360L^2} - \frac{Lx^3}{9} - \frac{L^2x^2}{8}\right] + \frac{q_0L^2}{\alpha GA}\left[\frac{1}{12}\left(\frac{x}{L}\right)^4 - \frac{1}{2}\left(\frac{x}{L}\right)^2 + \frac{\gamma}{q_0L^2}\cdot x\right]$$

E, G, q_0, L given

$$A = 2(0.050\,\text{in})(1\,\text{in}) + 2(0.050\,\text{in})(0.90\,\text{in})$$

$$A = 0.19\,\text{in}^2$$

$$I = 1/12\,(0.10\,\text{in})(0.90\,\text{in})^3 + 2(0.05\,\text{in})(1\,\text{in})(0.475\,\text{in})^2$$

$$I = 0.0286\,\text{in}^4$$

$$\alpha = \frac{tI}{A}\left[\frac{bh^2}{8} - \frac{h_1^2}{8}(b - t)\right]^{-1}$$ where $t = 2T = 0.10$ in

$b = 1$ in

$h = 1$ in

$h_1 = S - 2T = 0.90$ in

$$\alpha = 0.635$$

EM 339: MATERIALS

HOMEWORK #5

C. HOVELL
19 Oct. 07

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$q_o := 10$

$E := 30 \times 10^6$

$L := 20$

$G := 1.5 \times 10^6$

$T := 0.05$

$s := 1.0$

$A := 2 \cdot T \cdot s + 2 \cdot T \cdot (s - 2 \cdot T)$ $\quad A = 0.19$

$I := \frac{1}{12} \cdot 2 \cdot T \cdot (s - 2 \cdot T)^3 + 2 \cdot T \cdot s \cdot \left(\frac{s}{2} - \frac{T}{2}\right)^2$ $\quad I = 0.029$

$t := 2 \cdot T$

$b := s$

$h := s$

$h_1 := s - 2 \cdot T$

$\alpha := \frac{t \cdot I}{A} \cdot \left[\frac{b \cdot h^2}{8} - \frac{h_1}{8} \cdot (b - t)\right]^{-1}$ $\quad \alpha = 0.635$

$$v(x, \gamma) := \frac{-q_o}{E \cdot I} \cdot \left(\frac{x^4}{24} - \frac{x^6}{360 \cdot L^2} - \frac{L \cdot x^3}{9} - \frac{L^2 \cdot x^2}{8}\right) + \frac{q_o \cdot L^2}{\alpha \cdot G \cdot A} \cdot \left[\frac{1}{12}\left(\frac{x}{L}\right)^4 - \frac{1}{2}\left(\frac{x}{L}\right)^2 + \frac{\gamma}{q_o \cdot L^2} \cdot x\right]$$

a) $v(L, 0) = 0.358$ ✓

$\gamma := \frac{2}{3} \cdot q_o \cdot L$

b) $v(L, \gamma) = 0.373$ ✓

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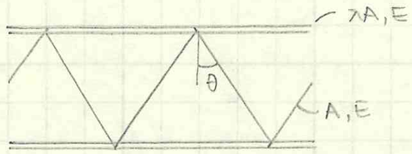
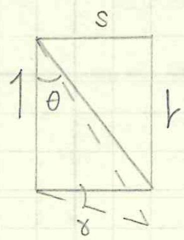
HOMEWORK #5

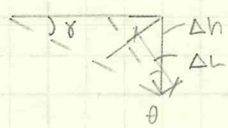
3.



$GA_s = EA\tan\theta\cos^3\theta = EA\sin\theta\cos^2\theta$

goal: $V = [\quad]\gamma$





$\Delta L = \Delta h\cos\theta$

$\Delta h = s\tan\gamma = s\gamma$ for small angles

$s = L\sin\theta$

$\Delta L = L\sin\theta\cdot\gamma\cdot\cos\theta$

Load in one strut axially:

$$F = \frac{V}{\cos\theta}$$

Axial deformation of a member

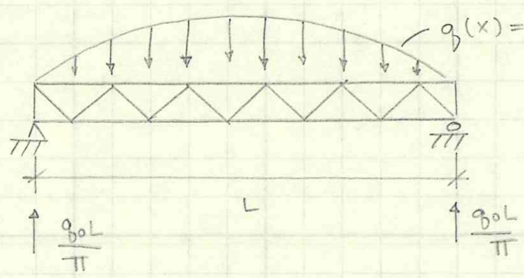
$$\Delta = \frac{PL}{EA}, \quad \text{or} \quad L\sin\theta\cdot\cos\theta\cdot\gamma = \frac{V\cdot L}{EA\cos\theta}$$

$$V = \underline{\sin\theta\cdot\cos^2\theta\cdot EA\cdot\gamma}$$

$$\underline{\underline{V = EA\sin\theta\cos^2\theta\cdot\gamma = EA\tan\theta\cdot\cos^3\theta\cdot\gamma}}$$

HOMEWORK #5

3. (cont'd)



$q(x) = q_0 \sin\frac{\pi x}{L}$, $R = \int_0^{L/2} q_0 \sin\frac{\pi x}{L}\,dx = \frac{-L}{\pi} q_0 \cos\frac{\pi x}{L}\Big|_0^{L/2}$

$= \frac{q_0}{\pi} L$

$$-EI\frac{d^4 v_b}{dx^4} = q(x) = q_0 \sin\frac{\pi}{L}x$$

$$-EI\frac{d^3 v_b}{dx^3} = \frac{-L}{\pi}\cos\frac{\pi}{L}x\, q_0 + A \longrightarrow V(0) = \frac{q_0 L}{\pi},\ A = 0$$

$$EI\frac{d^2 v_b}{dx^2} = \frac{L^2}{\pi^2} q_0 \sin\frac{\pi}{L}x + B \longrightarrow M(0) = 0,\ B = 0$$

$$EI\frac{dv_b}{dx} = \frac{-L^3}{\pi^3} q_0 \cos\frac{\pi}{L}x + C \longrightarrow \frac{dv_b}{dx}(L/2) = 0,\ C = 0$$

$$-EI v_b = \frac{L^4}{\pi^4} q_0 \sin\frac{\pi x}{L} + D \longrightarrow v_b(0) = 0,\ D = 0$$

$$v_b(x) = \frac{-q_0}{EI}\left[\frac{L^4}{\pi^4}\sin\frac{\pi x}{L}\right]$$

$$\frac{dV}{dx} = -q(x) = EA\sin\theta\cos^2\theta\,\frac{d^2 v_s}{dx^2}$$

$$EA[\quad]\frac{dv_s}{dx} = q_0\frac{L}{\pi}\cos\frac{\pi x}{L} + A \longrightarrow \frac{dv_s}{dx}(0) = V(0) = \frac{q_0 L}{\pi},\ A = 0$$

$$EA[\quad]\,v_s = q_0\frac{L^2}{\pi^2}\sin\frac{\pi x}{L} + B \longrightarrow v_s(0) = 0,\ B = 0$$

$$v(x) = \frac{q_0}{EA}\left[\frac{4}{\lambda h^2}\frac{L^4}{\pi^4}\sin\frac{\pi x}{L} + \frac{L^2}{\pi^2}\frac{\sin^{\pi x}/_L}{\sin\theta\cdot\cos^2\theta}\right]$$

Maximum at $x = L/2$

$$v_b(L/2) = \frac{4 q_0 L^4}{EA\lambda h^2\pi^4}$$

$$v_s(L/2) = \frac{q_0 L^2}{EA\pi^2\sin\theta\cos^2\theta}$$

λ, h, L, θ influence the relative magnitudes.
$\theta, h, \lambda \uparrow$, shear $\uparrow$
$L \uparrow$, shear $\downarrow$

8.5/10

EM 339: MATERIALS

C. HOVELL
19 Oct 07

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HOMEWORK #5

An example set-up for Problem 3-b, with typical or simple (=1) values for the constants

Constants with influence:

$L_o := 50$ $\quad h := 10$

$\lambda := 2$ $\quad \theta := \frac{\pi}{6}$

$I := \frac{\lambda \cdot A_o}{2} \cdot h^2$

Constants without influence:

$q_o := 1$

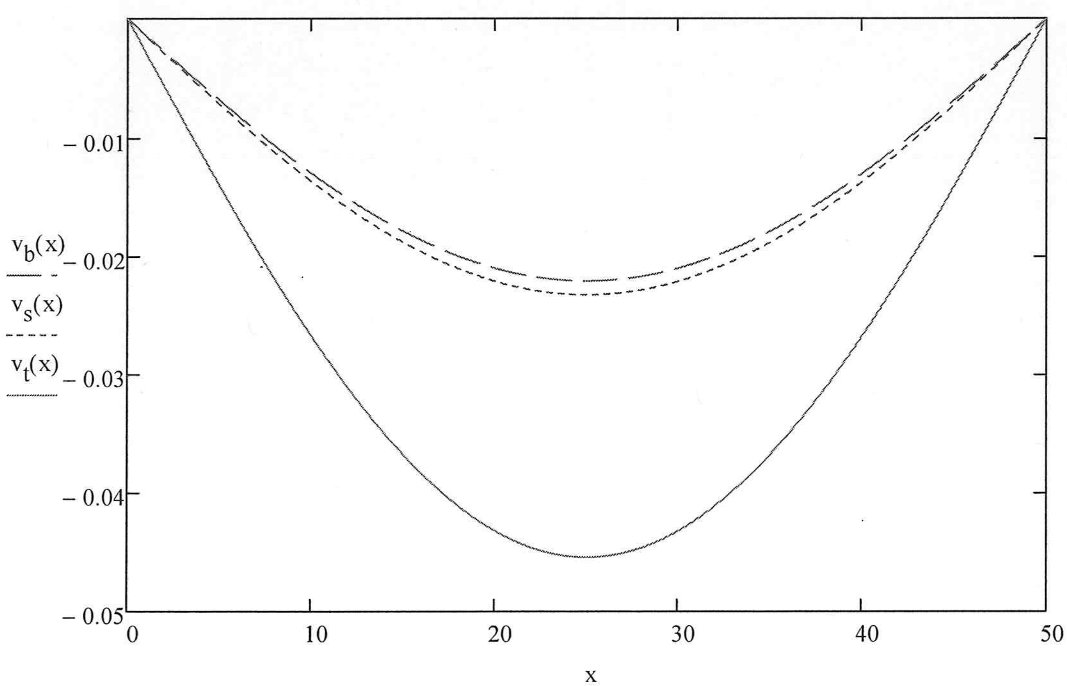
$A_o := 1$

$E := 29000$

$$v_b(x) := \frac{-q_o}{E \cdot I} \cdot \frac{L_o^{\,4}}{\pi^4} \cdot \sin\left(\frac{\pi}{L_o} \cdot x\right)$$

$$v_s(x) := \frac{-q_o}{E \cdot A_o \cdot \sin(\theta) \cdot (\cos(\theta))^2} \cdot \frac{L_o^{\,2}}{\pi^2} \cdot \sin\left(\frac{\pi}{L_o} \cdot x\right)$$

$$v_t(x) := v_s(x) + v_b(x)$$



ADVANCED STRENGTH OF MATERIALS (ASE/EM 339)

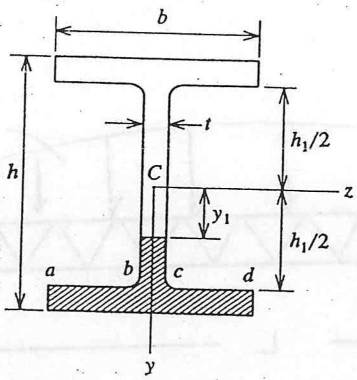
Homework #5

1. Show that the shear stress distribution in an I beam is given by:

$$\tau = \frac{V}{tI}\left\{\frac{b}{2}\left(\frac{h^2}{4} - \frac{h_1^2}{4}\right) + \frac{t}{2}\left(\frac{h_1^2}{4} - y_1^2\right)\right\}$$

Show that for this case the geometric correction factor α used for shear deformations in beams, is given by:

$$\alpha = \frac{tI}{A} \Big/ \left[\frac{bh^2}{8} - \frac{h_1^2}{8}(b-t)\right]$$



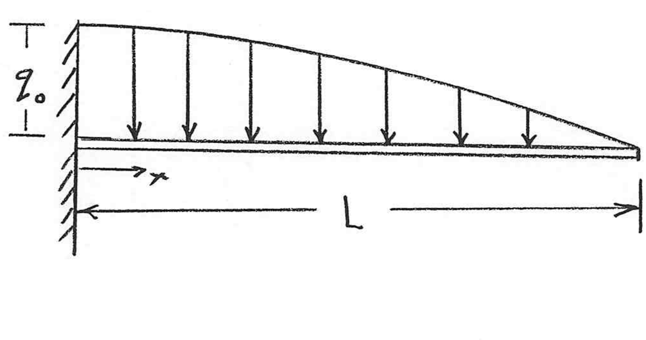
2. Consider a cantilevered beam with a distributed load as shown in the figure. The beam has length of 20 in. and a square cross section of wall thickness 0.050 in. and side of 1 in. The tube is made from an anisotropic material with Young's modulus of 30×10^6 psi and the shear modulus is 1.5×10^6 psi.

$$q(x) = q_0\left[1 - \left(\frac{x}{L}\right)^2\right], \qquad q_0 = 10 \text{ lbf/in}$$

Calculate the tip deflection assuming

(a) $v_b'(0) = v_s'(0) = 0$

(b) $v_b'(0) = 0, \quad v_s'(0) = \gamma$



EM339: MATERIALS

CATHERINE HOVELL
29 October 2007

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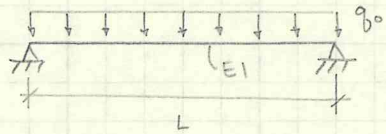
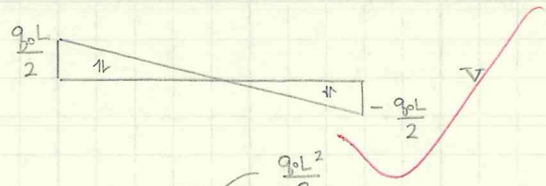
HOMEWORK #6

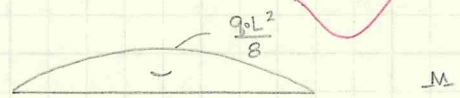
28/30

1\.



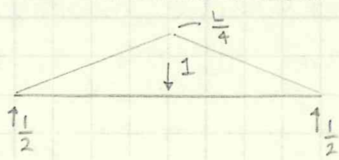
If one end is on rollers:





$$\theta = \frac{2}{3}\,\frac{q_0 L^2}{8EI}\cdot\frac{L}{2} = \frac{q_0 L^3}{24EI}$$

$$\underline{\Delta = \theta\cdot\frac{5}{8}L = \frac{5 q_0 L^4}{384EI}}$$ using moment area method



$$\Delta = 2\left[\frac{q_0 L^3}{24EI}\cdot\frac{5}{8}\cdot\frac{L}{4}\right] = \underline{\frac{5 q_0 L^4}{384EI}}$$ using virtual work approach

$$-EI\frac{d^4\bar{v}}{dx^4} = q(x) = -q_0$$

$$EI\frac{d^3\bar{v}}{dx^3} = q_0 x + A, \quad A = \frac{-q_0 L}{2} \quad \text{(shear at } x=0\text{)}$$

$$EI\frac{d^2\bar{v}}{dx^2} = \frac{q_0}{2}x^2 - \frac{q_0 L}{2}x + B, \quad B = 0 \quad \text{(moment at } x=0\text{)}$$

$$EI\frac{d\bar{v}}{dx} = \frac{q_0}{6}x^3 - \frac{q_0 L}{4}x^2 + C, \quad C = \frac{q_0 L^3}{24} \quad \text{(slope at } (x=L/2)=0\text{)}$$

$$EI\,\bar{v}(x) = \frac{q_0}{24}x^4 - \frac{q_0 L}{12}x^3 + \frac{q_0 L^3}{24}x + D, \quad D=0 \quad \text{(displacement at } x=0\text{)}$$

check $v(L)=0: \frac{1}{24} - \frac{1}{12} + \frac{1}{24} = 0$

$$EI\,\bar{v}(x) = q_0 L^4\left[\tfrac{1}{24}\left(\tfrac{x}{L}\right)^4 - \tfrac{1}{12}\left(\tfrac{x}{L}\right)^3 + \tfrac{1}{24}\left(\tfrac{x}{L}\right)\right]$$

$$\underline{\bar{v}(L/2) = \frac{5 q_0 L^4}{384EI}}$$ using equilibrium, beam theory.

HOMEWORK #6

1. (cont'd)

$$\bar{v}(L/2) = \frac{5 q_o L^4}{384 EI}$$

$$\hat{q}_o = \frac{384 EI h}{5 L^4}$$

$$\Lambda = \frac{q_o}{\hat{q}_o} = \frac{\bar{v}(L/2)}{h} = \frac{5 q_o L^4}{384 EI \cdot h}$$

$$v(x) = A\sinh\lambda x + B\cosh\lambda x + Cx + D - \frac{q_o}{N_o}\cdot\frac{x^2}{2}, \quad \lambda^2 = \frac{N_o}{EI}$$

$v(0)=0$, $B+D=0$ or $B=-D$

$v''(0)=0$, $v''(x) = \lambda^2(A\sinh\lambda x + B\cosh\lambda x) - \frac{q_o}{N_o}$

$$0 = \lambda^2 B - \frac{q_o}{N_o}, \quad B = \frac{q_o}{N_o\lambda^2}, \quad D = \frac{-q_o}{N_o\lambda^2}$$

$v(L)=0$,

$$A\sinh\lambda L + \frac{q_o}{N_o\lambda^2}\cosh\lambda L + CL - \frac{q_o}{N_o\lambda^2} - \frac{q_o L^2}{2N_o} = 0$$

$$C = \frac{1}{L}\left[\frac{q_o}{N_o}\left(\frac{1}{\lambda^2} + \frac{L^2}{2} - \frac{1}{\lambda^2}\cosh\lambda L\right) - A\sinh L\right]$$

$v''(L)=0$,

$$\lambda^2\left(A\sinh\lambda L + \frac{q_o}{N_o\lambda^2}\cosh\lambda L\right) - \frac{q_o}{N_o} = 0$$

$$A = \frac{q_o}{N_o\lambda^2}\left(\frac{1-\cosh\lambda L}{\sinh L}\right)$$

$$C = \frac{q_o}{2N_o}L$$

subbing in constants,

$$v(x) = \frac{q_o}{N_o\lambda^2}\left[\frac{1-\cosh\lambda L}{\sinh\lambda L}\sinh\lambda x + \cosh\lambda x - 1\right] + \frac{q_o}{2N_o}x(L-x)$$

considering

$$2\sinh^2\frac{x}{2} + 1 = \cosh x,$$

$$\frac{1-\cosh\lambda L}{\sinh\lambda L} = \frac{1 - 2\sinh^2\frac{\lambda L}{2} - 1}{\sinh(\lambda L)} = \frac{-2\sinh^2\frac{\lambda L}{2}}{\sinh\lambda L}$$

and $\sinh 2x = 2\sinh x\cosh x$,

$$= \frac{-2\sinh^2\frac{\lambda L}{2}}{2\sinh\frac{\lambda L}{2}\cosh\frac{\lambda L}{2}} = \frac{-\sinh\frac{\lambda L}{2}}{\cosh\frac{\lambda L}{2}}$$

Now consider λx terms:

$$\frac{-\sinh\left(\frac{\lambda L}{2}\right)\sinh(\lambda x) + \cosh(\lambda x)\cosh\left(\frac{\lambda L}{2}\right)}{\cosh\left(\frac{\lambda L}{2}\right)}$$

HOMEWORK #6

1. (cont'd)

$$\cosh(x+y) = \sinh x \sinh y + \cosh x \cosh y$$

also, $\sinh(-x) = -\sinh(x)$

$\cosh(-x) = \cosh(x)$

$$-\sinh\left(\frac{\lambda L}{2}\right)\sinh(\lambda x) + \cosh(\lambda x)\cosh\left(\frac{\lambda L}{2}\right) = \cosh\left(\frac{\lambda L}{2} - \lambda x\right)$$

Now,

$$v(x) = \frac{q_o}{N_o\lambda^2}\left[\frac{\cosh\left(\frac{\lambda L}{2} - \lambda x\right)}{\cosh\frac{\lambda L}{2}} - 1\right] + \frac{q_o}{2N_o}x(L-x)$$

$$\frac{dv}{dx} = \frac{q_o\lambda}{N_o\lambda^2}\left[\frac{-\sinh\left(\frac{\lambda L}{2} - \lambda x\right)}{\cosh\frac{\lambda L}{2}}\right] + \frac{q_o}{2N_o}(L-2x)$$

$$N_o = EA\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2\right], \quad N = EA\varepsilon \text{ or } \Delta = \frac{PL}{EA}, \text{ rearranged}$$

$$\frac{du}{dx} = \frac{N_o}{EA} - \frac{1}{2}\left(\frac{dv}{dx}\right)^2$$

integrating,

$$u(L) - u(0) = \frac{N_o L}{EA} - \frac{1}{2}\int_0^L\left(\frac{dv}{dx}\right)^2 dx = 0 \quad \text{no elongation of the N.A.}$$

This gives us:

$$2\frac{N_o L}{EA} = \int_0^L\left(\frac{dv}{dx}\right)^2 dx = \left(\frac{q_o}{N_o\lambda^2}\right)^2\int_0^L\left[\frac{-\lambda\sinh\left(\frac{\lambda L}{2} - \lambda x\right)}{\cosh\frac{\lambda L}{2}} + \frac{\lambda^2}{2}(L-2x)\right]^2 dx$$

$$\text{Define } \xi = \frac{\lambda L}{2}\left(1 - \frac{2x}{L}\right) = \frac{\lambda L}{2} - \lambda x = \frac{\lambda}{2}(L-2x)$$

$$d\xi = -\lambda dx$$

$$\frac{2N_o^3\lambda^4 L}{EAq_o^2} = \int_{\frac{\lambda L}{2}}^{-\frac{\lambda L}{2}}\left[\frac{-\lambda\sinh\xi}{\cosh\frac{\lambda L}{2}} + \lambda\xi\right]^2 \cdot \frac{-1}{\lambda}d\xi = -\lambda\int_{\frac{\lambda L}{2}}^{-\frac{\lambda L}{2}}\left[\frac{-\sinh\xi}{\cosh\frac{\lambda L}{2}} + \xi\right]^2 d\xi$$

$\xi(x=0) = \frac{\lambda L}{2}$

$\xi(x=L) = -\frac{\lambda L}{2}$

$$\frac{-2N_o^3\lambda^3 L}{EAq_o^2} = \int\left[\frac{\sinh^2\xi}{\cosh^2\frac{\lambda L}{2}} - 2\xi\frac{\sinh\xi}{\cosh\frac{\lambda L}{2}} + \xi^2\right]d\xi$$

$$= \frac{\xi^3}{3} + \frac{2\sinh\xi - 2\xi\cosh\xi}{\cosh\frac{\lambda L}{2}} + \frac{1}{2}\frac{\sinh\xi\cosh\xi - \xi}{\cosh^2\frac{\lambda L}{2}}\Bigg|_{\frac{\lambda L}{2}}^{-\frac{\lambda L}{2}}$$

(integral solved in mathcad)

HOMEWORK #6

1. (cont'd)

$$\frac{-2N_0^3\lambda^3 L}{EAq_0^2} = \frac{1}{3}\left(\frac{-\lambda L}{2}\right)^3 + \frac{2\sinh\left(-\frac{\lambda L}{2}\right) + \lambda L\cosh\left(-\frac{\lambda L}{2}\right)}{\cosh\left(\frac{\lambda L}{2}\right)} + \frac{1}{2}\,\frac{\sinh\left(-\frac{\lambda L}{2}\right)\cosh\left(-\frac{\lambda L}{2}\right) + \frac{\lambda L}{2}}{\cosh^2\left(\frac{\lambda L}{2}\right)}$$

$$-\left[\frac{1}{3}\left(\frac{\lambda L}{2}\right)^3 + \frac{2\sinh\left(\frac{\lambda L}{2}\right) - \lambda L\cosh\left(\frac{\lambda L}{2}\right)}{\cosh\frac{\lambda L}{2}} + \frac{1}{2}\,\frac{\sinh\left(\frac{\lambda L}{2}\right)\cosh\left(\frac{\lambda L}{2}\right) - \frac{\lambda L}{2}}{\cosh^2\left(\frac{\lambda L}{2}\right)}\right]$$

$$= \frac{-2}{3}\left(\frac{\lambda L}{2}\right)^3 + \frac{-4\sinh\left(\frac{\lambda L}{2}\right)}{\cosh\left(\frac{\lambda L}{2}\right)} + 2\lambda L + \frac{1}{2}\,\frac{\left(-\sinh\left(\frac{\lambda L}{2}\right)\times 2\right)}{\cosh\frac{\lambda L}{2}} + \frac{1}{2}\,\frac{\lambda L}{\cosh^2\frac{\lambda L}{2}}$$

$$= \frac{-2}{3}\left(\frac{\lambda L}{2}\right)^3 - 5\tanh\left(\frac{\lambda L}{2}\right) + 2\lambda L + \frac{\lambda L/2}{\cosh^2\frac{\lambda L}{2}}$$

Letting $\mu = \frac{\lambda L}{2}$, $\left(\lambda = \frac{2\mu}{L}\right)$

☆
$$\frac{16N_0^3\mu^3}{EAq_0^2L^2} = \frac{2}{3}\mu^3 + 5\tanh\mu - 4\mu + \frac{\mu}{\cosh^2\mu}$$

Now, consider:

$q_0 = \Lambda\hat{q}_0$

$\lambda^2 = \frac{N_0}{EI}$

$$> \quad \frac{16N_0^3\mu^3}{EAq_0^2L^2} = \frac{64N_0^2\mu^5 I}{q_0^2L^4A} = \frac{64N_0^2\mu^5 I}{\Lambda^2\hat{q}_0^2L^4A}$$

For a rectangular cross-section,

$$A = bh, \quad I = \tfrac{1}{12}bh^3$$

$$= \frac{64N_0^2\mu^5\,\tfrac{1}{12}bh^3}{\Lambda^2\hat{q}_0^2L^4bh} = \frac{64N_0^2\mu^5}{12\Lambda^2\hat{q}_0^2L^4}\cdot h^2$$

Earlier, we defined

$$h = \frac{5q_0L^4}{384EI\Lambda} = \frac{5\hat{q}_0L^4}{384EI}$$

Continuing,

$$= \frac{64N_0^2\mu^5}{12\Lambda^2\hat{q}_0^2L^4}\cdot\left(\frac{5}{384}\right)^2\frac{\hat{q}_0^2L^8}{(EI)^2} = \frac{16}{3}\left(\frac{5}{384}\right)^2\frac{N_0^2\mu^5L^4}{(N_0/\underset{\uparrow\,\frac{4\mu^2}{L^2}}{\lambda^2})^2\Lambda^2}$$

$$= \frac{1}{3}\left(\frac{5}{24}\right)^2\frac{\mu^9}{\Lambda^2}$$

Returning to Eq. ☆, above,

b) $$\frac{1}{3}\left(\frac{5}{24}\right)^2\frac{\mu^9}{\Lambda^2} - \frac{2}{3}\mu^3 - 5\tanh\mu + 4\mu - \frac{\mu}{\cosh^2\mu} = 0$$

where $\mu = \frac{L}{2}\sqrt{\frac{N_0}{EI}}$

HOMEWORK #6

1. (cont'd)

$$v(x) = \frac{q_o}{N_o \lambda^2}\left[\frac{\cosh\left(\frac{\lambda L}{2} - \lambda x\right)}{\cosh \frac{\lambda L}{2}} - 1\right] + \frac{q_o}{2N_o} x(L-x)$$

at $x = \frac{L}{2}$,

$$= \frac{q_o}{N_o \lambda^2}\left[\frac{\cosh\overset{1}{(0)}}{\cosh \frac{\lambda L}{2}} - 1\right] + \frac{q_o}{2N_o}\cdot\frac{L}{2}\cdot\frac{L}{2}$$

$$= \frac{q_o L^2}{4N_o \mu^2}\left[\frac{1}{\cosh\mu} - 1\right] + \frac{q_o L^2}{8N_o}$$

$$= \frac{q_o L^2}{8N_o \mu^2}\left[\frac{2 - 2\cosh\mu}{\cosh\mu} + \frac{\mu^2\cosh\mu}{\cosh\mu}\right]$$

$$= \frac{q_o L^2}{8N_o \mu^2}\left[\frac{2(1-\cosh\mu) + \mu^2\cosh\mu}{\cosh\mu}\right]$$

Now divide by

$$h = \frac{5\hat{q}_o L^4}{384EI}$$

$$\frac{v(L/2)}{h} = \frac{384EI\, q_o L^2}{40 N_o \mu^2 \hat{q}_o L^4}\left[\quad \cdots \quad\right]$$

$\frac{N_o}{4\mu^2}\cdot L^2$

Λ

$$= \frac{384}{40}\cdot\frac{N_o \cdot L^2}{N_o\cdot 4\mu^2}\cdot\frac{L^2}{L^4}\cdot\frac{q_o}{\hat{q}_o}\cdot\frac{1}{\mu^2} = \frac{12}{5}\frac{\Lambda}{\mu^4}\left[\qquad\right]$$

c) $$\frac{v(L/2)}{h} = \frac{12}{5}\frac{\Lambda}{\mu^4}\left[\frac{2(1-\cosh\mu) + \mu^2\cosh\mu}{\cosh\mu}\right]$$

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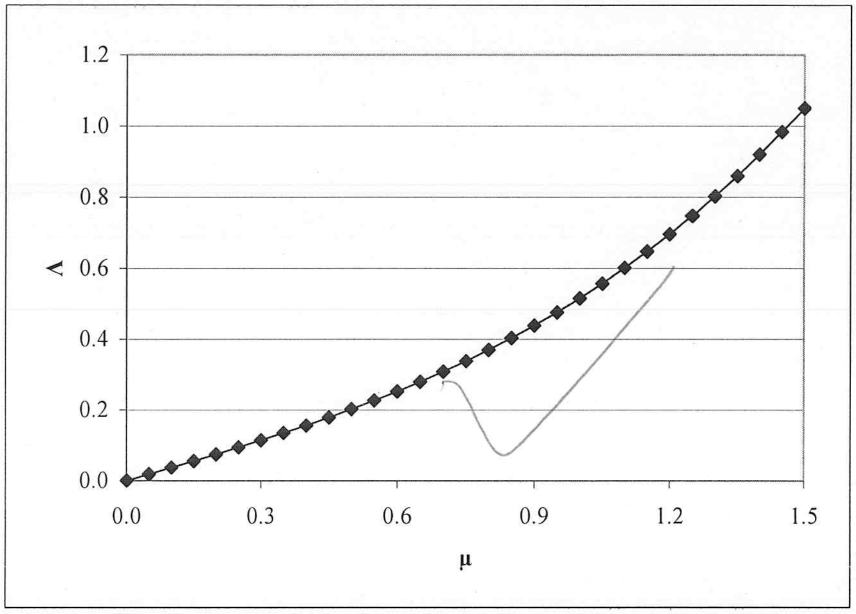
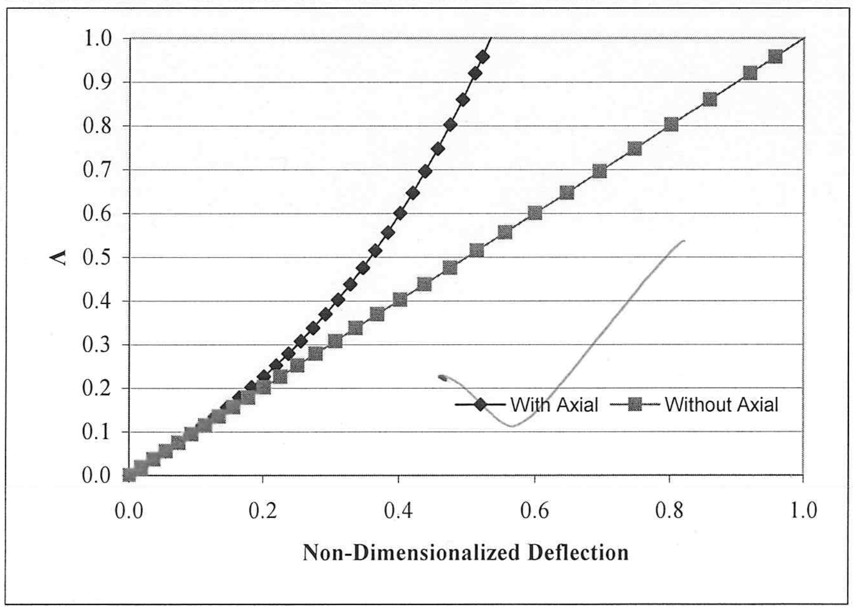
EM339: MATERIALS

CATHERINE HOVELL
29 October 2007

HOMEWORK #6

μ	Λ	v/h	v-bar/h
0.00	0.0000	0.0000	0.0000
0.05	0.0183	0.0183	0.0183
0.10	0.0368	0.0366	0.0368
0.15	0.0554	0.0549	0.0554
0.20	0.0744	0.0732	0.0744
0.25	0.0938	0.0915	0.0938
0.30	0.1138	0.1098	0.1138
0.35	0.1345	0.1281	0.1345
0.40	0.1559	0.1464	0.1559
0.45	0.1783	0.1647	0.1783
0.50	0.2016	0.1830	0.2016
0.55	0.2260	0.2013	0.2260
0.60	0.2517	0.2196	0.2517
0.65	0.2787	0.2378	0.2787
0.70	0.3072	0.2561	0.3072
0.75	0.3372	0.2744	0.3372
0.80	0.3688	0.2926	0.3688
0.85	0.4023	0.3109	0.4023
0.90	0.4376	0.3291	0.4376
0.95	0.4750	0.3474	0.4750
1.00	0.5145	0.3656	0.5145
1.05	0.5561	0.3838	0.5561
1.10	0.6002	0.4021	0.6002
1.15	0.6466	0.4203	0.6466
1.20	0.6957	0.4385	0.6957
1.25	0.7474	0.4567	0.7474
1.30	0.8018	0.4749	0.8018
1.35	0.8592	0.4930	0.8592
1.40	0.9196	0.5112	0.9196
1.43	0.9573	0.5221	0.9573
1.46	1.0005	0.5342	1.0005
1.45	0.9831	0.5294	0.9831
1.50	1.0498	0.5475	1.0498





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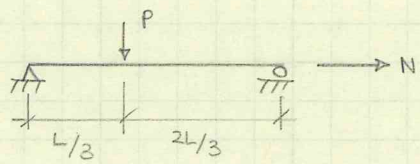
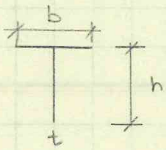
HOMEWORK #6

2.





$E = 10 \times 10^6$ psi

$h = 1$ in

$b = 1$ in

$t = 0.1$ in.

$L = 24$ in

From class,

$$v_1(x) = \frac{-P}{N_o k} \frac{\sinh \frac{2}{3}kL}{\sinh kL} \cdot \sinh kx + \frac{2P}{3N_o} \cdot x \qquad 0 < x < L/3$$

For this case, $a = 2L/3 = 16$ in

$$k^2 = \frac{N_o}{EI}$$

For the T-section,

$$\bar{y} = \frac{ht\, h/2 + bth}{ht + bt} = \frac{(1\text{in})^2 \frac{1}{2} + (1\text{in})^2}{2\text{ in}} = \frac{3}{4}\text{ in}$$

$$I = 1/12\,(th^3 + bt^3) + th\,(h/4)^2 + bt\,(h/4)^2$$

$$= 0.0209 \text{ in}^4$$

$$v_2(x) = \frac{P}{N_o k} \cdot \frac{\sinh kL/3}{\tanh(kL)} \left[\sinh kx - \tanh kL \cosh kx \right] + \frac{2P}{3N_o} x - \frac{P}{N_o}\left(x - \frac{L}{3}\right)$$

$$x > L/3$$

using mathcad,

$$\frac{d^2 v_2}{dx^2} = \frac{P}{N_o k} \frac{\sinh\left(\frac{1}{3}kL\right)}{\tanh(kL)} \left[\sinh(kx) \cdot k^2 - \tanh(k \cdot L) \cosh(k \cdot x)\, k^2 \right]$$

adding axial load does not change the location of maximum moment; $x = L/3$

$$M_{max} = \frac{P \cdot k}{N_o} \cdot \frac{\sinh(1/3\, kL)}{\tanh(kL)} \left[\sinh\left(\frac{1}{3}kL\right) - \tanh(kL) \cosh\left(\frac{1}{3}kL\right) \right]$$

HOMEWORK #6

2. (cont'd)

Maximum deflection occurs at $x > L/3$

$P = 500\,lb$, $N: [0, 1000]$

occurs when:

$$\frac{dv_2}{dx} = \frac{P}{N_o}\frac{\sinh(\frac{1}{3}kL)}{\tanh(kL)}\left[\cosh kx - \tanh kL \cdot \sinh kx\right] - \frac{P}{3N_o} = 0$$

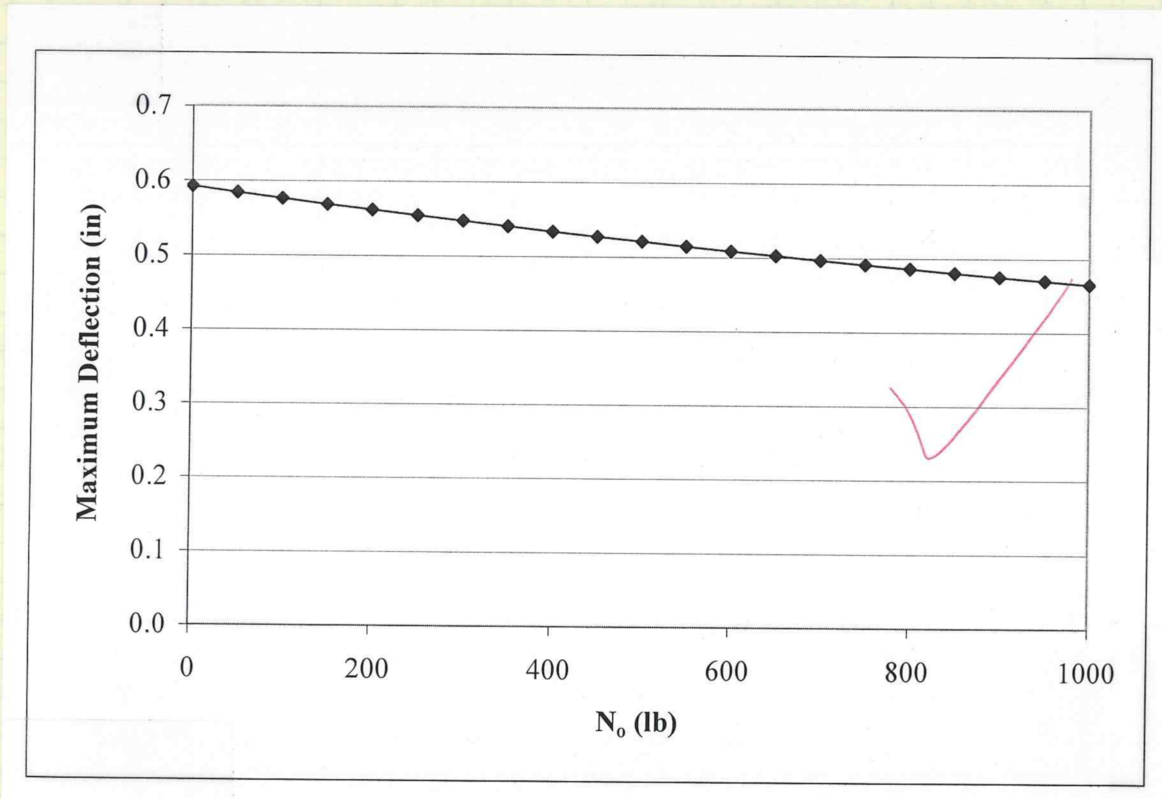
when $N_o = 0$,

$$\Delta_{max} = \frac{P}{27(EI)L}\cdot\frac{L}{3}\cdot\frac{2L}{3}\left(2L/3 + 2L/3\right)\sqrt{2L(2L/3 + 2L/3)} = 0.59\text{ in}$$

from AISC beam deflection tables

$$\text{at } x = \left[\frac{2L/3\,(2L/3 + 2L/3)}{3}\right]^{1/2} = 13.1\text{ in}$$

values as $N_o \to 1000\,lb$ found using goalseek in Excel



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HOMEWORK #6

2. (cont'd)

$$\Delta = \frac{-N_o L}{EA} + \frac{1}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx$$

$$\frac{dv_1}{dx} = \frac{P}{N_o}\left[\frac{-\sinh \frac{2}{3}kL}{\sinh kL}\cdot \cosh kx + \frac{2}{3}\right]$$

$$\frac{dv_2}{dx} = \frac{P}{N_o}\frac{\sinh \frac{1}{3}kL}{\tanh kL}\left[\cosh kx - \tanh kL \cdot \sinh kx\right] - \frac{P}{3N_o}$$

From here, solve approximately

$\frac{dv_1}{dx}(0.5) = 0.083$ for $N_o = 100$ lb

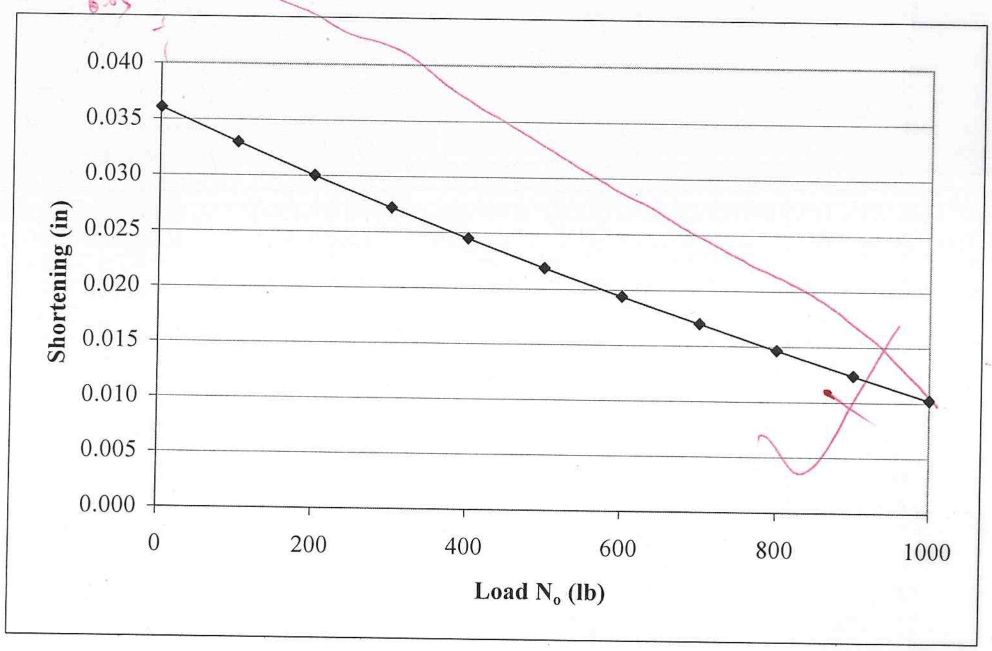
square value, then multiply by 1 in

repeat calc. at each inch (1.5, 2.5, 3.5...) and sum

values to get $\int_0^L \left(\frac{dv}{dx}\right)^2 dx$

delta positive for shortening

N_o	Δ
1	0.0360
100	0.0330
200	0.0300
300	0.0272
400	0.0245
500	0.0219
600	0.0194
700	0.0170
800	0.0146
900	0.0124
1000	0.0102



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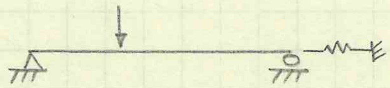
HOMEWORK #6

3.



$$\Delta = \frac{-N_o L}{EA} + \frac{1}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx = \frac{N_o}{k_s}, \quad k_s = \frac{EA}{15L}$$

use same method as in prob. 2:

- v_1', v_2' equations known.
- find value every inch, square it
- add up contributions from each inch (integral approx.)
- use Excel goalseek function to calculate when

$$\sum \Delta = \frac{N_o}{k_s}$$

N_o= 170.99087
k= 0.0286

x	v'	v'2	∑Δ
0.5	0.081	0.007	0.0012
1.5	0.080	0.006	0.0044
2.5	0.077	0.006	0.0074
3.5	0.072	0.005	0.0099
4.5	0.066	0.004	0.0121
5.5	0.058	0.003	0.0138
6.5	0.049	0.002	0.0150
7.5	0.038	0.001	0.0157
8.5	0.026	0.001	0.0161
9.5	0.015	0.000	0.0162
10.5	0.004	0.000	0.0162
11.5	-0.006	0.000	0.0162
12.5	-0.015	0.000	0.0163
13.5	-0.023	0.001	0.0166
14.5	-0.031	0.001	0.0171
15.5	-0.038	0.001	0.0178
16.5	-0.044	0.002	0.0187
17.5	-0.049	0.002	0.0199
18.5	-0.053	0.003	0.0214
19.5	-0.057	0.003	0.0230
20.5	-0.060	0.004	0.0248
21.5	-0.062	0.004	0.0267
22.5	-0.064	0.004	0.0288
23.5	-0.065	0.004	0.0308

N_o/k_s= 0.03078
goalseek 0.0001

$N_o \sim 171$ lb

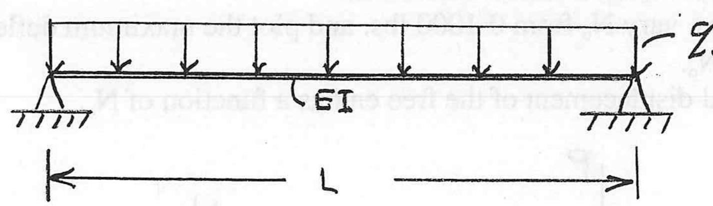
$N_o \sim 220$ lb.

8.5

ADVANCED STRENGTH OF MATERIALS (ASE/EM 339)

Homework #6 29 October 2007

1. Follow the step-by-step procedure described below and find the load-deflection response of a pinned-pinned beam loaded with a uniform load q_o/unit length.



(a) Show that for the corresponding linear beam (i.e. end supports on rollers) the deflection is

$$\bar{v}\left(\frac{L}{2}\right) = \frac{5}{384}\frac{q_o L^4}{EI}$$

Define $\bar{q}_o$ as the value of q_o for which $\bar{v}\left(\frac{L}{2}\right) = h$. Define Λ as follows:

$$\Lambda = \frac{q_o}{\bar{q}_o} = \frac{\bar{v}\left(\frac{L}{2}\right)}{h}$$

(b) Use your class notes to show that for the non-linear beam the following relationship holds (rectangular section of unit width and depth h).

$$\frac{1}{3}\left(\frac{5}{24}\right)^2 \frac{\mu^9}{\Lambda^2} + \frac{\mu}{\cosh^2\mu} - \frac{2}{3}\mu^3 + 4\mu - 5\tanh\mu = 0$$

where

$$\mu = \frac{L}{2}\sqrt{\frac{N_o}{EI}}$$

Plot Λ against $\mu (0 < \mu \leq 1.5)$

(c) Show that the deflection $v\left(\frac{L}{2}\right)$ reduces to

$$\frac{v\left(\frac{L}{2}\right)}{h} = \frac{12}{5}\frac{\Lambda}{\mu^4}\left[\frac{2(1-\cosh\mu) + \mu^2\cosh\mu}{\cosh\mu}\right]$$

(d) Use the plot from part (b) to plot $\frac{v\left(\frac{L}{2}\right)}{h}$ against $\Lambda (0 < \Lambda \leq 1)$.

(e) Plot Λ against $\frac{\bar{v}\left(\frac{L}{2}\right)}{h}$ on the same plot and compare. What do you conclude?

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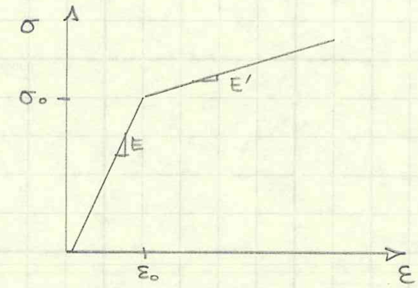
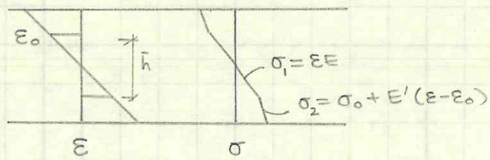
HOMEWORK #7

1. $$\frac{M}{M_0} = \frac{1}{\alpha}\left(\frac{K}{K_0}\right) - \frac{1}{2}\left(1-\frac{1}{\alpha}\right)\left(\frac{K_0}{K}\right)^2 + \frac{3}{2}\left(1-\frac{1}{\alpha}\right)$$



$$M_0 = \frac{\sigma_0 b h^2}{6}, \quad K_0 = \frac{2\varepsilon_0}{h}, \quad \alpha = \frac{E}{E'}$$



$$M = 2b\left[\int_0^{\bar{h}/2} \sigma_1 y\,dy + \int_{\bar{h}/2}^{h/2} \sigma_2 y\,dy\right]$$

$$\sigma_1 = \varepsilon E$$
$$\sigma_2 = \sigma_0 + E'\varepsilon - E'\varepsilon_0$$
$$= \sigma_0 + E'\varepsilon - E'\,\sigma_0/E$$
$$= \sigma_0(1-1/\alpha) + E'\varepsilon$$

$$\sigma = \varepsilon E = EKy$$

$$\frac{M}{2b} = EK\int_0^{\bar{h}/2} y^2\,dy + \int_{\bar{h}/2}^{h/2} \sigma_0(1-1/\alpha)\,y\,dy + \int_{\bar{h}/2}^{h/2} E'Ky^2\,dy$$

$$= \frac{EK}{24}\bar{h}^3 + \frac{E'K}{24}(h^3-\bar{h}^3) + \sigma_0(1-1/\alpha)\frac{1}{8}(h^2-\bar{h}^2)$$

$$\frac{12M}{bEK} = \bar{h}^3 + \frac{1}{\alpha}h^3 - \frac{1}{\alpha}\bar{h}^3 + \frac{3\sigma_0}{EK}(1-1/\alpha)(h^2-\bar{h}^2)$$

$$K = \frac{\varepsilon_0}{\bar{h}/2} = \frac{2\sigma_0}{E\bar{h}}$$

$$\frac{3\sigma_0}{E}\cdot\frac{E\bar{h}}{2\sigma_0} = \frac{3}{2}\bar{h}$$

$$\frac{12M}{bE}\cdot\frac{E\bar{h}}{2\sigma_0} = \frac{6M}{\sigma_0 b}\cdot\bar{h}$$

$$\frac{6M}{\sigma_0 b}\frac{1}{h^2} = \frac{1}{h^2\bar{h}}\left[\bar{h}^3(1-1/\alpha) + 1/\alpha\, h^3 + \frac{3}{2}\bar{h}(1-1/\alpha)(h^2-\bar{h}^2)\right]$$

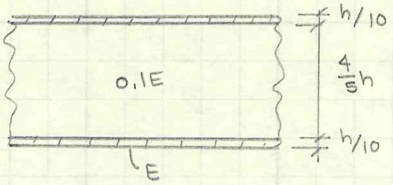
$$\frac{M}{M_0} = \frac{\bar{h}^2}{h^2}(1-1/\alpha) + \frac{1}{\alpha}\frac{h}{\bar{h}} + \frac{3}{2}(1-1/\alpha) - \frac{3}{2}(1-1/\alpha)\left(\frac{\bar{h}}{h}\right)^2$$

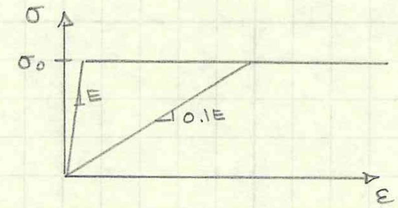
$$K_0 = \frac{\varepsilon_0}{h/2} = \frac{2\sigma_0}{Eh}, \quad \frac{K_0}{K} = \frac{\bar{h}}{h}$$

$$\frac{M}{M_0} = \frac{1}{\alpha}\left(\frac{K}{K_0}\right) - \frac{1}{2}\left(\frac{K_0}{K}\right)^2(1-1/\alpha) + 3/2\,(1-1/\alpha)$$

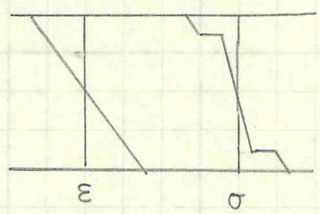
HOMEWORK #7

2.





In elastic range,



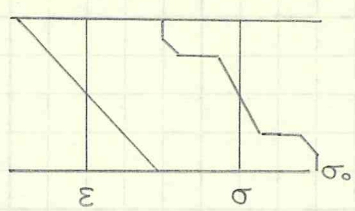
$$\frac{M}{2b} = \int_0^{2/5h} E'\varepsilon y\,dy + \int_{2/5h}^{h/2} E\varepsilon y\,dy$$

$$= \frac{1}{3} EK \frac{1}{8}\left[0.1(4/5h)^3 + h^3 - (4/5h)^3\right]$$

$$K = \frac{2\sigma_0}{Eh}, \quad M_0 = \frac{\sigma_0 b h^2}{6} \leftarrow \text{regular, single material beam response}$$

a. $\dfrac{M}{M_0} = \dfrac{337}{625}\dfrac{K}{K_0}$ until $\dfrac{K}{K_0} = 1$

one material partly plastic



$$\frac{M}{2b} = \int_0^{2/5h} E'Ky^2\,dy + \int_{2/5h}^{\bar{h}/2} EKy^2\,dy + \int_{\bar{h}/2}^{h/2} \sigma_0 y\,dy$$

$$\frac{M}{2b} = \frac{1}{3}KE'y^3\Big|_0^{2/5h} + \frac{1}{3}KEy^3\Big|_{2/5h}^{\bar{h}/2} + \frac{\sigma_0}{2}y^2\Big|_{\bar{h}/2}^{h/2}, \quad KE = \frac{2\sigma_0}{\bar{h}}$$

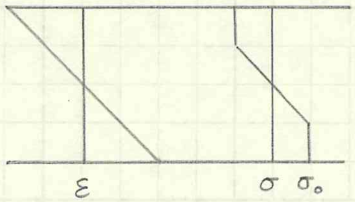
$$\frac{12M}{b} = \frac{2\sigma_0}{\bar{h}}(0.1)(4/5h)^3 + \frac{2\sigma_0}{\bar{h}}\left[\bar{h}^3 - (4/5h)^3\right] + 3\sigma_0 h^2 - 3\sigma_0\bar{h}^2$$

$$\frac{6M}{b\sigma_0 h^2} = \frac{32}{625}\frac{h}{\bar{h}} + \frac{\bar{h}}{h} - \frac{64}{125}\frac{h}{\bar{h}} + \frac{3}{2} - \frac{3}{2}\frac{\bar{h}^2}{h^2}$$

b. $\dfrac{M}{M_0} = \dfrac{3}{2} - \dfrac{288}{625}\left(\dfrac{K}{K_0}\right) - \dfrac{1}{2}\left(\dfrac{K_0}{K}\right)^2$ until $\dfrac{K}{K_0} = \dfrac{5}{4}$

HOMEWORK #7

2. Both materials are plastic



$$\frac{M}{2b} = \int_0^{\bar{h}/2} EKy^2\,dy + \int_{\bar{h}/2}^{h/2} \sigma_0 y\,dy$$

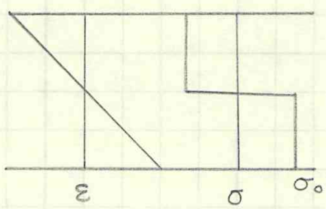
$$= 0.1EK\,\frac{1}{3}y^3\Big|_0^{\bar{h}/2} + \frac{1}{2}\sigma_0 y^2\Big|_{\bar{h}/2}^{h/2}$$

$$\frac{12M}{b} = 0.1\,\frac{2\sigma_0}{\bar{h}}\,\bar{h}^3 + 3\sigma_0 h^2 - 3\sigma_0 \bar{h}^2$$

$$\frac{6M}{bh^2\sigma_0} = 0.1\,\frac{\bar{h}^2}{h^2} + \frac{3}{2} - \frac{3}{2}\,\frac{\bar{h}^2}{h^2}$$

c. $$\frac{M}{M_0} = \frac{3}{2} - \frac{7}{5}\left(\frac{K_0}{K}\right)^2$$

Ultimate moment:



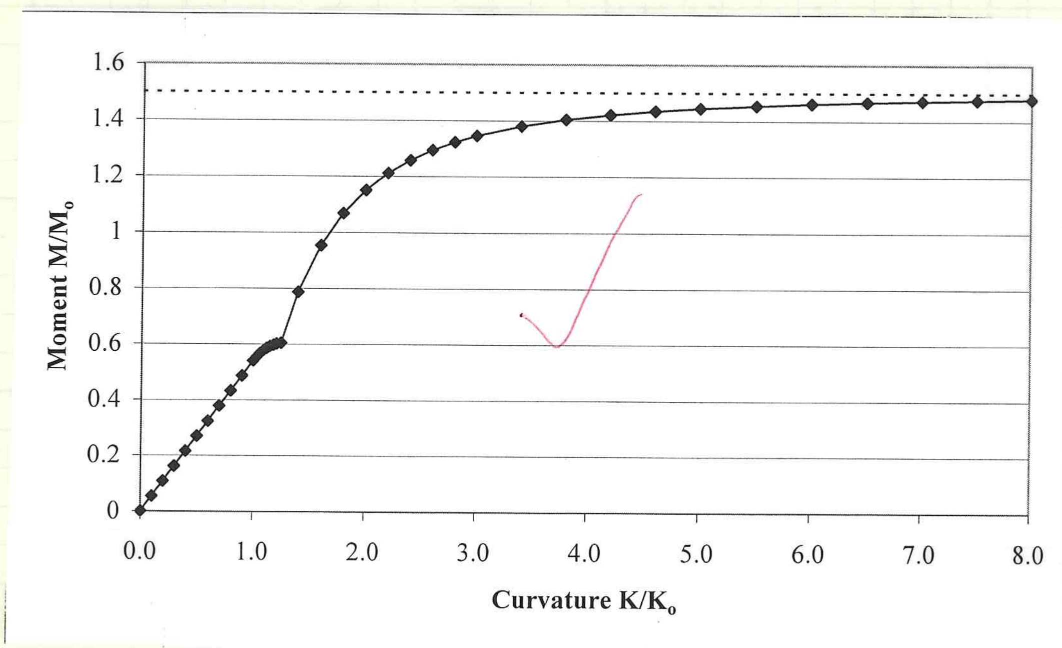
$$\frac{M}{2b} = \int_0^{h/2} \sigma_0 y\,dy = \frac{\sigma_0}{2}\,y^2\Big|_0^{h/2}$$

$$= \frac{\sigma_0}{2}\,\frac{h^2}{4}$$

$$M_p = \sigma_0\,\frac{h^2 b}{4} = \frac{3}{2}M_0$$

6

This same answer is found using the limit of equation c. as $K \to \infty$.



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09 NOV. 07

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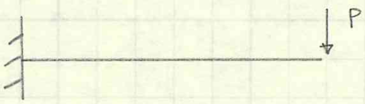
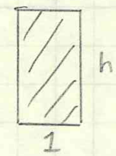
HOMEWORK #7

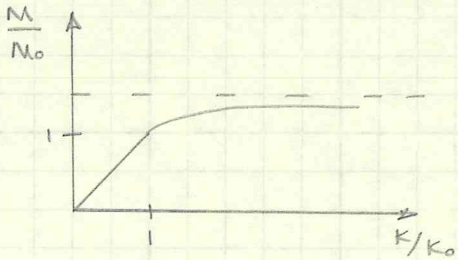
4.



$$\frac{L}{h} = 12 \qquad \frac{E}{\sigma_o} = \frac{1}{\varepsilon_o} = 600$$



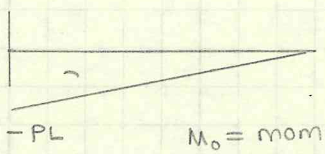
we already know that the $\frac{M}{M_o} \cdot \frac{K}{K_o}$ diagram is:



$$\frac{M}{M_o} = \frac{K}{K_o} \quad \text{for} \quad \frac{K}{K_o} \le 1$$

$$\frac{M}{M_o} = \frac{3}{2} - \frac{1}{2}\left(\frac{K_o}{K}\right)^2 \quad \frac{K}{K_o} \ge 1$$

Moment:



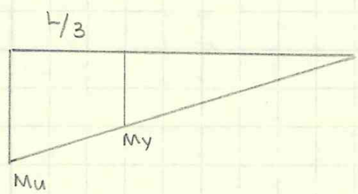
M_o = moment to cause yield

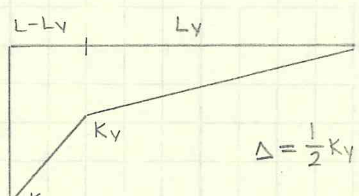
$$= \frac{\sigma_o b h^2}{6} = -P_y L$$

$$P_y = \frac{\sigma_o b h^2}{6L}$$

M_u = ultimate moment $= \frac{3}{2} M_o$

$$P_u = \frac{\sigma_o b h^2}{4L}$$





$$\Delta = \frac{1}{2} K_y L_y \cdot \frac{2}{3} L_y + K_y (L - L_y) \frac{1}{2} (L + L_y)$$

$$+ \frac{1}{2} (K - K_y)(L - L_y)\left(\frac{2}{3} L_y + \frac{1}{3} L\right)$$

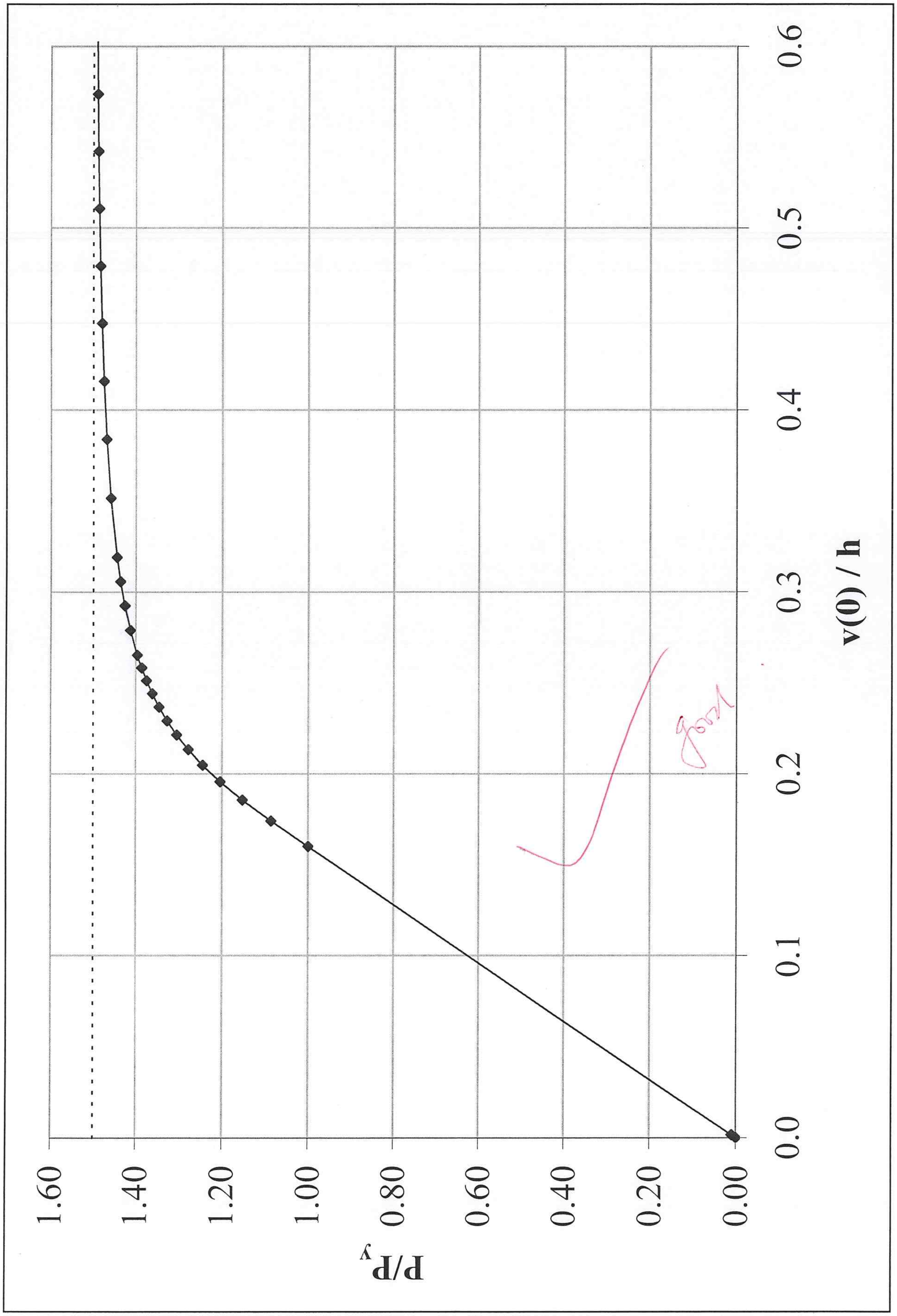
$$\frac{L_y}{L} = \frac{M_y}{M}, \quad L_y = \frac{M_y}{M} \cdot L, \quad K_y = K_o = \frac{2\varepsilon_o}{h}$$

Graph (a) follows, with $L/h = 12$ ratio assumed in ft/in

EM 339: MATERIALS

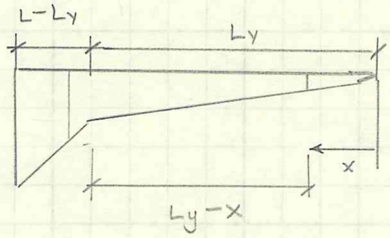
HOMEWORK #7

C. HOVELL
09 NOV. 07



HOMEWORK #7

4. (cont'd)



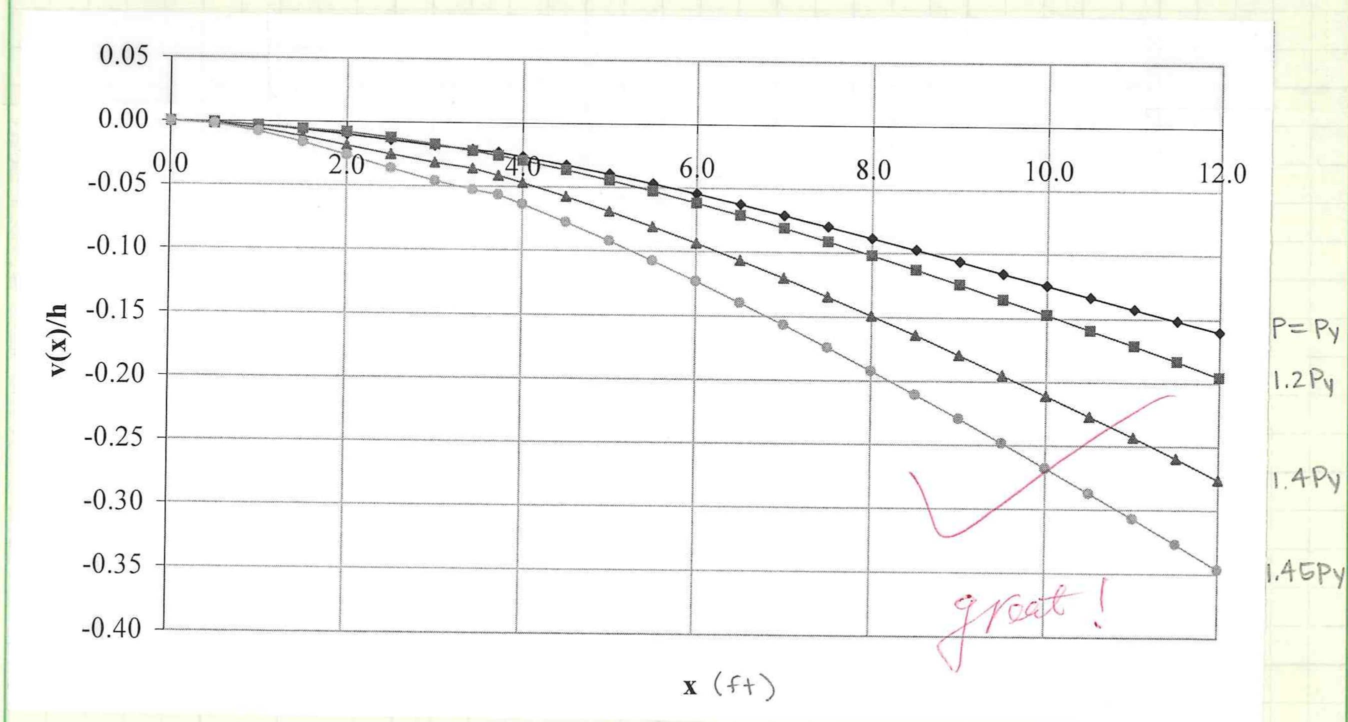
$x \le L_y$ $\quad \Delta_x = K_x \frac{1}{2}(L_y - x)^2 + \frac{1}{2}(K_y - K_x)\cdot\frac{2}{3}(L_y - x)^2 + K_y(L - L_y)\left(\frac{1}{2}L_y + \frac{1}{2}L - x\right) + \frac{1}{2}(K - K_y)(L - L_y)\left(\frac{2}{3}L + \frac{1}{3}L_y - x\right)$

$x \ge L_y$ $\quad \Delta_x = K_x\cdot(L - x)^2\cdot\frac{1}{2} + \frac{1}{2}(K - K_y)(L - x)^2\cdot\frac{2}{3}$

P	K
P_y	0.00028
$1.2P_y$	0.00036
$1.4P_y$	0.00062
$1.45P_y$	0.00088

$\frac{K_x}{K_y} = \frac{x}{L_y}, \quad x \le L_y$

$\frac{K_x - K_y}{K} = \frac{x - L_y}{L - L_y}, \quad x \ge L_y$



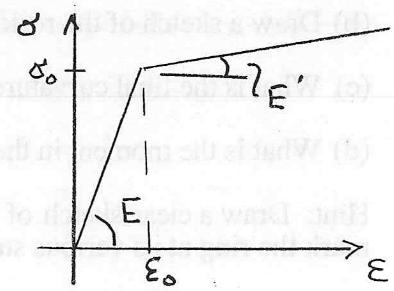
ADVANCED STRENGTH OF MATERIALS (ASE/EM 339)
Homework #7

1. Show that for a bilinear material stress-strain response the moment-curvature relationship for a rectangular beam $(b \times h)$ is given by

$$\frac{M}{M_o} = \left[\frac{1}{\alpha} \frac{\kappa}{\kappa_o} - \frac{1}{2}\left(1 - \frac{1}{\alpha}\right)\left(\frac{\kappa_o}{\kappa}\right)^2 + \frac{3}{2}\left(1 - \frac{1}{\alpha}\right) \right]$$

where

$$M_o = \frac{\sigma_0 b h^2}{6}, \quad \kappa_o = \frac{\varepsilon_o}{h/2}, \quad \alpha = \frac{E}{E'}$$

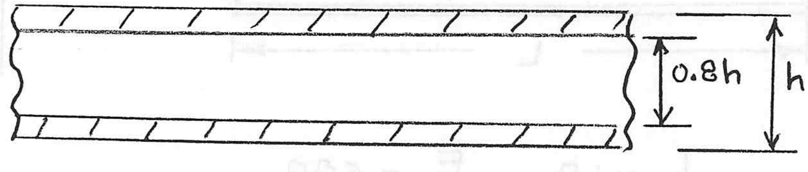


2. A composite beam consists of two outer beams with elastic modulus E and a 'core' beam with modulus $0.1E$. The two materials can be assumed to be elastic-perfectly plastic with yield stress σ_o.

Calculate the moment-curvature relationship for the composite beam
(a) in the elastic regime,
(b) when one of the materials is partly plastic,
(c) when both materials are plastic.

What is the ultimate moment of the beam section?

Sketch the stress distributions for all cases considered.

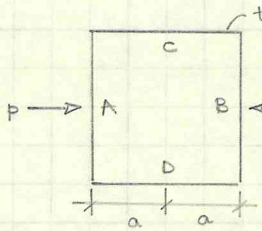


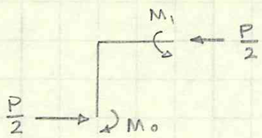
35/50

HOMEWORK #8

1.

$t \ll a$





$$M_1 = M_0 - \frac{P}{2}x \qquad x < a$$

$$M_2 = M_0 - \frac{P}{2}a \qquad x > a$$

$$U = \int_L \frac{M^2(x)}{2EI}\,dx$$

$$= \int_0^a \frac{M_1^2(x)}{2EI}\,dx + \int_a^{2a} \frac{M_2^2(x)}{2EI}\,dx$$

$$\frac{\partial U}{\partial M_0} = \int_0^a \frac{M_1(x)}{EI}\frac{\partial M_1}{\partial M_0}\,dx + \int_a^{2a} \frac{M_2(x)}{EI}\frac{\partial M_2}{\partial M_0}\,dx$$

$$\frac{\partial M_1}{\partial M_0} = 1, \quad \frac{\partial M_2}{\partial M_0} = 1$$

$$\frac{\partial U}{\partial M_0} = 0 = \int_0^a M_0 - \frac{P}{2}x\,dx + \int_a^{2a} M_0 - \frac{P}{2}a\,dx$$

$$= \left[M_0 x - \frac{P}{4}x^2\right]\Big|_0^a + \left[M_0 x - \frac{P}{2}ax\right]\Big|_a^{2a}$$

$$= M_0 a - \frac{P}{4}a^2 + M_0 a - \frac{P}{2}a^2 = 0$$

$$2M_0 a = \frac{3}{4}Pa^2$$

$$\underline{\underline{M_0 = \frac{3}{8}Pa}}$$

$$M(x) = \frac{3}{8}Pa - \frac{1}{2}Px, \qquad x \le a$$

$$M(x) = -\frac{1}{8}Pa, \qquad x > a$$

HOMEWORK #8

1. (cont'd)

Calculate deflections

$$\Delta_H = \frac{\partial U}{\partial P} = \frac{2}{EI}\left[\int_0^a \left(\frac{3}{8}Pa - \frac{1}{2}Px\right)\left(\frac{3}{8}a - \frac{1}{2}x\right)dx + \int_a^{2a}\left(-\frac{1}{8}Pa\right)\left(-\frac{1}{8}a\right)dx\right]$$

$$\left(\frac{9}{64}Pa^2 - \frac{3}{8}Pax + \frac{1}{4}Px^2\right) \qquad \left(\frac{1}{64}Pa^2\right)$$

$$= \frac{2}{EI}\left[\frac{9}{64}Pa^3 - \frac{3}{16}Pa^3 + \frac{1}{12}Pa^3\right] + \frac{2}{EI}\left[\frac{3}{64}Pa^3\right] = \frac{Pa^3}{6EI}$$

from each side

by symmetry, expansion between C and D is the same

$$\Delta AB = -\frac{Pa^3}{3EI} \quad \text{(smaller)}$$

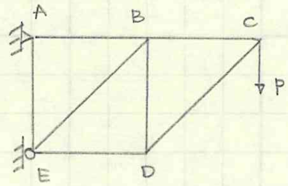
$$\Delta CD = \frac{Pa^3}{3EI} \quad \text{(bigger)}$$

6.5/10

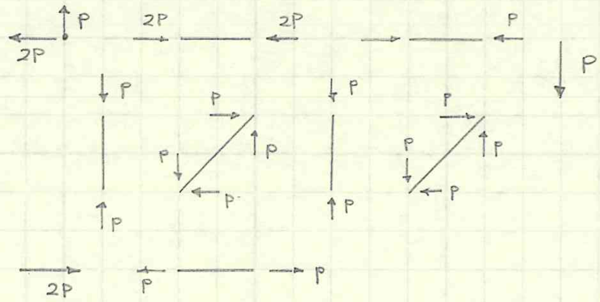
please see the solution. if need help, see Dr. 6 or your TA post.

HOMEWORK #8

2.



Applying a virtual load at C will create the same forces in the members, $N = n$



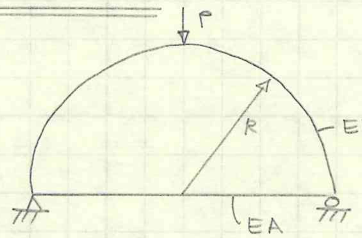
$$\Delta = \Sigma \frac{Nn}{EA} L$$

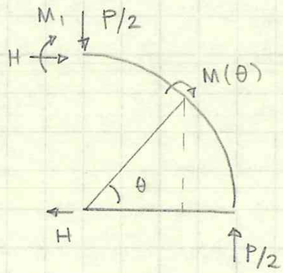
$$\Delta = \frac{PL}{EA}\left[1 + 1 + 1 + 2 + 1 + \sqrt{2}\cdot\sqrt{2} + \sqrt{2}\cdot\sqrt{2}\right] = 10\frac{PL}{EA}$$

$$\Delta_c = 10\frac{PL}{EA}$$

HOMEWORK #8

3.





$$M(\theta) = \frac{PR}{2}(1-\cos\theta) - HR\sin\theta$$

$$\Delta = \frac{-\partial U}{\partial H},$$ (−) because H and Δ are in opposite directions

$$-\Delta = 2\int_0^{\pi/2} \frac{M(\theta)}{EI} \cdot \underbrace{\frac{\partial M}{\partial H}}_{-R\sin\theta} R\,d\theta$$

$$= \frac{-2}{EI}\int_0^{\pi/2}\left[\frac{PR}{2}(1-\cos\theta) - HR\sin\theta\right](-\sin\theta)R^2\,d\theta$$

$$= \frac{2R^2}{EI}\int_0^{\pi/2} \frac{PR}{2}\sin\theta - \frac{PR}{2}\sin\theta\cos\theta - HR\sin^2\theta\,d\theta$$

$$\Delta = \frac{2R^2}{EI}\left[-\frac{PR}{2}\cos\theta\Big|_0^{\pi/2} - \frac{PR}{2}\cdot\frac{1}{2}\sin^2\theta\Big|_0^{\pi/2} - HR\left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/2}\right]$$

$$= \frac{2R^2}{EI}\left[\frac{PR}{2} - \frac{PR}{4} - HR\left(\pi/4\right)\right] = \frac{HR}{EA}$$

$$\frac{PR}{4} - \frac{\pi}{4}HR = \frac{I}{A}\cdot\frac{H}{2R}, \quad H = \frac{PR/4}{\frac{I}{2AR} + \pi/4\,R} = \frac{PR}{\frac{2I}{AR} + \pi R}$$

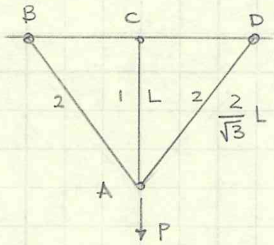
$$\Delta = \frac{HR}{EA} = \frac{R}{EA}\cdot\frac{PR}{\frac{2I}{AR} + \pi R} = \frac{2PR^2}{\frac{4EI}{R} + EA\pi R}$$

$$\Delta = \frac{PR^3}{2EI + \pi EAR^2}$$

8.5

HOMEWORK #8

4.



$\sigma = b\sqrt{\varepsilon}$, $\varepsilon = \frac{\sigma^2}{b^2}$

$$U_0^c = \int_0^\sigma \varepsilon \, d\sigma = \int_0^\sigma \frac{\sigma^2}{b^2} \, d\sigma = \frac{\sigma^3}{3b^2}$$

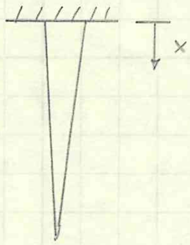
$$U^c = \int_0^L \int_A U_0^c \, dA \, dx$$

ΔL_y must be the same

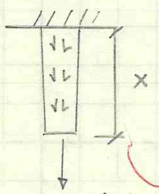
please see the solution post.

HOMEWORK #8

5.



$$A(x) = A_0\left(1 - \frac{x}{L}\right)$$



$$N(x) = P + \rho g \frac{1}{2} A(x)(L - x)$$

$$\Delta = \frac{\partial U}{\partial P} = \int_0^L \frac{N(x)}{EA} \frac{\partial N}{\partial P}\, dx$$

$$\frac{\partial N}{\partial P} = 1$$

$$\Delta = \int_0^L \frac{P + \rho g \frac{1}{2} A_0 (1 - x/L)(L - x)}{EA_0(1 - x/L)}\, dx$$

but $P = 0$, $A(x)$ cancels

$$\Delta = \int_0^L \frac{\rho g}{2E}(L - x)\, dx$$

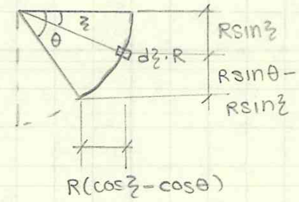
$$= \frac{\rho g}{2E}\left[Lx - \frac{x^2}{2}\right]_0^L = \frac{\rho g}{2E}\left(L^2 - \frac{1}{2}L^2\right)$$

$$\boxed{\Delta = \frac{\rho g L^2}{4E}}$$

6

HOMEWORK #8

6.



$$dM(\theta) = w\,d\xi\, R^2(\sin\theta - \sin\xi)$$

$$M(\theta) = wR^2\int_0^\theta \sin\theta - \sin\xi\, d\xi$$

$$= wR^2\left(\xi\sin\theta + \cos\xi\right)\Big|_0^\theta$$

$$= wR^2(\theta\sin\theta + \cos\theta - 1) - \underbrace{PR\sin\theta}$$

$$dT(\theta) = w\,d\xi\, R^2(\cos\xi - \cos\theta)$$

$$T(\theta) = wR^2\int_0^\theta \cos\xi - \cos\theta\, d\xi$$

$$= wR^2\left[\sin\xi - \xi\cos\theta\right]_0^\theta$$

$$= wR^2(\sin\theta - \theta\cos\theta) + \overbrace{PR(1-\cos\theta)}$$

From dummy load P

M(θ) and T(θ) is not correct.

$$\Delta = \int_0^{\pi/2} \frac{M(\theta)}{EI}\frac{\partial M}{\partial P} R\,d\theta + \int_0^{\pi/2}\frac{T(\theta)}{GJ}\frac{\partial T}{\partial P} R\,d\theta$$

$$\Delta = \frac{R^2}{EI}\int_0^{\pi/2}\left[wR^2(\theta\sin\theta + \cos\theta - 1) - PR\sin\theta\right](-\sin\theta)\,d\theta$$

$$+ \frac{R^2}{GJ}\int_0^{\pi/2}\left[wR^2(\sin\theta - \theta\cos\theta) + PR(1-\cos\theta)\right](1-\cos\theta)\,d\theta$$

set $P = 0$

$$\Delta = \frac{wR^4}{EI}\int_0^{\pi/2} -\theta\sin^2\theta - \sin\theta\cos\theta + \sin\theta\, d\theta$$

$$+ \frac{wR^4}{GJ}\int_0^{\pi/2} \sin\theta - \theta\cos\theta - \sin\theta\cos\theta + \theta\cos^2\theta\, d\theta$$

8/10

Using mathcad,

$$\underline{\underline{\Delta = \frac{wR^4}{EI}\left(\tfrac{1}{4} - \tfrac{\pi^2}{16}\right) + \frac{wR^4}{GJ}\left(\tfrac{\pi^2}{16} - \tfrac{\pi}{2} + \tfrac{5}{4}\right)}}$$

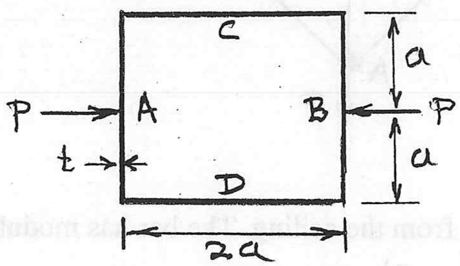
$$\int_0^{\frac{\pi}{2}} -\theta\cdot\sin(\theta)^2 - \sin(\theta)\cdot\cos(\theta) + \sin(\theta)\, d\theta \rightarrow \frac{1}{4} - \frac{\pi^2}{16}$$

$$\int_0^{\frac{\pi}{2}} \sin(\theta) - \theta\cdot\cos(\theta) - \sin(\theta)\cos(\theta) + \theta\cdot\cos(\theta)^2\, d\theta \rightarrow \frac{\pi^2}{16} - \frac{\pi}{2} + \frac{5}{4}$$

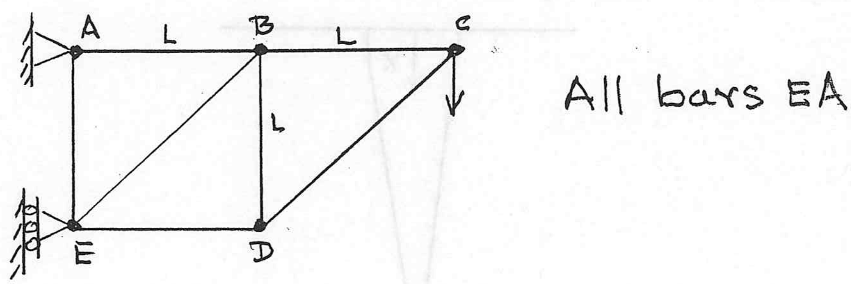
Advanced Strength of Materials (ASE/EM 339)

Homework #8

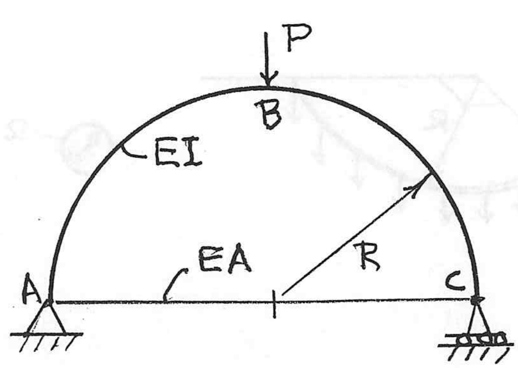
1. Use an energy method to calculate the changes in length between points *AB* and *CD* due to the applied load *P*. (Ring has wall thickness $t \ll a$)



2. Use the unit load method to calculate the vertical deflection at point *C* in the frame shown below.



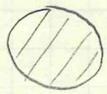
3. *ABC* is a thin semi-circular beam of radius *R* and bending rigidity *EI*. Points *A* and *C* are connected with a wire of axial rigidity *EA*. Calculate the deflection of point *C* due to the load *P* applied at *B*.



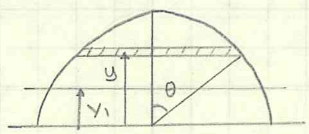
HOMEWORK #9

49.5/50

1. Shear form factor



Calculate Q:



$$Q = \int y\,dA = \int y\,b(y)\,dy$$

$b = R\sin\theta \cdot 2$

$y = R\cos\theta$

$dy = -R\sin\theta \cdot d\theta$

$$Q = \int_\theta^0 (R\cos\theta)(2R\sin\theta)(-R\sin\theta)\,d\theta = \int_\theta^0 -2R^3\sin^2\theta\cos\theta\,d\theta = \frac{2}{3}R^3\sin^3\theta$$

$$\beta = \frac{A}{I^2}\cdot 2\int_{\pi/2}^0 \left(\frac{Q}{b}\right)^2 dA$$

$dA = b(y)\,dy = 2R\sin\theta\cdot(-R\sin\theta)\,d\theta$

$$= \frac{2\pi R^2}{\frac{1}{16}\pi^2R^8}\int_{\pi/2}^0 -2\cancel{R^2}\cancel{\sin^2}\theta\,\frac{\cancel{4}/9\,R^6\sin^6\theta}{\cancel{4R^2\sin^2\theta}}\,d\theta$$

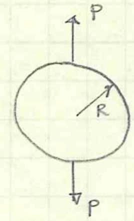
$$= \frac{32}{\pi R^6}\int_{\pi/2}^0 -\frac{2}{9}R^6\sin^6\theta\,d\theta = \frac{-64}{9\pi}\left[\frac{5\theta}{16} - \frac{15}{64}\cancel{\sin 2\theta} + \frac{3}{64}\cancel{\sin 4\theta} - \cancel{\frac{\sin 6\theta}{192}}\right]_{\pi/2}^0$$

$$\underline{\underline{\beta = \frac{10}{9}}}$$

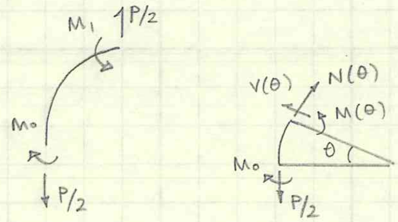
HOMEWORK #9

1. (cont'd)

$\beta = 10/9$



$$U_v = \int \beta \frac{V^2(x)}{2GA}\, dx$$



$$V(\theta)\sin\theta + N(\theta)\cos\theta = P/2$$

$$V(\theta)\cos\theta = N(\theta)\sin\theta$$

at $\theta = 0$, $V = 0$, otherwise,

$$V(\theta)\left[\sin\theta + \frac{\cos^2\theta}{\sin\theta}\right] = P/2$$

$$V(\theta) = \frac{P}{2\left[\sin\theta + \frac{\cos^2\theta}{\sin\theta}\right]} = \frac{P}{2}\sin\theta$$

$$u = \frac{\beta}{2GA}\int V^2(\theta)\, R\, d\theta$$

$$\frac{\partial u}{\partial P} = \frac{2\beta R}{GA}\int \frac{P}{2}\sin\theta \cdot \frac{1}{2}\sin\theta\, d\theta = \Delta$$

$$\Delta = \frac{10}{9}\cdot\frac{1}{2}\cdot\frac{PR}{GA}\int_0^{\pi/2} \sin^2\theta\, d\theta$$

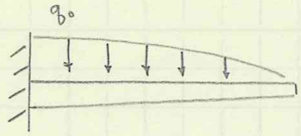
solving numerically,

$$\boxed{\Delta = 0.872\,\frac{PR}{GA}}$$

10

HOMEWORK #9

2.



$$q(x) = q_0\left[1 - (x/L)^2\right]$$

$$h(x) = h_0\left[1 - \tfrac{1}{2}\tfrac{x}{L}\right]$$

$$v(x) \approx a_1\left(\frac{x}{L}\right)^2 + a_2\left(\frac{x}{L}\right)^3 + a_3\left(\frac{x}{L}\right)^4$$

define $\xi = \frac{x}{L}$ $\qquad$ $dx = L\,d\xi$

$$\phi_1 = \xi^2 \qquad \phi_1' = 2\xi \qquad \phi_1'' = 2$$

$$\phi_2 = \xi^3 \qquad \phi_2' = 3\xi^2 \qquad \phi_2'' = 6\xi$$

$$\phi_3 = \xi^4 \qquad \phi_3' = 4\xi^3 \qquad \phi_3'' = 12\xi^2$$

$$v(\xi) = a_1\phi_1 + a_2\phi_2 + a_3\phi_3$$

$$v' = a_1\phi_1' + a_2\phi_2' + a_3\phi_3'$$

$$v'' = a_1\phi_1'' + a_2\phi_2'' + a_3\phi_3''$$

$$V = \int_0^1 \frac{EI(\xi)(v'')^2}{2L^4}\cdot L\,d\xi + \int_0^1 q(\xi)\,v(\xi)\,L\,d\xi$$

$$V := \int_0^1 \frac{E\cdot I(\xi)\cdot(\overset{v''}{v3})^2}{2\cdot Lo^3}\,d\xi - \int_0^1 q(\xi)\cdot \overset{v}{v1}\cdot Lo\,d\xi$$

$$\frac{d}{da_1}V \rightarrow \frac{E\cdot b\cdot h_o^3\cdot(350\cdot a_1 + 364\cdot a_2 + 392\cdot a_3)}{2240\cdot Lo^3} - \frac{2\cdot Lo\cdot q_o}{15}$$

$$\frac{d}{da_2}V \rightarrow \frac{E\cdot b\cdot h_o^3\cdot(364\cdot a_1 + 588\cdot a_2 + 768\cdot a_3)}{2240\cdot Lo^3} - \frac{Lo\cdot q_o}{12}$$

$$\frac{d}{da_3}V \rightarrow \frac{E\cdot b\cdot h_o^3\cdot(392\cdot a_1 + 768\cdot a_2 + 1116\cdot a_3)}{2240\cdot Lo^3} - \frac{2\cdot Lo\cdot q_o}{35}$$

Separating the terms and rearranging,

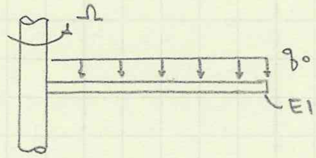
$$\frac{E\cdot b\cdot h_o^3}{L^3}\cdot\begin{pmatrix} \frac{5}{32} & \frac{13}{80} & \frac{7}{40} \\ \frac{13}{80} & \frac{21}{80} & \frac{12}{35} \\ \frac{7}{40} & \frac{12}{35} & \frac{279}{560} \end{pmatrix}\cdot\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} := q_o\cdot L\cdot\begin{pmatrix} \frac{2}{15} \\ \frac{1}{12} \\ \frac{2}{35} \end{pmatrix}$$

Try to do it by hand!

10

HOMEWORK #9

3.



$$v(x) = a_1\phi_1 + a_2\phi_2$$

$$\phi_1 = \left(\frac{x}{L}\right)^2, \quad \phi_2 = \left(\frac{x}{L}\right)^3$$

$$V = \int \frac{(v'')^2}{2EI}\,dx + \frac{1}{2}\int N(x)(v')^2\,dx - \int q(x)v(x)\,dx$$

$$N(x) = \int_x^L m\Omega^2 R\,dR$$

$$N(\xi) = \frac{m\Omega^2 L^2}{2}(1-\xi^2)$$

$$V := \int_0^1 \frac{EI\cdot(\overset{v''}{v3})^2}{2\cdot Lo^3}\,d\xi + \int_0^1 \frac{N(\xi)}{2\cdot Lo}\cdot(\overset{v'}{v2})^2 Lo\,d\xi - \int_0^1 q(\xi)\cdot \overset{v}{v1}\cdot Lo\,d\xi$$

$$\frac{d}{da_1}V \rightarrow \frac{2\cdot EI\cdot(2\cdot a_1 + 3\cdot a_2)}{Lo^3} - \frac{Lo\cdot q_o}{3} + \frac{Lo^2\cdot\Omega^2\cdot m\cdot(112\cdot a_1 + 105\cdot a_2)}{420}$$

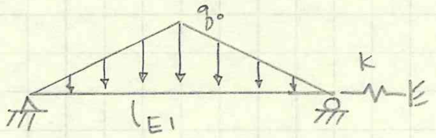
$$\frac{d}{da_2}V \rightarrow \frac{2\cdot EI\cdot(3\cdot a_1 + 6\cdot a_2)}{Lo^3} - \frac{Lo\cdot q_o}{4} + \frac{Lo^2\cdot\Omega^2\cdot m\cdot(105\cdot a_1 + 108\cdot a_2)}{420}$$

$$\frac{EI}{L^3}\cdot\begin{pmatrix}4 & 6\\ 6 & 12\end{pmatrix}\cdot\begin{pmatrix}a_1\\ a_2\end{pmatrix} + L^2\cdot\Omega^2\cdot m\cdot\begin{pmatrix}\frac{4}{15} & \frac{1}{4}\\ \frac{1}{4} & \frac{9}{35}\end{pmatrix}\cdot\begin{pmatrix}a_1\\ a_2\end{pmatrix} := q_o\cdot L\cdot\begin{pmatrix}\frac{1}{3}\\ \frac{1}{4}\end{pmatrix}$$

10

HOMEWORK #9

4.



$v(\xi) = a_1 \sin \pi\xi + a_2 \sin 3\pi\xi$

$q(\xi) = 2q_o\xi$ to $\xi = 0.5$

$\phi_1 = \sin\pi\xi, \ \phi_2 = \sin 3\pi\xi$

$N(\xi)$:

$$\Delta = \frac{1}{2}\int_0^1 \frac{1}{L^2}\left(\frac{dv}{d\xi}\right)^2 L\,d\xi - \frac{N_o L}{EA}$$

$$N(\xi) = \frac{k}{2L}\int_0^1 (v')^2\,d\xi$$

Assume $N = N_o$, constant

$$V = \int_0^1 \frac{EI(v'')^2}{2L^3}\,d\xi + \int_0^1 \frac{N_o}{2L}(v')^2\,d\xi - 2\int_0^{1/2} q(\xi)v(\xi)L\,d\xi$$

$$\frac{\partial V}{\partial a_1} = \frac{EI}{L^3}\int_0^1 -\pi^2(a_1 \sin\pi\xi + 9a_2 \sin 3\pi\xi)\cdot(-\pi^2 \sin\pi\xi)\,d\xi$$

(1)
$$= \frac{EI}{L^3}\pi^4\left[\frac{-a_1}{4\pi}(\sin 2\pi\xi - 2\pi x) + \frac{9a_2}{\pi}\cos\pi x \sin^3\pi x\right]_0^1$$

$$= \frac{EI}{L^3}\cdot\frac{\pi^4}{2}a_1 \qquad \frac{\partial V}{\partial a_2} = \frac{EI}{L^3}\,\frac{81\pi^4}{2}\,a_2$$

$$\frac{\partial V}{\partial a_1} = \frac{N_o}{L}\int_0^L \pi^2\left[a_1\cos\pi\xi + 3a_2\cos 3\pi\xi\right](\cos\pi\xi)\,d\xi$$

(2)
$$= \frac{N_o\pi^2}{2L}, \qquad \frac{\partial V}{\partial a_2} = \frac{9N_o\pi^2}{2L}$$

$$\frac{\partial V}{\partial a_1} = 2q_oL\int_0^{1/2}(a_1\sin\pi\xi + a_2\sin 3\pi\xi)(2\xi)\,d\xi$$

(3)
$$= 4q_oL\left[\frac{\sin\pi\xi}{\pi^2}a_1 + \frac{\sin 3\pi\xi}{9\pi^2}\right]_0^{1/2} \qquad (\xi\cos\pi\xi \text{ terms} = 0)$$

$$\frac{\partial V}{\partial a_1} = 0.36, \quad \frac{\partial V}{\partial a_2} = 0.36$$

$$\frac{4}{\pi^2}\left(\frac{8}{9}\right)$$

HOMEWORK #9

4. (cont'd)

with the axial load constant,

$$\frac{EI}{L^3}\cdot\frac{\pi^4}{2}\begin{bmatrix}1 & 0\\0 & 81\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix} + \frac{N_0\pi^2}{2L}\begin{bmatrix}1 & 0\\0 & 9\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix} = \frac{32}{9\pi^2}q_0L\begin{bmatrix}a_1\\a_2\end{bmatrix}$$

But,

$$N(\xi) = \frac{K\pi^2}{2L}\int_0^1 (a_1\sin\pi\xi + a_2\sin3\pi\xi)^2\,d\xi$$

$$= \frac{K\pi^2}{2L}\int_0^1 \left[a_1^2\sin^2\pi\xi + 2a_1a_2\sin\pi\xi\sin3\pi\xi + a_2^2\sin3\pi\xi\right]d\xi$$

$$= \frac{K\pi^2}{2L}\left[a_1^2\left(\frac{\xi}{2} - \frac{\sin2\pi\xi}{4\pi}\right) + \frac{2}{\pi}a_1a_2\left(\cos\pi\xi\sin^3\pi\xi\right) + a_2^2\left(\frac{6\pi x - \sin6\pi x}{12\pi}\right)\right]_0^1$$

$$= \frac{K\pi^2}{4L}(a_1^2 + a_2^2)$$

$$N(\xi) = \frac{K\pi^2}{4L}(a_1^2 + a_2^2)$$

solve numerically for a_1, a_2 to get equation for $v(x)$.

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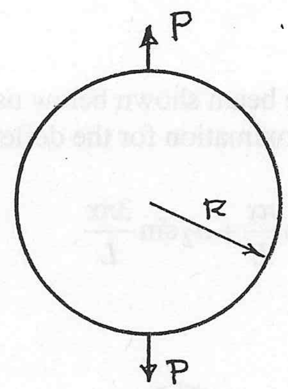
ADVANCED STRENGTH OF MATERIALS (ASE/EM 339)

Homework #9

1. Consider the circular ring loaded with point loads problem. In class we calculated the deflection of the points at which P is applied to be

$$\Delta = \frac{PR^3}{EI}\left[\frac{\pi}{4} - \frac{2}{\pi}\right]$$

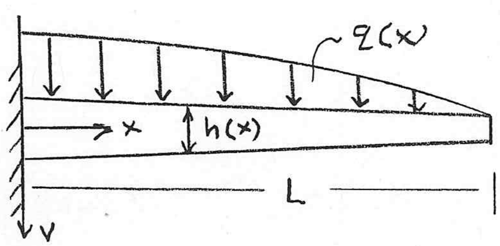
Use an energy method to evaluate the additional deflection that will result if the effect of shear is included. Assume the beam to have a circular cross section diameter 2R (first show that for a circular cross section beam the form shear factor is 10/9)



2. The wing of a light aircraft is approximated as a tapered beam with width b and depth $h(x) = h_o\left[1 - \frac{1}{2}\left(\frac{x}{L}\right)\right]$. The beam is loaded with a distributed load $q(x) = q_o\left[1 - \left(\frac{x}{L}\right)^2\right]$.

Use the Rayleigh-Ritz method to estimate the deflected shape using

$$v(x) \approx a_1\left(\frac{x}{L}\right)^2 + a_2\left(\frac{x}{L}\right)^3 + a_3\left(\frac{x}{L}\right)^4.$$



HOMEWORK #10

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1.

$$v(x) = A\sin kx + B\cos kx + \frac{R}{P}(L-x)$$

$$v'(x) = Ak\cos kx - Bk\sin kx - \frac{R}{P}$$

$$v''(x) = -Ak^2\sin kx - Bk^2\cos kx$$

Boundaries:

$v(0)=0, \quad B = \frac{-R}{P}L$

$v'(0)=0, \quad Ak = \frac{R}{P}, \; A = \frac{R}{Pk}$

$v(L)=0, \quad A\sin kL + B\cos kL = 0$

$$\frac{R}{P}\left[\frac{1}{k}\sin kL - L\cos kL\right] = 0$$

either $\frac{R}{P} = 0$, $\tan kL = kL$

true at: $kL = 4.49 \qquad P = \frac{2.045\pi^2 EI}{L^2}$

$kL = 7.73 \qquad P = \frac{6.05\pi^2 EI}{L^2}$

Mode shapes:

$$v(x) = A\sin\left(4.49\frac{x}{L}\right) + B\cos\left(4.49\frac{x}{L}\right) + \frac{R}{P}(L-x)$$

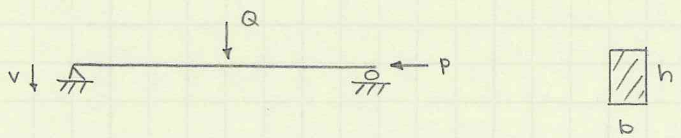
$$A = \frac{RL}{4.49P}, \quad B = \frac{-R}{P}L$$

$$v(x) = A\sin\left(7.73\frac{x}{L}\right) + B\cos\left(7.73\frac{x}{L}\right) + \frac{R}{P}(L-x)$$

$$A = \frac{RL}{7.73P}, \quad B = \frac{-R}{P}L$$

HOMEWORK #10

2.



Deflection from load:

$$-EIv_0'''' = q(x) = 0$$
$$-EIv_0''' = A$$
$$-EIv_0'' = Ax + B$$
$$-EIv_0' = \frac{A}{2}x^2 + Bx + C$$
$$-EIv_0 = \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D$$

Apply boundaries:

$$v_0(0) = 0, \quad D = 0$$
$$v_0''(0) = 0, \quad B = 0$$
$$-EIv_0'''(0) = \frac{-Q}{2}, \quad A = \frac{-Q}{2}$$
$$v_0'(L/2) = 0, \quad \frac{-Q}{4}(L/2)^2 + C = 0$$
$$C = \frac{Q}{16}L^2$$

$$\longrightarrow v_0(x) = \frac{QL^3}{EI}\left[\frac{-1}{12}\left(\frac{x}{L}\right)^3 + \frac{1}{16}\left(\frac{x}{L}\right)\right]$$
$$v_0'(x) = \frac{QL^3}{EI}\left[\frac{-1}{4}\cdot\frac{x^2}{L^3} + \frac{1}{16L}\right]$$
$$v_0''(x) = \frac{QL}{EI}\left[\frac{-1}{2}\frac{x}{L}\right] = \frac{-Qx}{2EI}$$

Now, $v'' + k^2 v = v_0''$

$$v(x) = A\sin kx + B\cos kx - \frac{Qx}{2EI}\cdot\frac{1}{k^2}$$
$$v(0) = 0, \quad B = 0$$
$$v'(x) = Ak\cos kx - \cancel{Bk\sin kx} - \frac{Q}{2P}$$
$$v'(L/2) = 0, \quad Ak\cos kL/2 = \frac{Q}{2P}$$
$$A = \frac{Q}{2EIk^3\cos kL/2}$$

$$v(x) = \frac{Q}{2EIk^2}\left[\frac{\sin kx}{k\cos kL/2} - x\right], \text{ max at } x = L/2$$

$$\boxed{v_{max} = \frac{Q}{2EI\cdot k^2}\left[\frac{1}{k}\tan\frac{kL}{2} - \frac{L}{2}\right]}$$

HOMEWORK #10

2. (cont'd)

$$v_{max} = \frac{Q}{2EI\,k^3}\left[\tan\frac{kL}{2} - \frac{kL}{2}\right]$$

$$kL = \left[\frac{PL^2}{EI}\right]^{1/2} = \pi\sqrt{P/P_c}$$

$L = 20\,in$
$Q = 50\,lb$
$\sigma_o = 50\,ksi$
$h = b = 1\,in$, $I = 1/12\,in^4$
$E = 30\times10^6\,psi$

$$v_{max} = \frac{QL^3}{2EI\lambda^3}\left(\tan\lambda/2 - \lambda/2\right), \quad \lambda = kL$$

M_{max} at $L/2$,

$$v''(x) = \frac{Q}{2EI\cdot k^3}\left[\frac{-\sin kx}{\cos kL/2}\right]\cdot k^2 = \frac{-Q}{2EIk}\cdot\frac{\sin kx}{\cos kL/2}$$

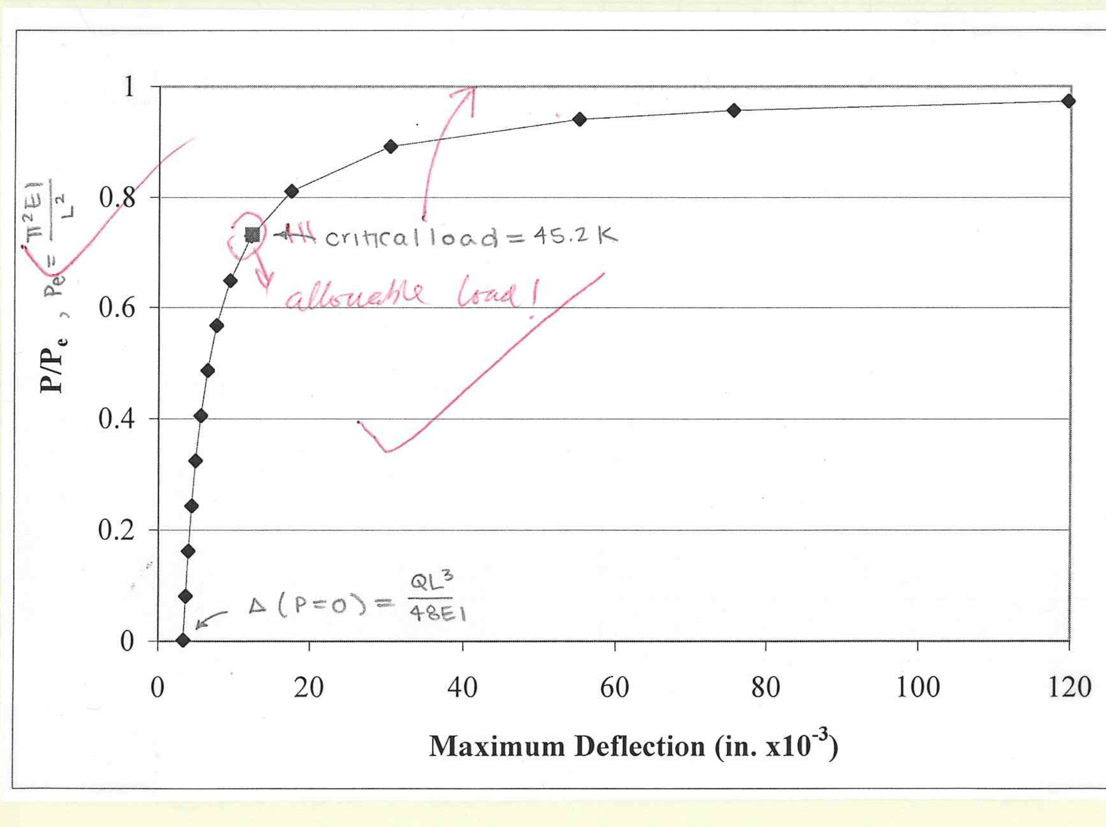
$$v''(L/2) = \frac{-Q}{2\cancel{E}Ik}\cdot\tan\frac{kL}{2} = \frac{-M_{max}}{\cancel{E}I}$$

$$\sigma_{max} = \frac{P}{A} + \frac{M_{max}c}{I} = \sigma_o$$

$$\sigma_o = \frac{P}{bh} + \frac{Q}{2k}\tan\frac{kL}{2}\cdot\frac{h/2}{1/12\,bh^3}$$

$$\sigma_o = \frac{P}{bh} + \frac{3Q}{kbh^2}\tan\frac{kL}{2}$$

solving in Excel,
$P_{cr} = 45.2\,K$



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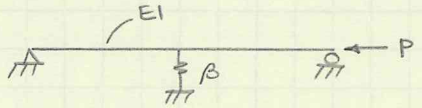
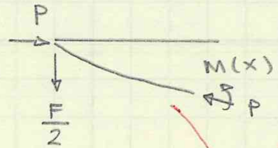
HOMEWORK #10

3.

Note: I have redifined the spring stiffness as β, so as to not confuse it with $K^2 = \frac{P}{EI}$





F = spring force

$$M(x) = Pv - \frac{F}{2}x = -EIv''$$

$$EIv'' + Pv = \frac{F}{2}x = \frac{1}{2}\beta v_o \cdot x$$

$v_o = v(L/2)$

$F = \beta v_o$

$$v(x) = A\sin Kx + B\cos Kx + \frac{\beta v_o}{2K^2}\cdot\frac{x}{EI}$$

$v(0) = 0, \quad B = 0$

$v(L/2) = v_o, \quad v_o = A\sin K\frac{L}{2} + \frac{\beta v_o}{2K^2}\cdot\frac{L}{2}\cdot\frac{1}{EI}$

$$v_o = \frac{A\sin KL/2}{1 - \beta L/4EIK^2}$$

$$v'(x) = AK\cos Kx + \frac{\beta v_o}{2K^2}$$

$v'(L/2) = 0, \quad 0 = AK\cos\frac{KL}{2} + \frac{\beta v_o}{2EIK^2}$

$$v_o = -\frac{2AK^3}{\beta}\cos\frac{KL}{2}\cdot EI$$

Equating,

$$\frac{-2AEIK^3}{\beta}\cos\frac{KL}{2}\left[1 - \beta L/4EIK^2\right] = A\sin\frac{KL}{2}$$

$$\frac{-2}{\beta}EI\cdot K^3 + \frac{1}{2}KL = \tan\frac{KL}{2}$$

$$\boxed{\frac{KL}{2} - \frac{EI}{L^3}\cdot\frac{2(KL)^3}{\beta} = \tan\frac{KL}{2}}$$

If $\bar{K} = \frac{16EI}{\beta L^3}$, $\xi = \frac{KL}{2}$, $\quad \underline{\xi - \bar{K}\xi^3 = \tan\xi}$

HOMEWORK #10

3. (cont'd)

Approximate solution:

$$V = \int_0^L \frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx + \int_0^L \frac{P}{2}\left(\frac{dv}{dx}\right)^2 dx + \underbrace{\tfrac{1}{2}\beta v(L/2)^2}_{\substack{\text{potential energy of a spring} \\ =\frac{1}{2}kx^2}} - \underbrace{\beta v(L/2)^2}_{\text{work done by spring}}$$

Assume

$$v(x) = a_1 \sin\frac{\pi x}{L}, \quad v(0)=0,\ v(L)=0$$

$$v'(x) = \frac{\pi}{L} a_1 \cos\pi\frac{x}{L}$$

$$v''(x) = -a_1\left(\frac{\pi}{L}\right)^2 \sin\frac{\pi x}{L}$$

$$v(L/2) = a_1 \sin \pi/2 = a_1$$

$$\frac{\partial V}{\partial a_1} = EI \cdot a_1\left(\frac{\pi}{L}\right)^4 \int_0^L \left(-\sin\frac{\pi x}{L}\right)^2 dx + Pa_1\left(\frac{\pi}{L}\right)^2 \int_0^L \left(\cos\frac{\pi x}{L}\right)^2 dx - \beta a_1$$

$$\int_0^L \sin^2\frac{\pi x}{L}\,dx = \frac{x}{2} - \frac{L\sin\left(\frac{2\pi x}{L}\right)}{4\pi}\Bigg|_0^L = \frac{L}{2}$$

$$\int_0^L \cos^2\frac{\pi x}{L}\,dx = \frac{x}{2} + \frac{L\sin\left(\frac{2\pi x}{L}\right)}{4\pi}\Bigg|_0^L = \frac{L}{2}$$

$$\frac{EI\cancel{a_1}\pi^4}{L^4}\cdot\frac{L}{2} + P\cancel{a_1}\frac{\pi^2}{2L} = \beta\cancel{a_1}$$

$$P = \frac{\beta + EI\pi^4/2L^3}{\pi^2}\cdot 2L = \frac{2\beta L}{\pi^2}\left(1 + \frac{\pi^4}{2}\cdot\frac{EI}{\beta L^3}\right)$$

if you like, please see my solution post.

Comparing to exact, approximation accuracy varies with choice of constants. But, increasing β increases the load, as does decreasing L and increasing I.

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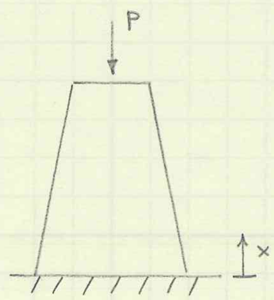
HOMEWORK #10

4.



$b(x) = b$
$h(x) = 2h(1 - x/2L)$

$A(x) = 2bh(1 - x/2L)$
$I(x) = \frac{2}{3}bh^3(1 - x/2L)^3$
$I_w(x) = \frac{1}{6}hb^3(1 - x/2L)$

$$V(x) = \int_0^L \frac{EI}{2}(v'')^2\,dx - \int_0^L \frac{1}{2}P(v')^2\,dx$$

Approximate:

$$v(x) = a_1\phi_1 + a_2\phi_2$$

$$\phi_1 = 1 - \cos\frac{\pi x}{2L} = 1 - \cos\frac{\pi}{2}\xi$$

$$\phi_2 = 1 - \cos\frac{3\pi x}{2L} = 1 - \cos\frac{3}{2}\pi\xi$$

$$\frac{\partial V}{\partial a_1} = \frac{Ehb^3}{6L^3}\int_0^1 (1 - \xi/2)(a_1\phi_1'' + a_2\phi_2'')\phi_1''\,d\xi - P\int_0^1 (a_1\phi_1' + a_2\phi_2')\phi_1'\,d\xi$$

$$\phi_1' = \frac{\pi}{2}\sin\frac{\pi}{2}\xi \qquad \phi_2' = \frac{3}{2}\pi\sin\frac{3}{2}\pi\xi$$

$$\phi_1'' = \frac{\pi^2}{4}\cos\frac{\pi}{2}\xi \qquad \phi_2'' = \frac{9}{4}\pi^2\cos\frac{3}{2}\pi\xi$$

Sample calculations:

$$\frac{Ehb^3}{6L^3}\begin{bmatrix} \int_0^1 (1-\xi/2)(\pi^2/4\cos\pi/2\,\xi)^2\,d\xi & \int_0^1 (1-\xi/2)\pi^4(\frac{1}{4}\cos\frac{\pi}{2}\xi)(9/4\cos\frac{3}{2}\pi\xi)\,d\xi \\ \int_0^1 (1-\xi/2)(\pi^2/4\cos\pi/2\,\xi)(\frac{9}{4}\pi^2\cos\frac{3}{2}\pi\xi)\,d\xi & \int_0^1 (1-\xi/2)(9/4\pi^2\cos\frac{3}{2}\pi\xi)^2\,d\xi \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\int_0^1 \xi\cos^2\frac{\pi}{2}\xi = \frac{1}{4\pi^2}\Big[2\cos\pi\xi + \pi\xi(\pi\xi + 2\sin(\pi\xi))\Big]_0^1 = \frac{1}{4} - \frac{1}{\pi^2}$$

$$\int_0^1 \cos^2\frac{\pi}{2}\xi = \frac{\xi}{2} + \frac{\sin\pi\xi}{2\pi}\Big|_0^1 = \frac{1}{2}$$

$$\int_0^1 \xi\cos\frac{\pi}{2}\xi\cos\frac{3\pi}{2}\xi = \frac{-1}{2\pi^2}$$

$$\int_0^1 \cos\frac{\pi}{2}\xi\cos\frac{3\pi}{2}\xi = \frac{\sin\pi\xi}{2\pi} + \frac{\sin 2\pi\xi}{4\pi}\Big|_0^1 = 0$$

$$\int_0^1 \xi\cos^2\frac{3}{2}\pi\xi = \frac{-1}{9\pi^2} + \frac{1}{4}$$

HOMEWORK #10

4. (cont'd)

$$\frac{Ehb^3}{6L^3}\begin{bmatrix} \frac{\pi^4}{16}\left(\frac{3}{8}+\frac{1}{2\pi^2}\right) & \frac{9\pi^2}{64} \\ \frac{9\pi^2}{64} & \frac{81\pi^4}{16}\left(\frac{3}{8}+\frac{1}{18\pi^2}\right)\end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = P\begin{bmatrix} \pi^2/8 & 0 \\ 0 & 9\pi^2/8 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\underset{\sim}{K} \qquad\qquad \underset{\sim}{B}$$

$$\lambda = \frac{6PL^3}{Ehb^3}$$

$$|\underset{\sim}{K} - \lambda \underset{\sim}{B}| = 0$$

$$\det\begin{bmatrix} \frac{\pi^4}{16}\left(\frac{3}{8}+\frac{1}{2\pi^2}\right) - \lambda\frac{\pi^2}{8} & \frac{9\pi^2}{64} \\ \frac{9\pi^2}{64} & \frac{81\pi^4}{16}\left(\frac{3}{8}+\frac{1}{18\pi^2}\right) - \lambda\frac{9\pi^2}{8}\end{bmatrix} = 0$$

solving for the roots,

$$\lambda = 16.9,\ 2.09 = \frac{6PL^3}{Ehb^3}$$

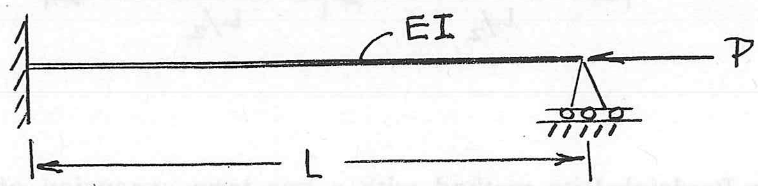
$$P_{cr} = \frac{2.09\,Ehb^3}{6L^3}, \quad \text{or,} \quad I_o = 1/12\,(2h)\,b^3 = 1/6\,hb^3$$

$$P_{cr} = \frac{0.21\,\pi^2 E I_o}{L^3}$$

10

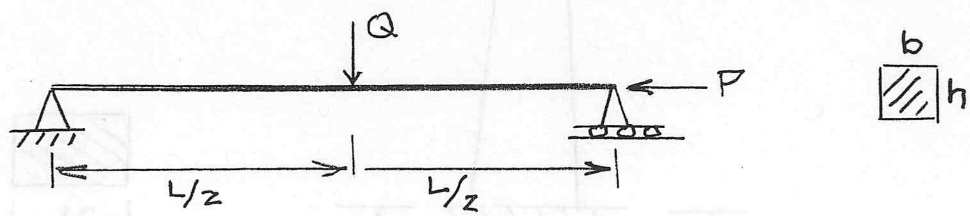
ADVANCED STRENGTH OF MATERIALS (ASE/EM 339)
Homework #10

1. Calculate the first two eigenvalues and corresponding eigenmodes for the column shown in the figure.



2. A simply supported column is loaded with a transverse load Q and a compressive axial load P.

   (a) Find an expression for the maximum deflection

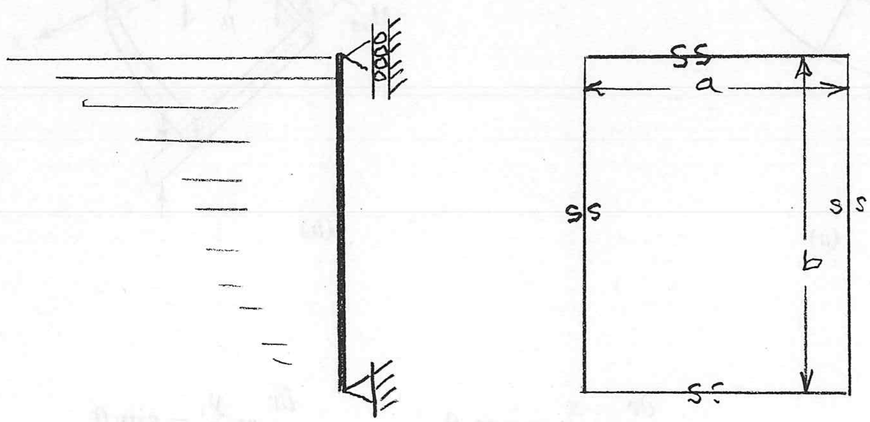


   (b) For the variables given below, calculate the maximum allowable axial load. Plot P versus $v(L/2)$ and identify the critical load.

$$L = 20 \text{ in}$$
$$Q = 50 \text{ lb}$$
$$\sigma_o = 50 \times 10^3 \text{ psi}$$
$$h = b = 1 \text{ in}$$
$$E = 30 \times 10^6 \text{ psi}$$

ADVANCED STRENGTH OF MATERIALS (ASE/EM 339)
Homework #11

1. A small dam can be represented as a simply supported plate as shown. Find an expression for the deflection of the plate.



2. The lateral deflection of a rectangular plate with built-in edges of lengths a and b and subjected to a uniform load q_o, is given by

$$w = c_0\left(\frac{x^4}{a^4} - 2\frac{x^3}{a^3} + \frac{x^2}{a^2}\right)\left(\frac{y^4}{b^4} - 2\frac{y^3}{b^3} + \frac{y^2}{b^2}\right)$$

where c_0 is a constant. Determine: *(a)* whether this deflection satisfies the boundary conditions of the plate; *(b)* the maximum plane stress components σ_x and τ_{xy} at the center, for $a = b$.

3. A square spacecraft panel is subjected to uniformly distributed twisting moment $M_{xy} = M_o$ along all four edges. Determine an expression for the deflection surface w.

4. A circular plate radius R is clamped at the boundary and loaded with a uniform distributed load q_o per unit area. Plot the stresses that develop (i.e., σ_r, σ_θ) on the top and bottom surfaces as a function of r. Discuss your results.

5. Calculate an expression for the deflected shape ($w(r)$) of a circular plate radius R that is simply supported at the edges and is loaded by a point load P at the center.

ANALYSIS OF STRESS

Pg 1-19a

Equations

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \sigma_{xy}\sin 2\theta$$

$$\tau_n = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \sigma_{xy}\cos 2\theta$$

Principal stresses

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \sigma_{xy}^2}$$

$$\tan 2\theta_p = \frac{2\sigma_{xy}}{\sigma_x - \sigma_y}$$

handout pg 9a

Failure theories

$$\sigma_o^2 = \frac{1}{2}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2)\right]$$ pg 9b

$$\left|\frac{\sigma_1 - \sigma_2}{2}\right| = \frac{\sigma_o}{2}$$, etc. pg 13

Strain Equations pg 17a $\varepsilon_n = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy}\sin\theta\cos\theta$

σ/ε relationships pg 19a

BEAM THEORIES

pg 20-47

Equations

$$\frac{d^2M}{\partial x^2} = \frac{dV}{dx} = -q(x) \,, \qquad q(x) = EI\frac{d^4v}{dx^4}$$

$$\varepsilon = yK$$

Half. range functions – pg 27, 27a

$$\langle x-a \rangle = \begin{cases} 0 & \text{for } (x-a) < 0 \\ x-a & \text{for } (x-a) \geq 0 \end{cases}$$

continuous supports – pg 29-40

$$\frac{d^4v}{dx^4} + \alpha^4 v = \frac{q(x)}{EI} \,, \qquad \alpha^4 = \frac{k}{EI}$$

$$v_H = \underbrace{e^{\mu x}\left[c_1 \sin\mu x + c_2 \cos\mu x\right]}_{\text{0 for an infinitely long beam, as } v(\infty)=0} + e^{-\mu x}\left[c_3 \sin\mu x + c_4 \cos\mu x\right]$$

Remember particular solution! (dist. load)

$$\mu^4 = \frac{k}{4EI}$$

page of equations — 32a

Shear Effects – pg 40

$$V = \alpha A \tau_{max} = \alpha AG\frac{dv_s}{dx} \,, \qquad -q(x) = \alpha GA\frac{d^2v_s}{dx^2}$$

$$\frac{d^2v}{dx^2} = \frac{-1}{EI}\left[M(x) + \frac{EI}{\alpha GA}q(x)\right]$$

$v_s'(0) \neq 0$ at a fixed end! (pg 46)

$$v_s'(0) = \frac{V(0)}{\alpha GA}$$

BEAM THEORIES

pg 48 - 63

Axially loaded beams - pg 48-57

$$EI\frac{d^4v}{dx^4} - N\frac{d^2v}{dx^2} = q(x)$$

$$\Delta = \frac{-N_0L}{EA} + \frac{1}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx$$ - pg 55

generally taken as 0

$$v(x) = A\sinh kx + B\cosh kx + Cx + D$$

Inelastic Behavior - pg 58-63

$M_0 = \frac{1}{6} b h_0^2 \sigma_0$ for rectangular section

$$K_0 = \frac{\varepsilon_0}{h/2} = \frac{2\sigma_0}{Eh}$$

$M = \int_{-h/2}^{h/2} E\varepsilon y\,dy \cdot b$, constant b

$(\sigma \cdot y\,dA)$

ENERGY METHODS

pg 64–88

Introduction – pg 64

$$U_0 = \int_0^{\varepsilon_x} \sigma_x \, d\varepsilon_x$$

$$U_0^c = \int_0^{\sigma_x} \varepsilon_x \, d\sigma_x$$

– equal each other in the linearly elastic case

$$U_0 = U_0^c$$

$$U = \int_0^L \frac{1}{2} P \frac{\partial \delta}{dx} dx = \int_0^L \frac{1}{2} EA \left(\frac{d\delta}{dx}\right)^2 dx = \int_0^L \frac{1}{2} \frac{P^2}{EA} dx \quad \text{– pg 66}$$

$$U = \int_0^L \frac{1}{2} MK \, dx = \int_0^L \frac{1}{2} EIK^2 dx = \int_0^L \frac{1}{2} \frac{M^2}{EI} dx \quad \text{– pg 67}$$

$$U = \int_0^L \frac{1}{2} T \frac{d\phi}{dx} dx = \int_0^L \frac{1}{2} GJ \left(\frac{d\phi}{dx}\right)^2 dx = \int_0^L \frac{T^2}{2GJ} dx \quad \text{– pg 68}$$

Principle of virtual work – pg 69

$$\Delta = \frac{\partial U^c}{\partial P} = \frac{\partial U}{\partial P} \quad \text{if linear·elastic}$$

add dummy load at point of interest, if necessary. Then set load = 0.

Shear deformation – pg 78

$$U = \int_0^L \beta \frac{V^2}{2GA} dx$$

$$\beta = \frac{A}{I^2} \int_A \frac{Q^2}{b^2} dA$$

$\beta = 6/5$ for a rectangle
$= 10/9$ for a ring

Unit load method

$$\Delta = \int \frac{mM}{EI} dx + \int \frac{nN}{EA} dx + \int \frac{tT}{JG} dx + \int \beta \frac{vV}{GA} dx \quad \text{– pg 83}$$

$$\Delta = \Sigma \frac{nNL}{EA} \quad \text{– pg 85}$$

Rayleigh·Ritz Method – pg 85a

$$V = \int_0^L \frac{EI}{2} \left(\frac{d^2v}{dx^2}\right)^2 dx + \underbrace{\int_0^L \frac{k}{2} (v)^2 dx}_{\text{elastic foundation}} + \int_0^L \frac{N}{2} \left(\frac{dv}{dx}\right)^2 dx + \int_0^L - q(x) \, v \, dx \quad \text{– pg 86}$$

$$\frac{\partial V}{\partial a_1} = 0, \quad \frac{\partial V}{\partial a_2} = 0 \qquad v(x) \sim a_1 \phi_1(x) + a_2 \phi_2(x) \ldots$$

BUCKLING

pg 89 - 99

Introduction

$$EI\frac{d^4v}{dx^4} + P\frac{d^2v}{dx^2} = 0$$

solve for P that makes this true

$$v(x) = A\sin kx + B\cos kx + Cx + D, \quad k^2 = \frac{P}{EI}$$

$$P_c = \frac{\pi^2 EI}{L^2} \cdot n^2$$

Eccentric loading - pg 95

$$v'' + k^2 v = -k^2 e$$

$$\sigma_{max} = \frac{P}{A} + \frac{M_{max} \cdot c}{I} = \sigma_0$$

onset of first yield

imperfect columns - pg 96

$$v'' + k^2 v = v_0''$$

PLATE BENDING

pg 100-108

Introduction

$$\underline{M} = \frac{E}{1-\nu^2} \cdot \frac{t^3}{12} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{bmatrix} K_x \\ K_y \\ 2K_{xy} \end{bmatrix}$$

$$D\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right] = q(x,y)$$

EM 339: MATERIALS | | 18 December 07

RANDOM INFO

integrals

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin 2x}{4}$$

$$\int \sin^2 ax \, dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$\int \cos^2 ax \, dx = \frac{x}{2} + \frac{\sin 2ax}{4}$$

$$\int \sin ax \cdot \cos ax = \frac{-\cos^2 ax}{2a}$$

$$\int \sin ax \cdot \cos bx \, dx = \frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)}$$

$$\int \sinh ax \, dx = \frac{1}{a} \cosh ax$$

$$\int \cosh ax \, dx = \frac{1}{a} \sinh ax$$ ← no negative!

ASE/EM 339

REFERENCE BOOKS

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ADVANCED STRENGTH OF MATERIALS (ASE/EM 339)

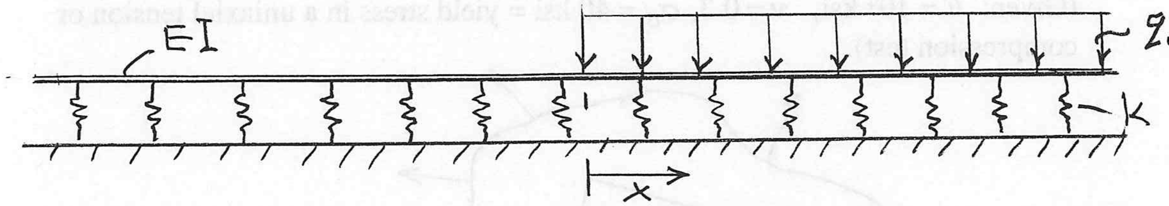
QUIZ #1

Friday, October 14, 2005; 2:00 – 3:00 p.m., WRW312

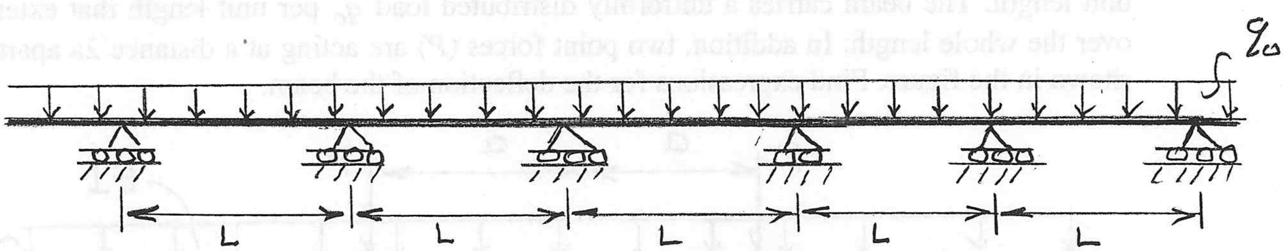
Closed Book Exam

1. An infinite beam of bending rigidity EI is resting on an elastic foundation of stiffness k per unit length. The beam carries a uniformly distributed load q_o per unit length that extends from $x = 0$ to $x \rightarrow \infty$.

   (a) Find expressions for the deflection of the beam.
   (b) Find the position and value of the maximum deflection.



2. A long beam like structure carries a uniformly distributed load q_o per unit length. The beam is simply supported at intervals of span L as shown in the figure. The beam is made of a composite material with elastic modulus E and a shear modulus $G = E/20$. The cross section is rectangular ($b \times h$). Calculate the maximum deflection of a typical span by including the effect of shear. Find the change in the maximum deflection due to shear deformation for $L/h = 10$



ADVANCED STRENGTH OF MATERIALS (ASE/EM 339)

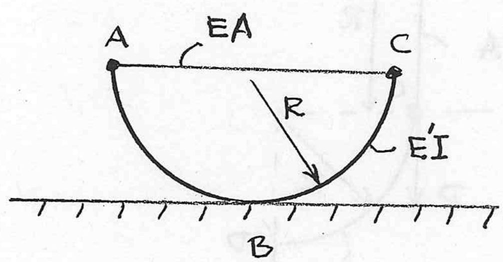
QUIZ #2

Wednesday, November 15, 2006; 2:00 – 3:00 pm, WRW312

Closed Book Exam

1. A thin-walled semi-circular cross section channel is full of water. At the top it is reinforced by struts of axial rigidity EA. Idealize the problem as a thin-walled semi-circular ring of bending rigidity $E'I$ (corresponds to rigidity of a unit width) with a single strut connecting A and C. The structure is loaded by fluid pressure γy ($\gamma = \rho g$ per unit width, ρ is the density of water).

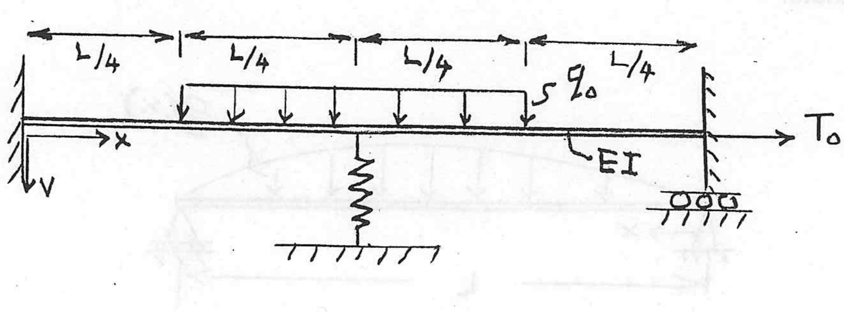
   Use Castigliano's theorems to calculate the force in the strut.



2. A beam caries a uniformly distributed load q_o over part of its span as well as a tensile axial force T_o. The deflection of the beam is to be approximated by $v(x) = a\left[1 - \cos\left(\frac{2\pi x}{L}\right)\right]$. Use the Rayleigh-Ritz method to find the best estimate of a in terms of the parameters of the problem.

   a. Is the suggested function suitable? Explain your answer.

   b. Can you suggest a second function whose inclusion to the solution will improve the approximate solution? How will the improvement influence the maximum deflection?



ADVANCED STRENGTH OF MATERIALS (ASE/EM 339)

FINAL EXAM

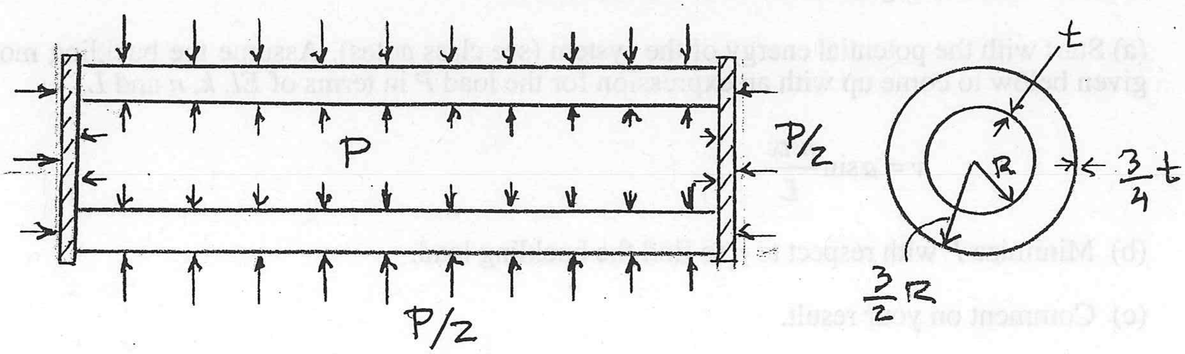
Monday, December 19, 2005; 9:00 – 11:30 p.m., WRW 312

You can use your class notes and homework but no other material.

1. A long double walled cylindrical vessel carries internal pressure (P) in the inner cylinder and external pressure ($P/2$) in the outer cylinder. The Two cylinders are connected to rigid end-plates as shown in the figure. The two cylinders are made of the same material with elastic modulus E, Poisson's ratio ν and yield stress σ_o. Given the geometric characteristics shown in the figure find:

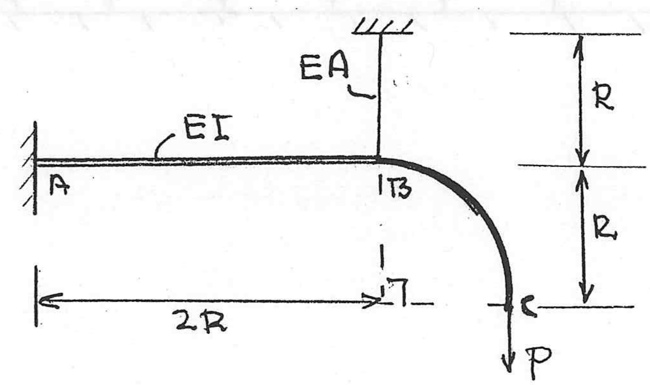
   (a) The stresses in each cylinder in terms of P, R and t (use $\nu = 1/4$).

   (b) Find the maximum allowable value of PR/t in terms of σ_o



2. Beam ABC consists of straight section (AB) of length 2R and a circular arc section (BC) of radius R. The beam has bending rigidity EI, is fixed at A and is supported by a cable of axial rigidity EA at B as shown in the figure. Use an energy method to calculate the vertical displacement of point C in terms of P, EI, EA and R.

   Hint: At the final stage of your calulations use $EA = 3EI/R^2$.



ADVANCED STRENGTH OF MATERIALS (ASE/EM 339)

FINAL EXAM

Friday, December 15, 2006; 2:00 – 4:30 p.m., WRW 113

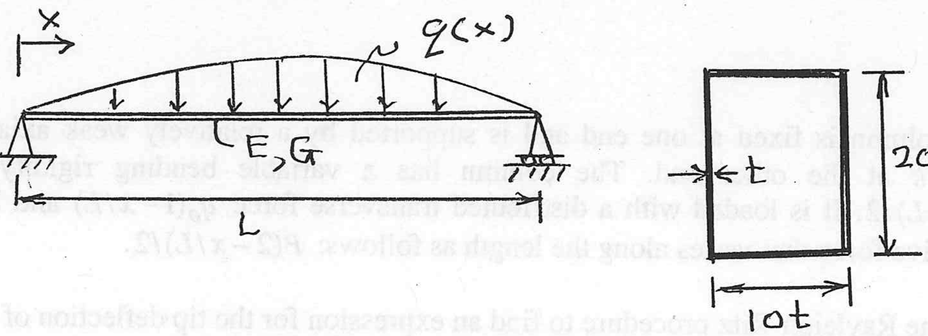
You can use your class notes and homework but no other material.

1. A simply supported beam is loaded by a distributed force $q_o \sin\frac{\pi x}{L}$. The beam is made out of a composite and has a rectangular cross section with the dimensions shown in the figure.

   (a) Calculate the deflection at mid-span including the effect of shear given that the shear modulus of the composite $G = E/20$ and $I = 2778t^4$.

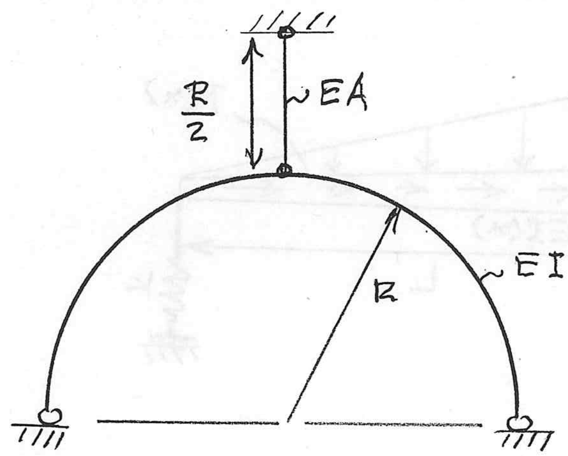
   (b) Given that $L = 300t$ what is the percent change in the maximum displacement from the effect of shear?

   Hint:: You are expected to establish the shear correction factor α for the given cross section.



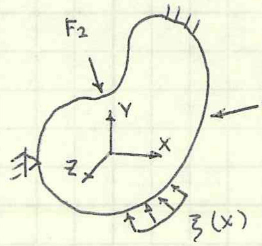
2. A thin semi-circular heavy beam of radius R and bending rigidity EI, is simply supported at the ends and in addition it is supported by a strut of length $R/2$ and axial rigidity EA. Use Castigliano's theorems to find the deflection at the middle support given that the ring weight is $\rho g A$ per unit length.

   Hint: At the final stage of your calculations use $EA = 3EI/R^2$.



ANALYSIS OF STRESS

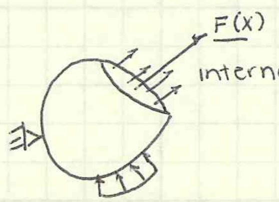
on a body



consider a solid body acted upon by external and body forces

goal: develop a systematic way of representing the internal forces

cut body to find forces



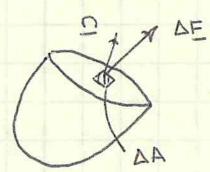
internal forces – keep the half shown in equilibrium with other half (equal & opposite)

The force on any internal point, O, can be evaluated by cutting the body with a plane passing through point O and equilibrating the internal and external forces.

$$\sum F_{int} = \sum F_{ext}, \text{ etc.}$$

It is preferable that we talk in terms of internal stresses, rather than forces.

Traction vector



= point O

$\Delta \underline{F}$: local force acting on an elemental area ΔA around point O

Local traction vector:

$$\underset{\sim}{\overset{n}{T}} = \lim_{\Delta A \to 0} \frac{\Delta \underline{F}}{\Delta A} = \frac{d\underline{F}(x)}{dA}$$

vector

$$\overset{n}{\underline{T}} = \begin{Bmatrix} \overset{n}{T}_x \\ \overset{n}{T}_y \\ \overset{n}{T}_z \end{Bmatrix}$$

units: $\left[\overset{n}{\underline{T}} \right] = [\text{stress}]$

clearly, a complete description of the internal state of stress at point O requires that we know $\overset{n}{\underline{T}} \; \forall \, \underline{n}$ "for all"

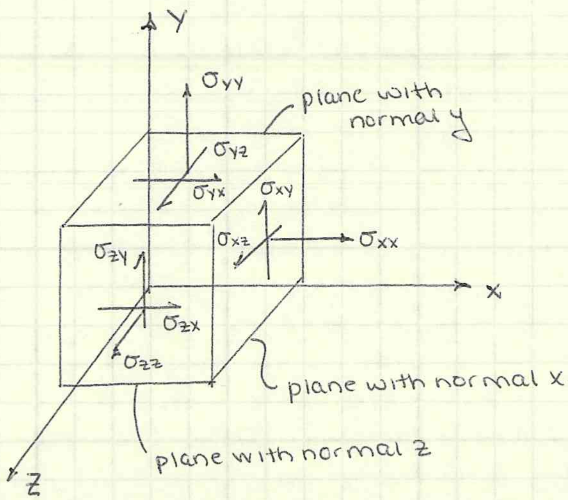
infinite number of planes $\longrightarrow$ infinite number of $\overset{n}{\underline{T}}$

ANALYSIS OF STRESS

on a generic body (cont'd)

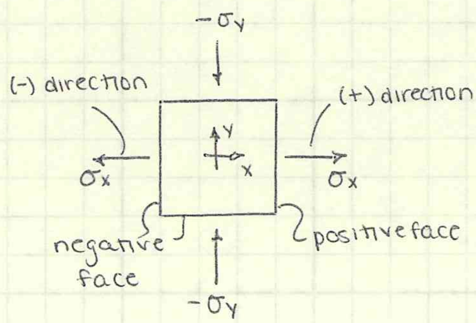
we will show that it is sufficient to know the components of stress on three mutually perpendicular planes through O

components of stress



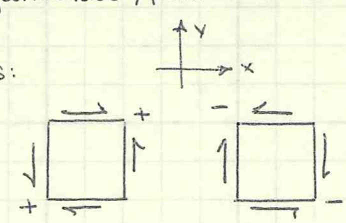
consider the special case when the internal planes considered are aligned with the axes (x, y, z)

σ_{ij} : i = plane with normal i
j = component acting in direction j



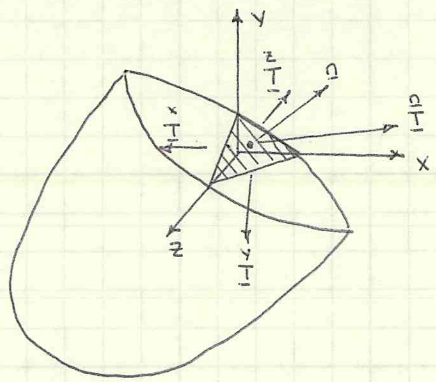
Sign convention
- tension is positive
- (+) positive face / positive direction; negative face / negative direction
- (-) positive face / negative direction; negative face / positive direction

Shears:



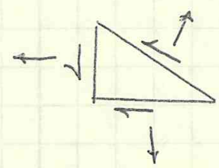
ANALYSIS OF STRESS

Stresses in a body



shaded area – elemental pyramid
centered around point O,
lining up with axes
back sides of pyramid have normals
x, y, and z

3D version of:

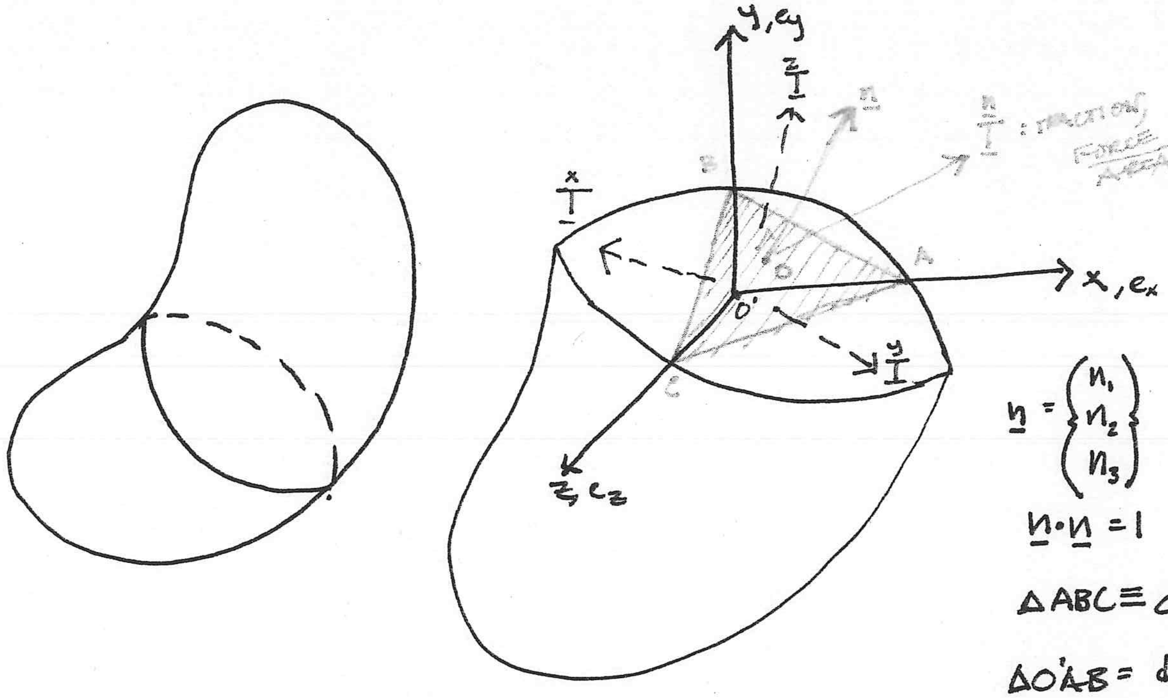


4 tractions must be in equilibrium

$$A_n \overset{n}{\underline{T}} = A_x \overset{x}{\underline{T}} + A_y \overset{y}{\underline{T}} + A_z \overset{z}{\underline{T}}$$

areas should be dA, not just A

$$\overset{n}{\underline{T}}\, dA_n = \overset{x}{\underline{T}}\, dA_x + \overset{y}{\underline{T}}\, dA_y + \overset{z}{\underline{T}}\, dA_z$$



$$\underline{n} = \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}$$

$$\underline{n}\cdot\underline{n} = 1$$

$$\Delta ABC \equiv dA_n$$

$$\Delta O'AB = dA_z$$

$$\Delta O'BC = dA_x$$

$$\Delta O'AC = dA_y$$

LET THE TRACTIONS ON EACH AREA BE $\overset{n}{\underline{T}}, \overset{x}{\underline{T}}, \overset{y}{\underline{T}}, \overset{z}{\underline{T}}$

EQUILIBRIUM OF FORCES REQUIRES THAT:

$$\overset{n}{\underline{T}}\, dA_n = \overset{x}{\underline{T}}\, dA_x + \overset{y}{\underline{T}}\, dA_y + \overset{z}{\underline{T}}\, dA_z \quad (*)$$

BUT, $dA_x = dA_n \underbrace{\underline{n}\cdot\underline{e}_{x}} = dA_n \underbrace{n_1}$ (PROJECTION OF A_n ON PLANE O'BC)

FIRST COMPONENT

$dA_y = dA_n\, \underline{n}\cdot e_y = dA_n n_2$ (**)

$dA_z = dA_n \underline{n}\cdot e_y = dA_n n_3$

SUBSTITUTE (**) INTO (*): $\overset{n}{\underline{T}}\, \cancel{dA_n} = (\overset{x}{\underline{T}} n_1 + \overset{y}{\underline{T}} n_2 + \overset{z}{\underline{T}} n_3)\, \cancel{dA_n}$

$$\therefore\ \overset{n}{\underline{T}} = \overset{x}{\underline{T}} n_1 + \overset{y}{\underline{T}} n_2 + \overset{z}{\underline{T}} n_3 \qquad (2)$$

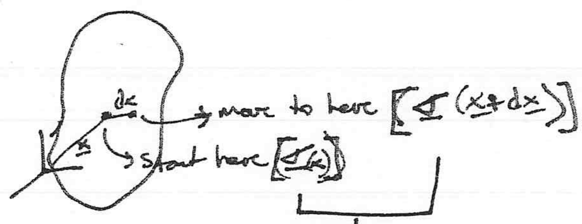
$$\begin{Bmatrix} \overset{n}{T}_x \\ \overset{n}{T}_y \\ \overset{n}{T}_z \end{Bmatrix} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xz} \end{Bmatrix} n_1 + \begin{Bmatrix} \sigma_{yx} \\ \sigma_{yy} \\ \sigma_{yz} \end{Bmatrix} n_2 + \begin{Bmatrix} \sigma_{zx} \\ \sigma_{zy} \\ \sigma_{zz} \end{Bmatrix} n_3$$

$$\text{OR} = \begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \dots \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} \quad (3)$$

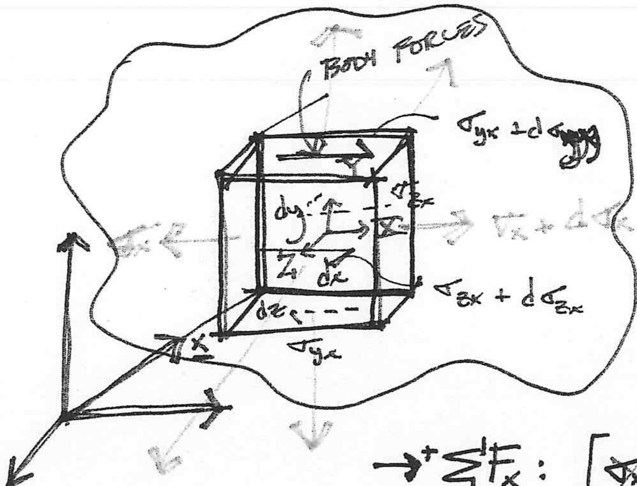
: EQN'S (2) & (3) EXIST IN THE LIMIT AS $dA_n \to \emptyset$, AND THE FOUR PLANES PASS THROUGH O.

- SINCE $\underline{n}$ IS ARBITRARY, THE STRESSES ON ANY PLANE THROUGH O CAN BE EVALUATED FROM EQUATION (3) IF $[\underline{\sigma}]$ (NINE NUMBERS) ARE KNOWN.

EQUILIBRIUM EQUATIONS



RULES FOR HOW STRESSES RELATED AS YOU MOVE THROUGH THE SOLID!



$$\xrightarrow{+}\sum F_x: \left[\cancel{\sigma_x} + \frac{\partial \sigma_x}{\partial x}\cancel{dx}\right]\cancel{dz\,dy} - \cancel{\sigma_x}\,\cancel{dz}\,dy +$$

$$\left[\cancel{\sigma_{yx}} + \frac{\partial \sigma_{xy}}{\partial y}\cancel{dy}\right]\cancel{dx\,dz} - \cancel{\sigma_{yx}}\,\cancel{dx}\,dz +$$

$$\left[\cancel{\sigma_{zx}} + \frac{\partial \sigma_{zx}}{\partial z}\cancel{dz}\right]\cancel{dx\,dy} - \cancel{\sigma_{zx}}\,\cancel{dx}\,dy +$$

$$X\,\cancel{dx\,dy\,dz} = \emptyset$$

(4) EQUILIBRIUM EQUATIONS ⇒

$$\therefore \boxed{\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + X = \emptyset}$$

THIS EQ. MUST BE SATISFIED;

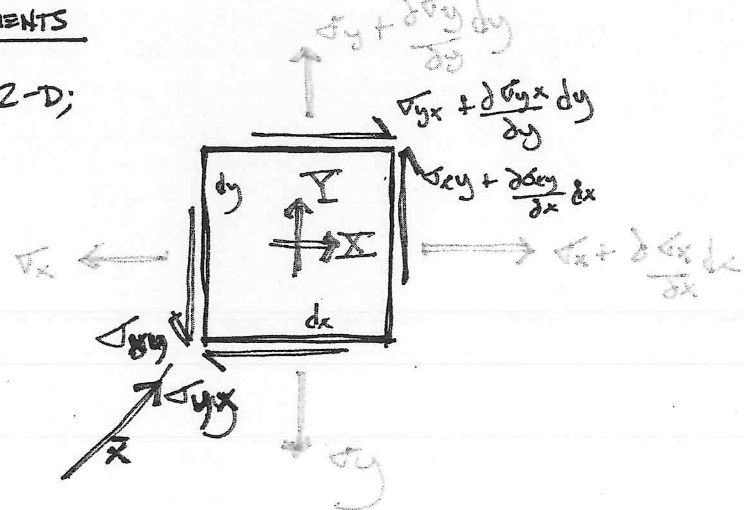
LIKEWISE, AS THERE ARE OTHER SURFACES...

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + Y = \emptyset$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + Z = \emptyset$$

EQUILIBRIUM OF MOMENTS

SPECIAL CASE, 2-D;



- TAKE MOMENTS ABOUT ANY POINT YOU CAN SHOW $\boxed{\sigma_{xy} = \sigma_{yx}}$ (SEE HOMEWORK)
- SIMILARLY FOR THE XZ AND YZ PLANES, WE GET $\sigma_{xz} = \sigma_{zx}$

$$\sigma_{yz} = \sigma_{zy}$$

$\therefore$ THE STRESS MATRIX IN EQ. (3) IS SYMMETRIC!

STRESS STATES

Two Dimensions

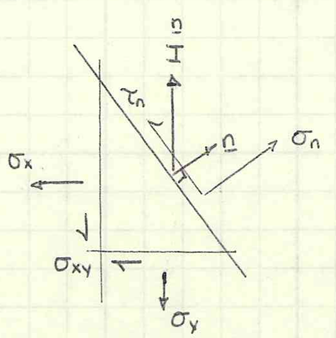
$$\left.\begin{array}{l} \overset{n}{\underline{T}} = \underline{\sigma}\,\underline{n} \\ \underline{\sigma} = \underline{\sigma}^T \end{array}\right\} \quad \begin{Bmatrix} \overset{n}{T}_x \\ \overset{n}{T}_y \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix}$$

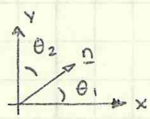
Equilibrium

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + X = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + Y = 0$$

Transformation of stresses (2D)





$$\underline{n} = \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} \cos\theta_1 \\ \cos\theta_2 \end{Bmatrix}$$

$\theta_1 = \theta, \ \theta_2 = 90 - \theta_1$

if you know the stresses in the directions of the axes (x,y), how do you get normal, shear for an arbitrary plane?

$$\overset{n}{\underline{T}} = \underline{\sigma}\,\underline{n}$$

$$\begin{Bmatrix} \overset{n}{T}_x \\ \overset{n}{T}_y \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} \sigma_x n_1 + \sigma_{xy} n_2 \\ \sigma_{xy} n_1 + \sigma_{yy} n_2 \end{Bmatrix}$$

$$\sigma_n = \overset{n}{\underline{T}} \cdot n = \left[\sigma_x n_1 + \sigma_{xy} n_2 \, , \, \sigma_{xy} n_1 + \sigma_{yy} n_2\right] \cdot \left[n_1, n_2\right]$$

$$= \sigma_x n_1^2 + \sigma_y n_2^2 + 2\sigma_{xy} n_1 n_2$$

$$n_1^2 + n_2^2 = \cos^2\theta_1 + \cos^2\theta_2 = \cos^2\theta + \sin^2\theta = 1$$

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\sigma_{xy} \sin\theta\cos\theta$$

$$\boxed{\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \sigma_{xy}\sin 2\theta}$$

$$\underline{\tau}_n = \overset{n}{\underline{T}} - \sigma_n n \longrightarrow \tau_n^2 = \overset{n}{\underline{T}} \cdot \overset{n}{\underline{T}} - \sigma_n^2$$

it can be shown that $\tau_n = -\sigma_x \cos\theta\sin\theta + \sigma_y \sin\theta\cos\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta)$

OR,

$$\boxed{\tau_n = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \sigma_{xy}\cos 2\theta}$$

STRESS STATES

Stress Transformations

$$\sigma_n = \frac{\sigma_x+\sigma_y}{2} + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta + \sigma_{xy}\sin 2\theta$$

$$\tau_n = -\left(\frac{\sigma_x-\sigma_y}{2}\right)\sin 2\theta + \sigma_{xy}\cos 2\theta$$

– sinusoidal functions with maximums and minimums

⟶ PRINCIPAL STRESSES

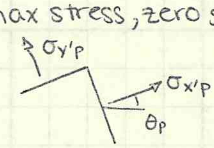
Principal stresses

compute by finding zeroes in derivatives

$$\tan 2\theta_p = \frac{2\sigma_{xy}}{\sigma_x-\sigma_y}, \quad \theta_p \text{ and } \theta_p + \pi/2$$

$$\sigma_{x'p,\,y'p} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \sigma_{xy}^2}$$

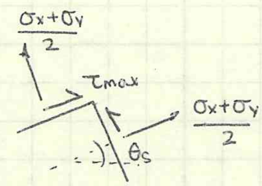
max stress, zero shear



$$\tan 2\theta_s = -\left(\frac{\sigma_x-\sigma_y}{2\sigma_{xy}}\right)$$

$$\theta_s = \theta_p + \pi/4$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \sigma_{xy}^2}$$

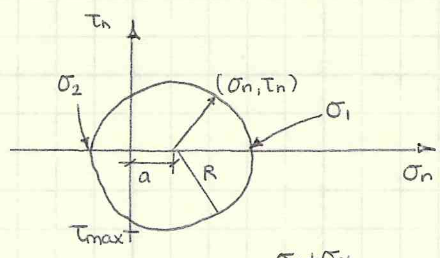


ALL EQUATIONS ARE SUMMARIZED ON HANDOUT.

considering σ_n, τ_n equations,

$$\left[\sigma_n - \frac{\sigma_x+\sigma_y}{2}\right]^2 + \tau_n^2 = \left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \sigma_{xy}^2$$

or $\quad (\sigma_n - a)^2 + \tau_n^2 = R \quad$ – equation for a circle



angle to given σ_x, σ_{xy} is $2\theta_p$ (not θ_p)

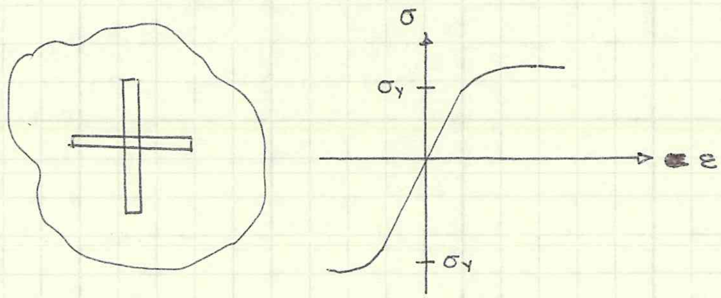
$\frac{\sigma_x+\sigma_y}{2}$ = offset from (0,0) at center

$$R = \left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \sigma_{xy}^2$$

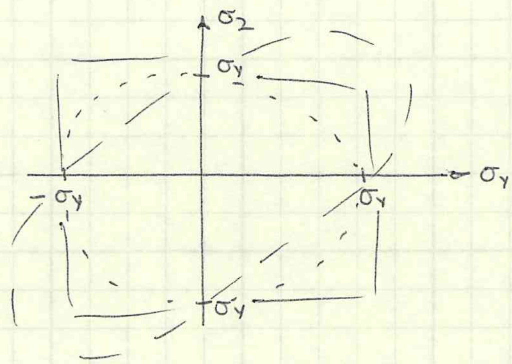
STRESS STATES

Homework #1 – deals with transformations

Yielding under multiaxial stress



two directions of load, two loads
(x, y, positive, negative)



how to connect σ_y points?
circle, box...?

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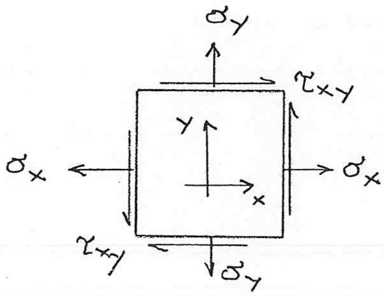
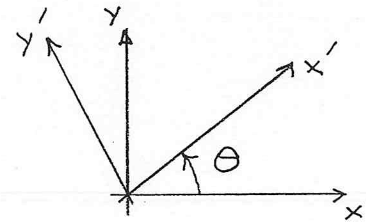
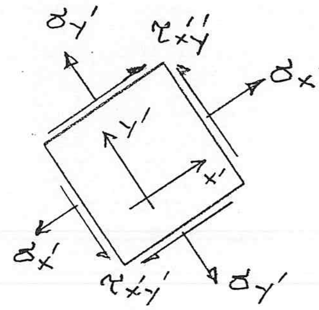
EM 319
TRANSFORMATION OF STRESSES







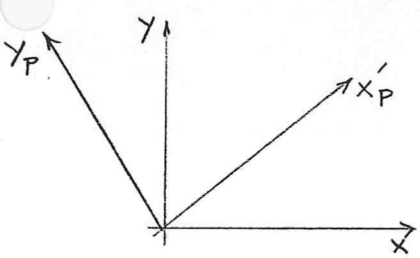
(1) TRANSFORMATION EQUATIONS

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta = \left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y'} = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta = \left(\frac{\sigma_x+\sigma_y}{2}\right) - \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) = -\left(\frac{\sigma_x-\sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

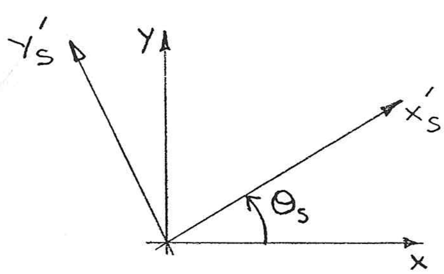
(2) PRINCIPAL VALUES



Principal Directions: $\boxed{\tan 2\theta_p = \left(\frac{2\tau_{xy}}{\sigma_x-\sigma_y}\right)}$

Principal Stresses: $\boxed{\sigma_{x'_p}\Big|_{\substack{\max\\ \min}} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}}$

(3) MAXIMUM SHEARING STRESS



Direction of Max. Shear: $\boxed{\tan 2\theta_s = -\left(\frac{\sigma_x-\sigma_y}{2\tau_{xy}}\right)}$

Maximum Shear Stress: $\boxed{\tau_{x'y'_s}\Big|_{\max} = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}}$

Direct Stress in θ_s frame: $\boxed{\sigma_{x'_s} = \sigma_{y'_s} = \left(\frac{\sigma_x+\sigma_y}{2}\right)}$

The Von Mises and Tresca Yield Criteria

Von Mises postulated (1913) that a *material point yields when the distortional energy reaches a critical value* (also Henky 1924). The criterion can be written in several forms

$$\frac{1}{6}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]=k^2, \tag{1}$$

$$\frac{1}{6}\left[(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2)\right]=k^2.$$

Calibrated to a uniaxial test with yield stress σ_o, $k=\dfrac{\sigma_o}{\sqrt{3}}$. The criterion leads to a circular cylindrical surface in the $(\sigma_1,\sigma_2,\sigma_3)$ space shown in Fig. 1. Its intersection with the Π-plane is a circle of radius $\sqrt{\dfrac{2}{3}}\sigma_o$ (see Fig. 2a). For thin-walled circular tubes plane stress suffices for which (1) reduces to

$$\left[\sigma_{11}^2-\sigma_{11}\sigma_{22}+\sigma_{22}^2+3\sigma_{12}^2\right]^{1/2}=\sigma_o \tag{2}$$

The principal stress version of (2) reduces to the ellipse shown in Fig. 2b.

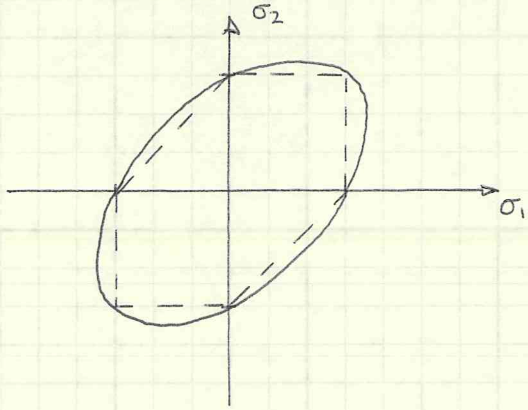
An alternative yield criterion due to Tresca (1864) is based on the postulate that *yielding occurs when the maximum shear reaches a critical value* which can be written as

$$\max\left\{\frac{\sigma_1-\sigma_2}{2}=\kappa,\ \frac{\sigma_2-\sigma_3}{2}=\kappa,\ \frac{\sigma_3-\sigma_1}{2}=\kappa\right\}. \tag{3}$$

Calibrated to the same uniaxial test, $\kappa=\dfrac{\sigma_o}{2}$. This yield cylinder is hexagonal with the cross section shown in the Fig. 2a. In the case of plane stress ($\sigma_3=0$), the yield surface reduces to the polygon shown in Fig. 2b. Both criteria are accepted in structural design. Tresca's simplicity makes it attractive in analysis while the continuous nature of the von Mises yield function has made it the dominant candidate for use in subsequent yielding (flow rules).

YIELDING UNDER STRESS

Multiaxial stresses



The von Mises yield criterion

von Mises postulated that a material point yields when the distortional strain energy reaches a critical value. The distortional shear energy is due to shear (excludes energy due to volume change).

The DSE can be written:

$$U_d = c\left\{\frac{1}{6}\left[(\sigma_x+\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2\right]+\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{zx}^2\right\}$$

in principal directions,

$$U_d = \alpha\left\{\frac{1}{6}\left[(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2\right]\right\}$$

or the material yields when

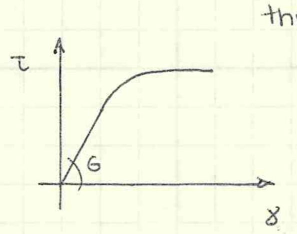
$\{\ \} = k_o^2$, which is found experimentally

Pure shear: test by applying torque to a thin-walled circular tube

$T = 2A\tau t$

$T = 2\pi R^2 t\tau$

little stress variation through wall thickness



$\tau_y^2 = k_o^2$, $\tau_y = k_o$

$\tau_y = \sigma_y/\sqrt{3}$

Direct tension test,

$k_o = \sigma_y/\sqrt{3}$

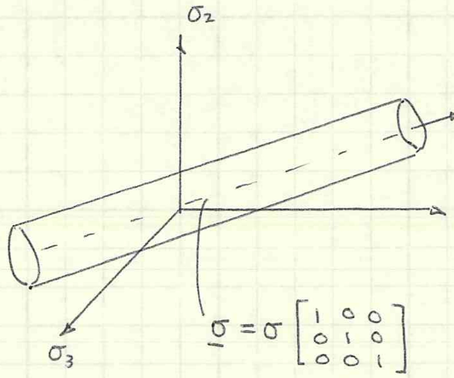
σ_y = yield stress, σ_o

In 2D, $\frac{1}{6}\left[(\sigma_x-\sigma_y)^2+\sigma_y^2+\sigma_x^2\right]+\sigma_{xy}^2 = \frac{\sigma_o^2}{3}$

$$\boxed{\sigma_o^2 = \sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\sigma_{xy}^2}$$

YIELD CRITERION

Von Mises in 3D



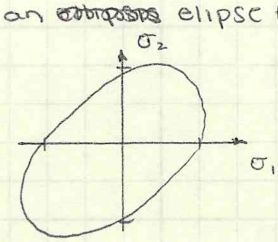
$\underline{n} = \frac{1}{\sqrt{3}} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$ same angle in all directions (from each axis)

within cylinder boundaries, elastic
on edges → first yielding occurs

$\underline{\sigma} = \sigma \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

in hydrostatic tension or compression,
no yield will occur. Shear is needed.

in 2D, think of the intersection of the cylinder
with a plane – an ~~[illegible]~~ elipse forms



YIELD CRITERION

Tresca Yield criterion

A material point will yield when the maximum shear reaches a critical value

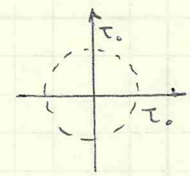
$$\left|\frac{\sigma_1-\sigma_2}{2}\right| = k_0 \qquad \left|\frac{\sigma_2-\sigma_3}{2}\right| = k_0 \qquad \left|\frac{\sigma_3-\sigma_1}{2}\right| = k_0$$

if any of the three equations is true, the material yields.

(1) uniaxial tension test

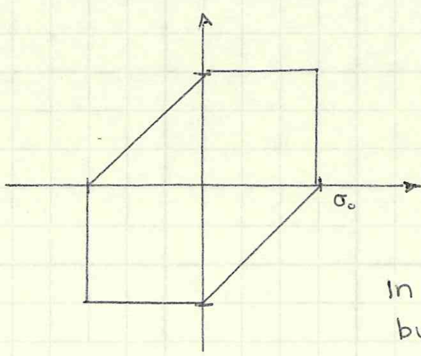
$$\frac{\sigma_0}{2} = k_0$$

(2) Pure shear test



$$\frac{\tau_0+\tau_0}{2} = \tau_0 = k_0$$

thus, $\frac{\sigma_0}{2} = \tau_0$



In 3D, similar cylinder to von Mises, but the cylinder is hexagonal

YIELD CRITERION

Tresca

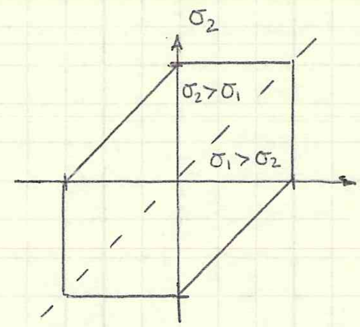
$$\left|\frac{\sigma_1-\sigma_2}{2}\right| \text{ or } \left|\frac{\sigma_2-\sigma_3}{2}\right| \text{ or } \left|\frac{\sigma_3-\sigma_1}{2}\right| = k_0$$

calibrate to uniaxial test $k_0 = \sigma_0/2$

pureshear test $k_0 = \sigma_0/2 = \tau_0$

In 2D, $\sigma_3 = 0$

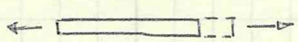
$$\left|\frac{\sigma_1-\sigma_2}{2}\right| = k_0, \quad \left|\frac{\sigma_2}{2}\right| = k_0, \quad \left|\frac{\sigma_1}{2}\right| = k_0$$



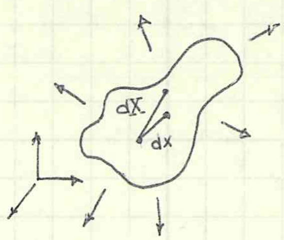
in 3D, it's a cylinder with a hexagonal cross-section

most materials behave more similarly to von Mises criterion than to Tresca

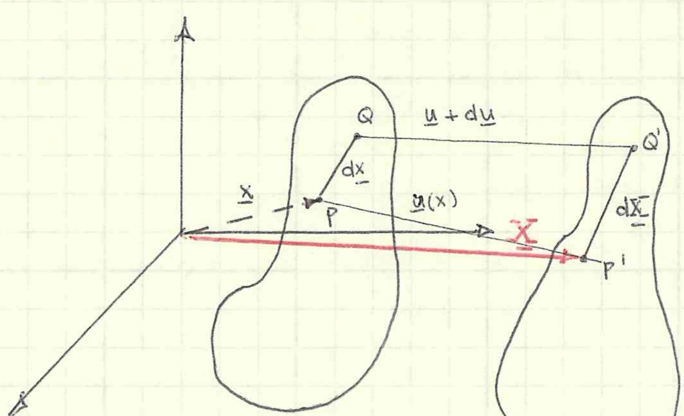
Analysis of strain



$$\varepsilon = \frac{\Delta L}{L}$$



dx - original segment
dX - segment post-movement



pre-stress (undeformed)

post-stress (deformed)

$\underline{x} = (x_1, y_1, z_1)$
$\underline{X} = (X_1, Y_1, Z_1)$

$\underline{u}$ - displacement from P to P'

P:	$\underline{x}$	$\underline{X}$
Q:	$\underline{x} + d\underline{x}$	$\underline{X} + d\underline{X}$

and $\underline{X} = \underline{x} + \underline{u}$

$\underline{u}(\underline{x})$ = displacement vector

STRAIN ANALYSIS

Big Potato example cont'd

$$|PQ|^2 = ds^2 = dx^2 + dy^2 + dz^2$$

$$|P'Q'|^2 = dS^2 = dX^2 + dY^2 + dZ^2$$

consider the change in length between PQ and P'Q'

$$d\underline{X} = d\underline{x} + d\underline{u} = \begin{Bmatrix} dx + du \\ dy + dv \\ dz + dw \end{Bmatrix}$$

$$\therefore dS^2 = (dx+du)^2 + (dy+dv)^2 + (dz+dw)^2$$

the change in length

$$dS^2 - ds^2 = (dx+du)^2 + (dy+dv)^2 + (dz+dw)^2 - (dx^2 + dy^2 + dz^2)$$

expanding and subtracting,

$$dS^2 - ds^2 = 2dx\,du + 2dy\,dv + 2dz\,dw + du^2 + dv^2 + dw^2$$

since $\underline{u} = \underline{u}(x)$,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz$$

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

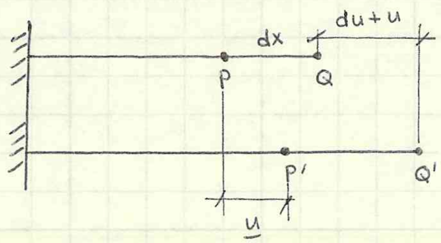
sub all of those into first equation $(dS^2 - ds^2)$:

resulting equation produces $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}$

$$\left[\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} + \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]\frac{1}{2} \\
\varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \\
\varepsilon_z &= \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] \\
\gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y} \\
\gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\cdot\frac{\partial w}{\partial z} \\
\gamma_{xz} &= \frac{\partial w}{\partial u} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial z}\cdot\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial z}
\end{aligned}\right.$$

STRAIN CALCULATIONS

one-dimensional example



$$dS^2 - ds^2 = dX^2 - dx^2$$

$$= (du + dx)^2 - dx^2$$

$$= 2\,du\,dx + du^2$$

$$= 2\left\{ \frac{du}{dx} + \frac{1}{2}\left(\frac{du}{dx}\right)^2 \right\} dx^2 = \varepsilon_x \cdot 2dx^2$$

$$\frac{dS^2 - ds^2}{2} = \varepsilon_x\, dx^2 \longleftarrow dx = ds$$

$$\varepsilon_x = \frac{1}{2}\left[\left(\frac{dS}{ds}\right)^2 - 1\right]$$

$$\frac{dS}{ds} = \left[2\varepsilon_x + 1\right]^{1/2}$$

if $\varepsilon_x << 1$, $\frac{dS}{ds} = \varepsilon_x + 1$

Taylor series expansion
$1 + \frac{1}{2}(2\varepsilon_x)$

STATES OF STRAIN

Equations

$$\frac{dS^2}{ds^2} - 1 = 2\varepsilon_x = 2\left\{\frac{du}{dx} + \frac{1}{2}\left[\frac{du}{dx}\right]^2\right\}$$

$$\frac{dS}{ds} = \left[1 + 2\varepsilon_x\right]^{1/2}$$

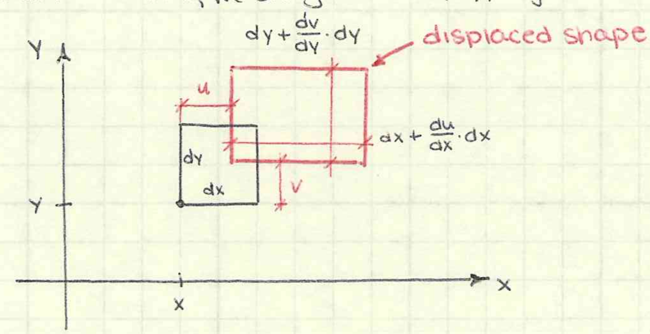
if $\varepsilon_x \ll 1$, $\frac{dS}{ds} = 1 + \varepsilon_x$, or $\underline{\varepsilon_x = \frac{dS - ds}{ds}}$, $\varepsilon_x = \frac{du}{dx}$

if we accept small strains in all dimensions, then the quadratic terms $\left(\left(\frac{du}{dx}\right)^2\right)$ can be neglected and the strains are reduced to:

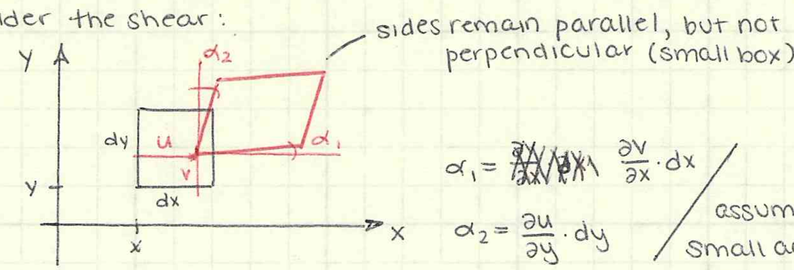
$$\varepsilon_x = \frac{du}{dx},\ \varepsilon_y = \frac{dv}{dy},\ \varepsilon_z = \frac{dw}{dz}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\ \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z},\ \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

in two dimensions, we only have $\varepsilon_x, \varepsilon_y, \gamma_{xy}$



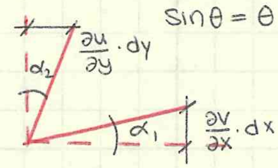
consider the shear:



$\alpha_1 = \cancel{\frac{\partial y}{\partial x}}\ \frac{\partial v}{\partial x} \cdot dx$

$\alpha_2 = \frac{\partial u}{\partial y} \cdot dy$

assuming small angles: $\sin\theta = \theta$



STATES OF STRAIN

Compatibility of Strains

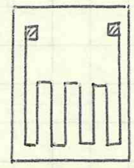
$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

strain gauges provide $\varepsilon_x, \varepsilon_y, \gamma_{xy}$
two unknowns $(\partial u, \partial v)$,
three equations

overconstrained!

The strain-displacement equations relate six strains to 3 displacements. Knowing $\underline{\varepsilon}(\underline{x})$, one can integrate and evaluate $\underline{u}$. But, this is an overdetermined system of equations. For the integration to yield unique solutions, additional conditions must be satisfied by $\underline{\varepsilon}(\underline{x})$.

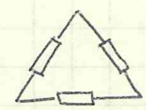
Side note on strain gauges



~~sop~~ serpentining wire
concealed in enamel
bonded to surface

elongation of body alters gauge, lengthening wire, changing the resistance of the wire

Rosettes – 3 directions



can find $\varepsilon_x, \varepsilon_y, \gamma_{xy}$

Compatibility Equations

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

+ 4 others, on handout

Two dimensions

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

ONLY EQUATION!

Strain Displacment Equations

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right],$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right],$$

$$\varepsilon_z = \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right],$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y},$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z},$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}.$$

Compatibility Equations

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y},$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z},$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x},$$

$$2\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right),$$

$$2\frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x}\right),$$

$$2\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y}\right).$$

STATES OF STRAIN

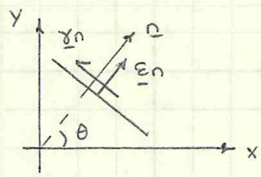
Transformation of Strains

Six components of strain:

$$\begin{bmatrix} \varepsilon_x & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{xy}/2 & \varepsilon_y & \gamma_{yz}/2 \\ \gamma_{xz}/2 & \gamma_{yz}/2 & \varepsilon_z \end{bmatrix}$$

[These components represent a tensorial set of order 2 (i.e., they are similar to the stress tensor). Thus, the strains transform in the same manner as the stresses, provided σ_{xy} is replaced by $\gamma_{xy}/2$.]

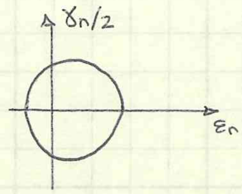
In 2D,



$$\left(\varepsilon_n = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy} \sin\theta\cos\theta \right)$$

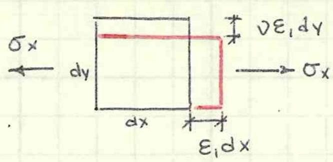
$$\frac{\gamma_{n'}}{2} = -\varepsilon_x \sin\theta\cos\theta + \varepsilon_y \sin\theta\cos\theta + \frac{\gamma_{xy}}{2}(\cos^2\theta - \sin^2\theta)$$

Mohr's circle is the same for strain as stress

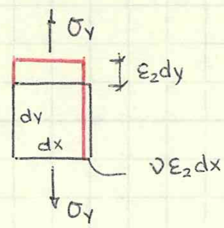


Stress-Strain Relationships

Plane stress $(\sigma_x, \sigma_y, \sigma_{xy})$

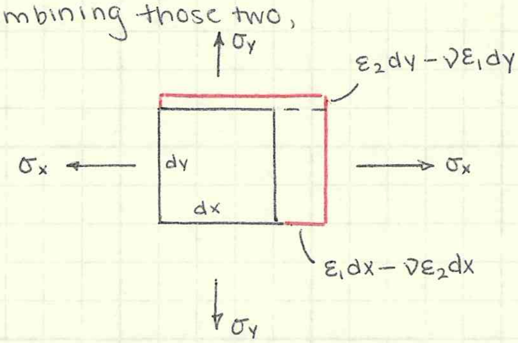


$$\varepsilon_1 = \frac{\sigma_x}{E}$$



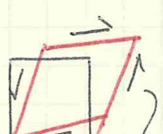
$$\varepsilon_2 = \frac{\sigma_y}{E}$$

combining those two,



$$\varepsilon_x = \varepsilon_1 - \nu\varepsilon_2 = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E}$$

$$\varepsilon_y = \varepsilon_2 - \nu\varepsilon_1 = \frac{\sigma_y}{E} - \frac{\nu\sigma_x}{E}$$



$$\sigma_{xy} = G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

shear stresses should be drawn on original box

STATES OF STRAIN

Stress-strain relationship

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \underset{\uparrow\,\gamma_{xy}}{\varepsilon_z} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}$$

or,

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

Next week: guest professor
no office hours
due 21 Sept 07

Stress-strain Relationships

$$\sigma_{xx} = \frac{E}{1+\nu}\left[\varepsilon_{xx} + \frac{\nu}{1-2\nu}\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right)\right]$$

$$\sigma_{yy} = \frac{E}{1+\nu}\left[\varepsilon_{yy} + \frac{\nu}{1-2\nu}\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right)\right]$$

$$\sigma_{zz} = \frac{E}{1+\nu}\left[\varepsilon_{zz} + \frac{\nu}{1-2\nu}\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right)\right]$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

$$\sigma_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz}$$

$$\sigma_{zx} = \frac{E}{2(1+\nu)}\gamma_{zx}$$

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right]$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{zz} + \sigma_{xx}\right)\right]$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right]$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\sigma_{xy}$$

$$\gamma_{yz} = \frac{2(1+\nu)}{E}\sigma_{yz}$$

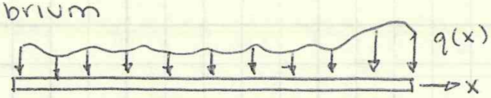
$$\gamma_{zx} = \frac{2(1+\nu)}{E}\sigma_{zx}$$

ELEMENTAL BEAM THEORY

Engineering (Bernoulli-Euler) Beam Theory
t_{1L}

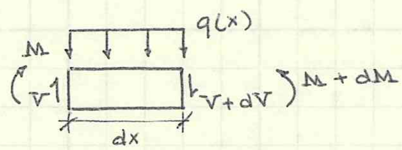
- Equilibrium
- Kinematics
- Constitution

(1) Equilibrium



height much smaller than length

small element:



ΣF_y: $V = q(x)dx + V + dV$

$dV = -q\,dx$

$$\boxed{\frac{dV}{dx} = -q(x)}$$

Rate of change of shear force is equal to $q(x)$
- point load, $q(x)=0$, V constant
- distributed load, slope of V related to $q(x)$

ΣM: $M = M + dM - (q(x)dx)\frac{1}{2}dx - (V+dV)dx$

$dM = \cancel{q(x)\frac{1}{2}dx^2} + V dx + \cancel{dV dx}$

second order terms $\rightarrow 0$

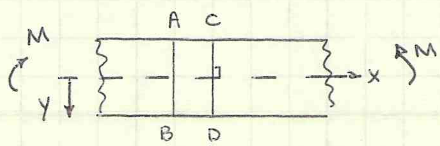
$$\boxed{\frac{dM}{dx} = V} \quad \text{or} \quad \frac{d^2M}{dx^2} = -q(x)$$

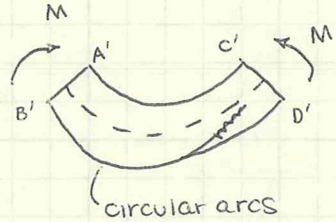
ELEMENTAL BEAM THEORY

Engineering Beam Theory

(2) Kinematics

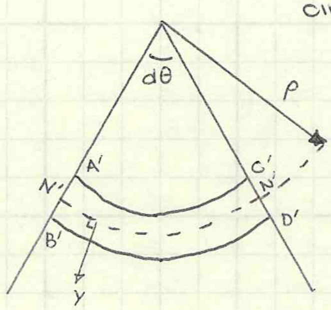
consider a section of a long, slender beam in pure bending
(not a function of x





circular arcs
A'C', B'D'
from coincentric
circles

Assumptions on shape:
- plane sections remain plane (normal to the axis of the beam)
  NO SHEAR STRAIN.



deformed axis at radius ρ & length ds

$ds = \rho\, d\theta$

$\frac{d\theta}{ds} = \frac{1}{\rho} = \kappa$

↳ curvature

strain varies from top to bottom
zero strain at neutral axis

clearly, the bending has put some fibers (A'C') in compression and some (B'D') in tension

Let the fiber N'N' (neutral axis) be the fiber that has retained its original length

$|N'N'| = ds = \rho\, d\theta$

Now consider another fiber P'Q' (between N.A. and B'D'), located a distance y from N'N'.

$|P'Q'| = d\theta(\rho + y)$

but $|PQ| = d\theta\rho$, as N'N'

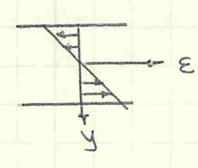
Thus the strain in P'Q' is

$\varepsilon = \frac{\Delta L}{L_o}$

$\varepsilon = \frac{(\rho+y)d\theta - \rho d\theta}{\rho d\theta}$

$\boxed{\varepsilon = \frac{y}{\rho}}$ or $\varepsilon = y\kappa$

strain is linearly distributed through the thickness



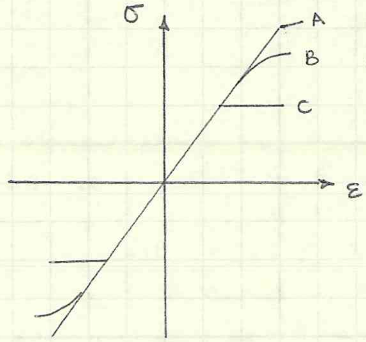
No mentions of material yet! Just small deformations.

ELEMENTAL BEAM THEORY

Engineering Beam Theory

(3) constitution

Possible stress-strain behaviors:



A perfectly elastic

B strain hardening

C elastic perfectly plastic

Then the stress distribution in the bent beam is $\sigma = \sigma(\varepsilon) = \sigma(yK)$

A: linearly elastic, isotropic

$\sigma = E\varepsilon$

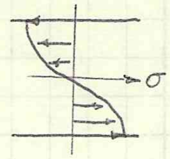
$\sigma(x) = E\,y\,K(x)$ ← non-uniform curvature

linear distribution of stress

B: strain-hardening material

$\sigma = C\varepsilon^{1/n}$

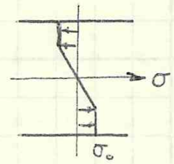
$\sigma(x) = c(x)\,y^{1/n}\,K(x)^{1/n}$



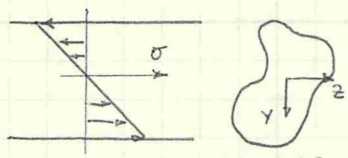
C: perfectly plastic

$\sigma = E\varepsilon \quad | \quad \varepsilon < \sigma_0/E \qquad \sigma(x) = E\,y\,K(x)$

$\sigma = 0 \quad | \quad \varepsilon \geq \sigma_0/E \qquad \sigma(x) = \sigma_0$



Position of the neutral axis



x-section

Net resultant of axial forces must be zero

$\int_A \sigma\, dA = 0$ ← in terms of x

For case A (linear elastic),

$$\int_A E\varepsilon(x)\, dA = EK(x) \int_A y\, dA = 0$$

to be true,

$\underline{\int_A y\, dA = 0}$ location of the centroid!

first moment of area about the N.A. = 0

ELEMENTAL BEAM THEORY

Summary of other day

$$\frac{dV}{dx} = -q(x)$$

$$\frac{dM}{dx} = V$$

Moment · Curvature Relationship

Consider a linear · elastic material

consider the moment due to σ acting on dA

$$dM = \int_A \sigma(x,y)\, y\, dA$$

$$M(x) = \int_A \sigma(x,y)\, y\, dA = \int_A E y K(x)\, y\, dA$$

$$= EK(x) \underbrace{\int_A y^2 dA}_{I,\ \text{moment of inertia}}$$

$$= EI\, K(x) = \frac{EI}{\rho(x)}$$

EI – flexural rigidity of the beam

$$\sigma(x) = \frac{Ey}{\rho(x)} = \frac{M(x)\, y}{I}$$

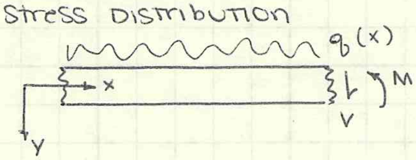
How will this fail? Maximum moment / stress, etc.

$$\sigma_{max} = \frac{M_{max}\, y_{max}}{I} = \frac{M_{max}\, c}{I}$$

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{\rho}$$

BEAM THEORY

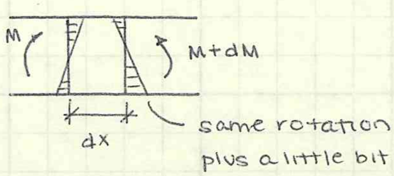
Shear Stress Distribution

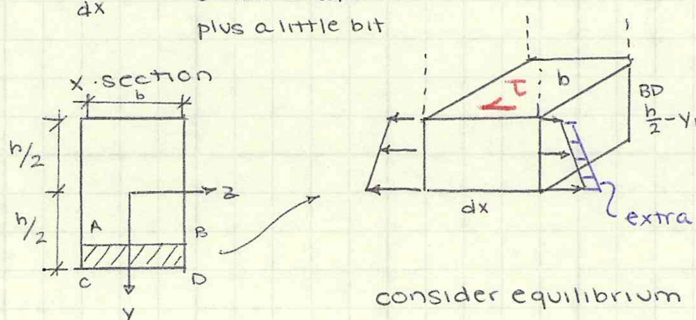


$$\frac{dV(x)}{dx} = -q(x), \quad V(x) = -\int_0^x q(x)\,dx \quad \text{— shear force}$$

$$\frac{dM}{dx} = V, \quad M(x) = \int_0^x V(x)\,dx \quad \text{— moment}$$

If $V=0$, $M=0$ (or constant nonzero)
but in general, shear force is not zero,
thus causing shear stresses in transversely
loaded beams





τ balances inequality

consider equilibrium

$$\sigma(x) = \frac{M(x)y}{I}, \quad \sigma(x+dx) = \frac{M(x+dx)y}{I}$$

Net force on element, $N(x)$:

$$N(x) = \int_A \sigma(x)\,dy$$

$$N(x) = \int_{y_1}^{h/2} \sigma(x)\,b\,dy \quad \text{on left side}$$

$$N(x+dx) = \int_{y_1}^{h/2} \sigma(x+dx)\,b\,dy \quad \text{on right face}$$

Equilibrium:

$$\int_{y_1}^{h/2} \frac{M(x)y}{I}\,b\,dy + \tau b\,dx = \int_{y_1}^{h/2} \frac{M+dM}{I}\,y\,b\,dy$$

$$\tau b\,dx = b\int_{y_1}^{h/2} \frac{dM}{I}\,y\,dy$$

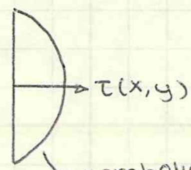
$$\tau(x,y) = \frac{dM(x)}{dx}\cdot\frac{1}{I}\int_{y_1}^{h/2} y\,dy = \frac{V}{I}\int_{y_1}^{h/2} y\,dy$$

BEAM THEORY

Shear stress distribution

$$\tau(x,y) = \frac{V}{I}\int_{y_1}^{h/2} y\,dy$$

$$= \frac{V(x)}{I}\left[\left.\frac{y^2}{2}\right|_{y_1}^{h/2}\right]$$



parabolic distribution of shear stress

$$\tau_{max} = \frac{V(x)}{I}\frac{h^2}{8}$$

Recall:

$$V = \int_A \tau\,dA$$

maximum at neutral axis

zero at extreme fibers

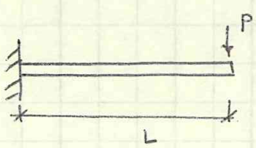
If τ is parabolic, γ should also be parabolic:

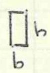
$$\tau = G\gamma$$

But, plane sections remain plane, so $\gamma = 0$

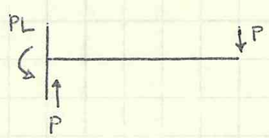
consequence of the assumptions that were made; it happens (contradictions)

Example:





when will beam fail?



$M_{max} = PL$

$V_{max} = P$

$$\sigma_{max} = \frac{PL\,h/2}{1/12\,bh^3} = \frac{6PL}{bh^2}$$

$$\tau_{max} = \frac{Ph^2}{8\cdot 1/12\,bh^3} = \frac{3P}{2bh}$$

$$\text{average shear stress} = \frac{V}{A}$$

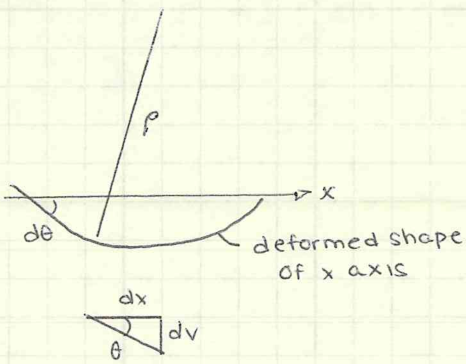
$$\frac{\sigma_{max}}{\tau_{max}} = \frac{6PL/bh^2}{3P/2bh} = \frac{4L}{h}$$

long, slender beam, $\frac{L}{h} \sim 20\text{–}30$

shorter – shear failure occurs at support

BEAM THEORY

Curvature



M (+)

Note: for this concave upwards shape (caused by $M>0$), θ decreases with increasing s

$$\frac{1}{\rho} = K = \frac{-d\theta}{ds} = -\frac{d\theta}{dx}\cdot\frac{dx}{ds}$$

$$ds = \sqrt{dx^2+dv^2} = dx\sqrt{1+\left(\frac{dv}{dx}\right)^2}$$

$$\frac{dx}{ds} = \left[1+v'^2\right]^{-1/2} \qquad v' = \frac{dv}{dx}$$

$$\tan\theta = \frac{dv}{dx}, \quad \frac{d}{dx}\tan\theta = \sec^2\theta\,\frac{d\theta}{dx} = \frac{d^2v}{dx^2}$$

$$\frac{d\theta}{dx} = \cos^2\theta\,\frac{d^2v}{dx^2} = \frac{v''}{1+(v')^2}$$

$$\frac{1}{\rho} = \frac{-v''}{\left[1+(v')^2\right]^{3/2}}$$

$$\text{If } \frac{dv}{dx} \ll 1, \quad K = \frac{-d^2v}{dx^2}$$

$$M = -EI\frac{d^2v}{dx^2}$$

$$V = -EI\frac{d^3v}{dx^3}$$

$$q = EI\frac{d^4v}{dx^4}$$

To summarize,

Assumptions:

- plane sections remain plane and normal...
- normal transverse stress is zero ($\sigma_y = 0$)
- beam section rotations (and deflections) are small

Equations

$$\frac{d^2M}{dx^2} = -q(x), \quad EI\frac{d^4v}{dx^4} = q(x)$$

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{\rho}, \quad M = EIK$$

$$K = -\frac{d^2v}{dx^2}$$

$$\tau = \frac{V}{2I}\left[h^2/4 - y_1^2\right]$$ ← for a rectangular cross-section

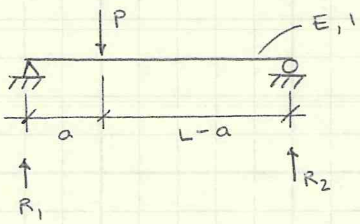
BEAM THEORY

Half Range functions

$$\langle x - a \rangle = \begin{cases} 0 & \text{for } (x-a) < 0 \\ x-a & \text{for } (x-a) > 0 \end{cases}$$

used for loads that do not span the whole beam

Examples with half range functions



$$R_1 + R_2 = P$$

$$R_2 = \frac{Pa}{L}, \quad R_1 = P(1 - a/L)$$

$$M(x) = \begin{cases} R_1 x & \text{for } x < a \\ R_1 x - P(x-a) & \text{for } x > a \end{cases}$$

half range function

$$M(x) = R_1 x - P\langle x - a \rangle$$

$$-EI \frac{d^2v}{dx^2} = R_1 x - P\langle x-a \rangle$$

$$-EI \frac{dv}{dx} = \frac{R_1}{2} x^2 - \frac{P}{2}\langle x-a \rangle^2 + A$$

$$-EIv = \frac{R_1}{6} x^3 - \frac{P}{6}\langle x-a \rangle^3 + Ax + B$$

consider boundary conditions

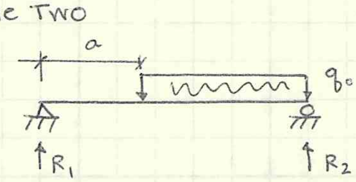
$$v(0) = 0, \quad v(L) = 0$$

$$B = 0 \qquad \frac{PaL^2}{6} - \frac{P}{6}(L-a)^3 + AL = 0$$

use normal ( ) because

x = L means x > a

Example Two



$$R_1 + R_2 = q_0 (L - a)$$

$$M(x) = -EI\frac{d^2v}{dx^2} = R_1 x - q_0 \langle x-a \rangle^2 \frac{1}{2}$$

$$-EI\frac{dv}{dx} = \frac{R_1}{2}x^2 - \frac{q_0}{6}\langle x-a \rangle^3 + A$$

$$-EI\,v = \frac{R_1}{6}x^3 - \frac{q_0}{24}\langle x-a\rangle^4 + Ax + \cancel{B}^{\,0}$$

$$A = \frac{1}{L}\left[\frac{q_0}{24}(L-a)^4 - \frac{R_1}{6}x^3\right]$$

HALF RANGE FUNCTIONS

Definition

$$\langle x-a\rangle = \begin{cases} 0 & \text{for } (x-a)<0 \\ (x-a) & \text{for } (x-a)>0 \end{cases}$$

Some Useful Relationships

$$\frac{d}{dx}\langle x-a\rangle^n = n\,\langle x-a\rangle^{n-1} \qquad (n\neq 0)$$

$$\frac{d}{dx}\langle x-a\rangle^1 = \langle x-a\rangle^0 \quad = \begin{cases} 0 & \text{for } x<a \\ 1 & \text{for } x>a \end{cases}$$

$$\frac{d}{dx}\langle x-a\rangle^0 = \langle x-a\rangle^{-1} \quad = \begin{cases} 0 & \text{for } x<a \text{ and } x>a \\ \infty & \text{for } x=a \end{cases}$$

$$\frac{d}{dx}\langle x-a\rangle^{-1} = -\langle x-a\rangle^{-2} \quad = \begin{cases} 0 & \text{for } x<a \text{ and } x>a \\ \pm\infty & \text{for } x=a \end{cases}$$

$$\int_0^x \langle x-a\rangle^{n-1}\,dx = \frac{1}{n}x - a^n \qquad (n\neq 0)$$

$$\int_0^x \langle x-a\rangle^{0}\,dx = \langle x-a\rangle^1$$

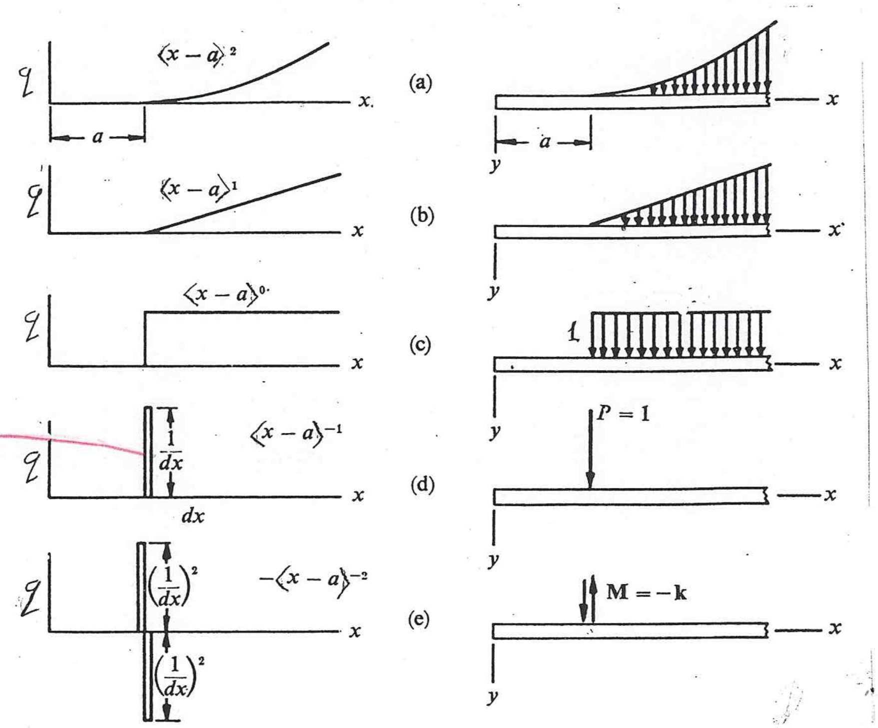
$$\int_0^x \langle x-a\rangle^{-1}\,dx = \langle x-a\rangle^0$$

$$\int_0^x \langle x-a\rangle^{-2}\,dx = -\langle x-a\rangle^{-1}$$

Use of Half Range Functions in Beams

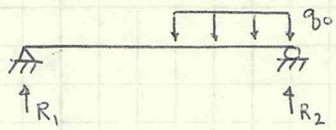
$$\text{Beam Equation} \quad EI\frac{d^4v}{dx^4} = q(x)$$

$\frac{1}{dx}$ as $dx \to 0$ $\to \infty$



HALF-RANGE FUNCTIONS

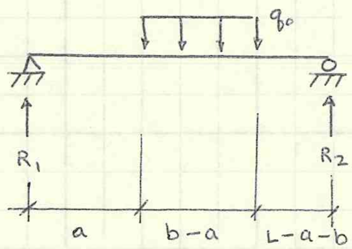
Example



$$M(x) = -EI\frac{d^2v}{dx^2} = R_1 x - \frac{q_0}{2}\langle x-a\rangle^2$$

as in notes last week

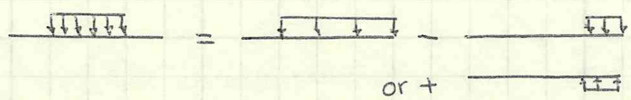
Example #2



$$M(x) = R_1 x - \frac{q_0}{2}\langle x-a\rangle^2$$

there is no "off" for the < >

can't go back to zero at $x=b$

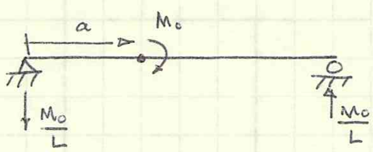


$$M(x) = R_1 x - \frac{q_0}{2}\langle x-a\rangle^2 - \frac{q_0}{2}\langle x-b\rangle^2$$

$$-EIv = \frac{R_1}{6}x^3 - \frac{q_0}{24}\langle x-a\rangle^4 - \frac{q_0}{24}\langle x-b\rangle^4 + Ax + B$$

use boundary conditions to get A, B

Example #3



$$M(x) = -EI\frac{d^2v}{dx^2} = -\frac{M_0}{L}x + M_0\langle x-a\rangle^0$$

$$-EIv = -\frac{M_0}{6L}x^3 + \frac{M_0}{2}\langle x-a\rangle^2 + Ax + B$$

again, use BCs to solve for A, B

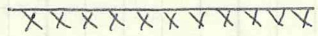
interior supports

load continues across

factor in when calculating reactions, writing equation for moment, and finding constants

ADVANCED BEAM THEORIES

continuous supports

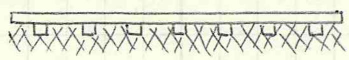


e.g. foundations – ground, floor surface
deformable material (soil)

approximation using springs
what is k?

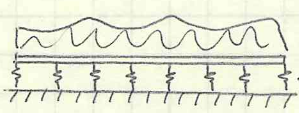
k, per unit L
k ≡ stiffness per unit length
$[F/L^2]$

Example



railroad tracks resting on ties
with gravel beneath

considering the generic spring-supported case:



$$EI\frac{d^4v}{dx^4} = \hat{q}(x) = q(x) - kv(x)$$

$$EI\frac{d^4v}{dx^4} + kv(x) = q(x)$$ complex differential equation

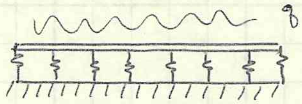
$$\frac{d^4v}{dx^4} + \frac{k}{EI}v = q(x)/EI$$

Let $k/EI = \alpha^4$

$$\frac{d^4v}{dx^4} + \alpha^4 v = q(x)/EI$$

CONTINUOUS SUPPORTS

Example problem



$$EI\frac{d^4v}{dx^4} + Kv = q(x)$$

$$\frac{d^4v}{dx^4} + \frac{K}{EI}v = \frac{q(x)}{EI}$$

define $\alpha = \left(\frac{K}{EI}\right)^{1/4}$

homogeneous solution

$$\frac{d^4v_H}{dx^4} + \alpha^4 v_H = 0$$

Let $v = Ae^{\lambda x}$

$$\lambda^4 + \alpha^4 = 0$$

$$\lambda^2 = \pm\alpha^2\sqrt{-1} = \pm i\alpha^2$$

$$\boxed{\lambda = \pm\sqrt{i}\,\alpha}$$

DeMoivre's Theorem

$$\sqrt{i} = \left[\cos\pi/2 + i\sin\pi/2\right]^{\frac{1}{2}} = \cos\left(\frac{\pi/2}{2} + \frac{2n\pi}{2}\right) + i\sin\left(\frac{\pi/2}{2} + \frac{2n\pi}{2}\right), n=0,1$$

$$\sqrt{-i} = \left[\cos\frac{3\pi}{2} + i\sin 3\pi/2\right]^{\frac{1}{2}} = \cos\left(\frac{3\pi}{4} + n\pi\right) + i\sin\left(\pi/4 + n\pi\right), n=0,1$$

Roots are:

$$\frac{1}{\sqrt{2}}(1+i),\ \frac{-1}{\sqrt{2}}(1+i),\ \frac{1}{\sqrt{2}}(-1+i),\ \frac{-1}{\sqrt{2}}(-1+i)$$

Let

$$\mu = \left(\frac{K}{4EI}\right)^{\frac{1}{4}}, \quad \text{then} \quad \lambda = \pm\mu(1\pm i)$$

Combining equations,

homogeneous solution

$$v_H = e^{\mu x}\left[c_1\sin\mu x + c_2\cos\mu x\right] + e^{-\mu x}\left[c_3\sin\mu x + c_4\cos\mu x\right]$$

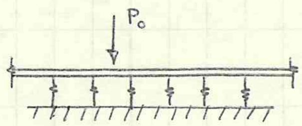
or

$$v_H = D_1\sinh\mu x\sin\mu x + D_2\cosh\mu x\cos\mu x + D_3\sinh\mu x\cos\mu x + D_4\cosh\mu x\sin\mu x$$

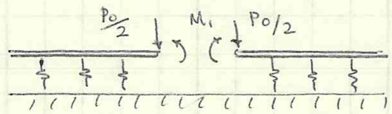
CONTINUOUS SUPPORTS

"Simple" example

beam of infinite length

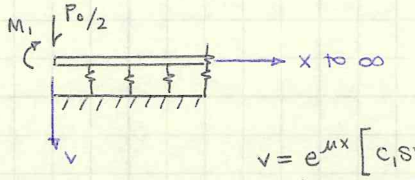


symmetric problem – even influence on either side of the point load P_0



Adding shear would make the two sides antisymmetric – one up, one down. Thus, no internal shear.

Focus on one side



$$v = \underbrace{e^{\mu x}\left[c_1 \sin\mu x + c_2 \cos\mu x\right]}_{\text{goes to } \infty} + e^{-\mu x}\underbrace{\left[c_3 \sin\mu x + c_4 \cos\mu x\right]}_{\text{decaying exponential function}}$$

$\sin x + \cos x =$ 

goes to ∞

$c_1 + c_2 = 0$

$v(\infty) = v'(\infty) = 0$

decaying exponential function

use boundary conditions

$v'(0) = 0$ – condition of symmetry

$-EIv'''(0) = \bar{V}(0) = -P_0/2$

becomes a math problem:

$$v'(x) = \mu e^{-\mu x}\left[-c_3 \sin\mu x - c_4 \cos\mu x + c_3 \cos\mu x \overset{(-)}{+} c_4 \sin\mu x\right]$$

$$v'(0) = 0, \quad -c_4 + c_3 = 0, \quad c_3 = c_4$$

$$v' = -2c_3 \mu e^{-\mu x} \sin\mu x$$

$$v'' = -2c_3 \mu^2 e^{-\mu x}\left[-\sin\mu x + \cos\mu x\right]$$

$$v''' = 4c_3 \mu^3 e^{-\mu x} \cos\mu x$$

So, ignore for now

$$-EI\left(4c_3 \mu^3 e^{-\mu x} \cos\mu x\right) = -P_0/2$$

$$\text{but } \mu^4 = \frac{\cancel{E}}{\cancel{4}k}\ \frac{4k}{EI}$$

$$-4EI\frac{\mu^4}{\mu} c_3 = -\cancel{4}\cancel{E}I\frac{1}{\mu}\frac{k}{\cancel{E}I}\cdot\frac{1}{\cancel{4}} c_3 = \frac{-P_0}{2}$$

$$\frac{-kc_3}{\mu} = \frac{-P_0}{2}, \quad c_3 = \frac{P}{2}\frac{\mu}{k}$$

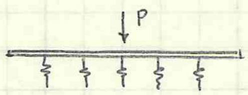
$$\boxed{v(x) = \frac{P\mu}{2k} e^{-\mu x}\left[\cos\mu x + \sin\mu x\right]}$$

CONTINUOUS SUPPORTS

Summary of equations

$$v_H(x) = e^{\mu x}(c_1 \sin \mu x + c_2 \cos \mu x) + e^{-\mu x}(c_3 \sin \mu x + c_4 \cos \mu x)$$

$$\mu = \left[\frac{k}{4EI}\right]^{1/4}$$



$$v(x) = \frac{P\mu}{2k} A(x) \qquad A(x) = e^{-\mu x}(\cos \mu x + \sin \mu x)$$

$$v'(x) = -\frac{P\mu^2}{k} B(x) \qquad B(x) = e^{-\mu x} \sin \mu x$$

$$V(x) = -\frac{P}{2} D(x) \qquad D(x) = e^{-\mu x} \cos \mu x$$

$$M(x) = \frac{P}{4\mu} C(x) \qquad C(x) = e^{-\mu x}(\cos \mu x - \sin \mu x)$$

Method 2:

$$v(x) = e^{-\mu x}(c_3 \sin \mu x + c_4 \cos \mu x)$$

$$v'(0) = 0, \quad c_3 = c_4, \quad v(x) = c_3 e^{-\mu x}(\sin \mu x + \cos \mu x)$$

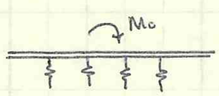
$$\int_0^\infty kv\, dx = \frac{P}{2}$$

$$2kc_3 \int_0^\infty e^{-\mu x}(\sin \mu x + \cos \mu x)\, dx = P$$

$$2kc_3 \frac{e^{-\mu x}}{\mu}\left[\sin \mu x + \cos \mu x\right]_0^\infty = P$$

$$c_3 = \frac{P\mu}{2k}$$

More equations



$$v(x) = \frac{M\mu^2}{k} B(x)$$

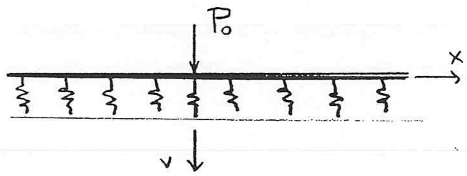
$$v'(x) = \frac{M\mu^3}{k} C(x)$$

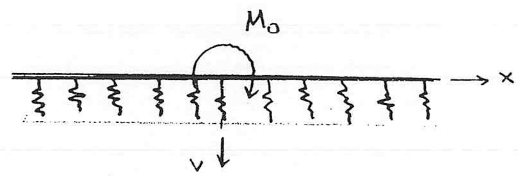
$$M(x) = \frac{M}{2} D(x)$$

$$V(x) = -\frac{M}{2} \mu A(x)$$

Summary

- $v_H(x) = e^{\mu x}(C_1 \sin \mu x + C_2 \cos \mu x) + e^{-\mu x}(C_3 \sin \mu x + C_4 \cos \mu x),\ \mu = \left(\frac{k}{4EI}\right)^{1/4}$
- Infinite Beams with Point "Loads"

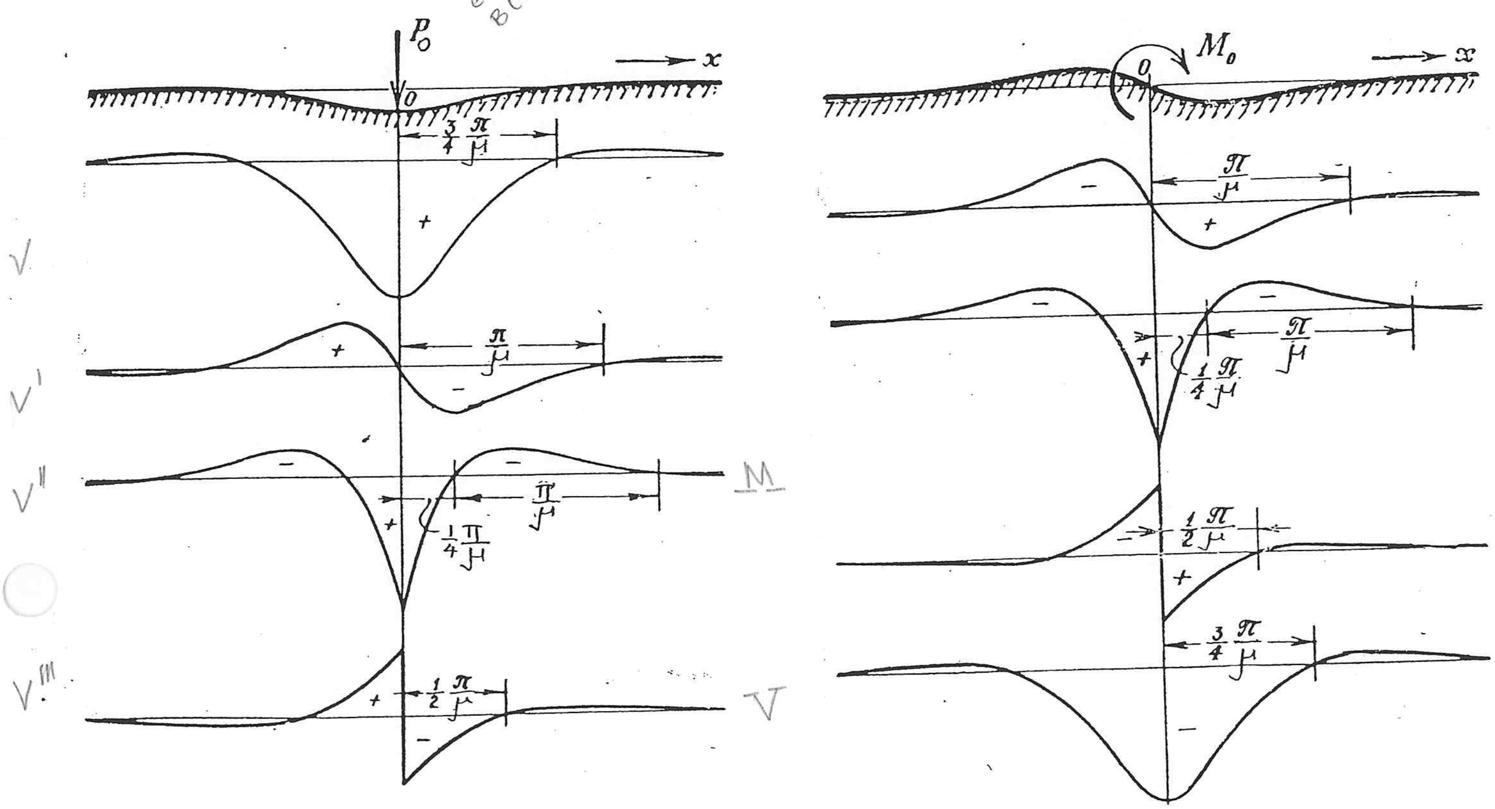




$$
\begin{aligned}
v(x) &= \frac{P_0\mu}{2k}A(x) & \qquad v(x) &= \frac{M_0\mu^2}{k}B(x) \\
v'(x) &= -\frac{P_0\mu^2}{k}B(x) & \qquad v'(x) &= \frac{M_0\mu^3}{k}C(x) \\
M(x) &= \frac{P_0}{4\mu}C(x) & \qquad M(x) &= \frac{M_0}{2}D(x) \\
V(x) &= -\frac{P_0}{2}D(x) & \qquad V(x) &= -\frac{M_0\mu}{2}A(x)
\end{aligned}
$$

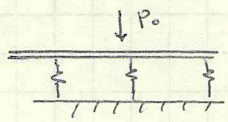
at $x=0$, fxns $=1$ except $B(0)=0$

$$
\left\{
\begin{aligned}
A(x) &= \bar{e}^{\mu x}[\cos \mu x + \sin \mu x] \\
B(x) &= \bar{e}^{\mu x}[\sin \mu x] \\
C(x) &= \bar{e}^{\mu x}[\cos \mu x - \sin \mu x] \\
D(x) &= \bar{e}^{\mu x}[\cos \mu x]
\end{aligned}
\right.
$$



CONTINUOUS FOUNDATION

Example problem



$$v(x) = \frac{P_0 \mu}{2k} A(x) \quad , \quad v'(x) = -\frac{P_0 \mu^2}{k} B(x)$$

E $A(x) = e^{-\mu x}(\cos\mu x + \sin\mu x)$ $\quad$ $A'(x) = -2\mu B(x)$

O $B(x) = e^{-\mu x}\sin\mu x$ $\quad$ $B'(x) = \mu C(x)$

E $C(x) = e^{-\mu x}(\cos\mu x - \sin\mu x)$ $\quad$ $C'(x) = -2\mu D(x)$

O $D(x) = e^{-\mu x}\cos\mu x$ $\quad$ $D'(x) = -2\mu A(x)$

$$\mu^4 = \frac{k}{4EI}$$

only valid for $x \geq 0$

$$v(x) = \frac{P_0\mu}{k} A(x)$$

$$v'(x) = -\frac{P_0\mu^2}{k} B(x)$$

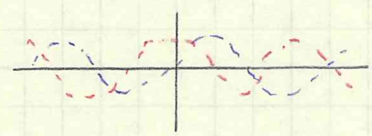
$$M = \frac{P_0\mu^3}{k} C(x) = \frac{P_0}{4\mu} C(x)$$

$$V(x) = \frac{-2P_0\mu^4}{k} D(x) = \frac{-P_0}{2} D(x)$$

even or odd functions

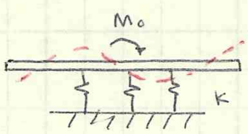
cosine – even or symmetric fxn
sine – odd or antisymmetric fxn

can be used to find response for points at $x < 0$



$\sin(x) = -\sin(-x)$
$\cos(x) = \cos(-x)$

Example two



response is not symmetric
– in fact, it is antisymmetric

$v(\infty) = 0$
$\rightarrow v(0) = 0 \leftarrow$
$M(0) = M_0/2 = -EIv''(0)$

solution is of the form

$$\frac{M_0\mu^2}{k}\sin\mu x$$

$$v_H(x) = e^{\mu x}(c_1\sin\mu x + c_2\cos\mu x) + e^{-\mu x}(c_3\sin\mu x + c_4\cos\mu x) = v(x)$$

no particular solution

$v(\infty) = 0, \quad c_1 = c_2 = 0$
else function explodes

$v(0) = 0, \quad c_4 = 0$
$$v(x) = e^{-\mu x} c_3 \sin\mu x$$

$$M(0) = \frac{M_0}{2} = -EIv''(0)$$

CONTINUOUS FOUNDATION

Example two, cont'd

considering boundaries,

$$-EIv''(0) = \frac{M_o}{2}$$

$$v(x) = c_3 e^{-\mu x} \sin \mu x$$

$$v''(x) = -c_3 2\mu^2 e^{-\mu x} \cos \mu x$$

$$\mu^4 = \frac{k}{4EI}$$

$$M(x) = -EIv''(x) = EI c_3 \frac{2\mu^4}{\mu^2} e^{-\mu x} \cos \mu x$$

$$= \frac{c_3 k}{2\mu^2} e^{-\mu x} \cos \mu x$$

$$M(0) = c_3 \frac{k}{\mu^2} = M_o, \quad c_3 = \frac{M_o \mu^2}{k}$$

so,

$$v(x) = \frac{M_o \mu^2}{k} e^{-\mu x} \cos \mu x$$

can always check general form by considering units $[\mu] = 1/L$

$v(x) = \frac{M_o \mu^2}{k} B(x)$ odd

$v'(x) = \frac{M_o \mu^3}{k} C(x)$ even

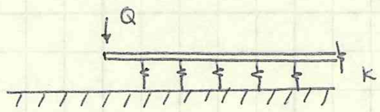
$M(x) = \frac{-M_o}{2} D(x)$ odd

$V(x) = \frac{-M_o \mu}{2} A(x)$ even

again, use even/odd to find values for system when $x < 0$

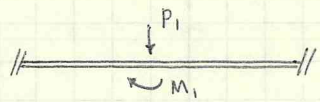
CONTINUOUS SUPPORTS

Examples considering superposition



$V(0) = -Q$
$M(0) = 0$

First, assume beam is in fact infinite



$$M(0) = \frac{P_1}{4\mu} + \frac{M_1}{2} = 0$$

er, M_1

$$V(0) = -\frac{P_1}{2} \cancel{H(0)} - \frac{M_0 \mu}{2} = Q(-1)$$

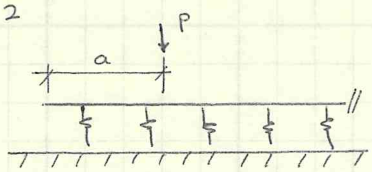
two equations, two unknowns

$$P_1 = -2\mu M_1$$
$$M_1 = -2\frac{Q}{\mu}, \quad P_1 = 4Q$$

calculate final displacement:

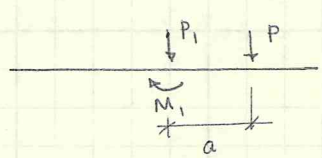
$$V(x) = \frac{P_1 \mu}{2k} A(x) + \frac{M_1 \mu^2}{k} B(x)$$
$$= \frac{Q\mu}{k} A(x) - \frac{2Q\mu}{k} B(x) \longleftarrow \text{solution}$$

Example 2



Boundary conditions

$M(0) = 0$
$V(0) = 0$



$$M(0) = \frac{P_1}{4\mu} + \frac{M_1}{2} + \frac{P_0}{4\mu} C(-a) = 0$$

M from P_1

M from M_1

M from P, origin is to the left of the original derived origin

$$V(0) = -\frac{P_1}{2} - \frac{M_1 \mu}{2} + \frac{P_0}{2} D(a) = 0$$

evaluate at a, multiply solution by (-1)

Solving,

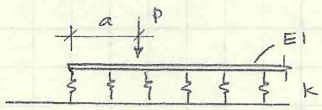
$$M_1 = \frac{P}{\mu}\left[C(a) + D(a)\right]$$

$$P_1 = P\left[C(a) + 2D(a)\right], \quad V(x) = \frac{P_1 \mu}{2k} A(x) + \frac{M_1 \mu^2}{k} B(x) + \frac{P\mu}{2k} A(x-a) \text{ if } x > a$$

$A[-(x-a)]$ if $x < a$

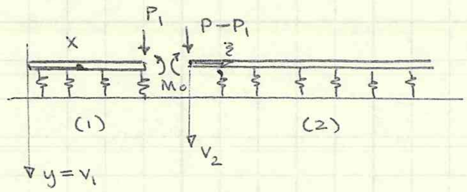
CONTINUOUS SUPPORTS

Continued examples



semi-infinite plate

Solution the second



Problem (2) – semi-infinite, load at end

$V_2(0) = P - P_1$

$M_2(0) = M_o$

$v_2(\infty) = 0$

$v_2'(\infty) = 0$

Problem (1) – finite length beam

$V_1(0) = 0$

$M_1(0) = 0$

$V_1(a) = P_1$

$M_1(a) = M_o$

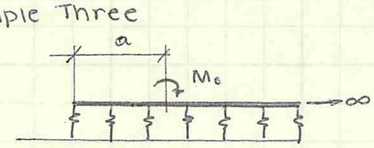
Combining,

$v_1(a) = v_2(0)$

$v_1'(a) = v_2'(0)$ ← not = 0 unless $a \to \infty$

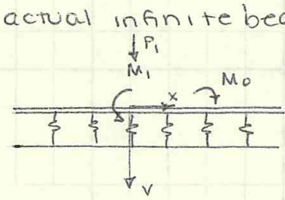
longer solution, but should match last week's

Example Three



similar to problem with load P at x=a

consider actual infinite beam, origin set x=a from moment



$$V(0) = 0 = \frac{-M_o\mu}{2}A(a) - \frac{M_1\mu}{2}A(0) - \frac{P_1}{2}D(0)$$

$$M(0) = 0 = \frac{M_o}{2}D(a)(-1) + \frac{M_1}{2}D(0) + \frac{P_1}{4\mu}C(0)$$

↑ antisym.

Knowing

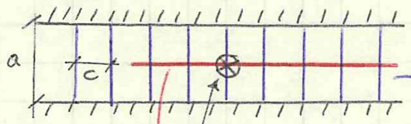
$A(0) = D(0) = C(0) = 1.0$

$\cancel{V(0)=0=\frac{-M_o\mu}{2}-\mu}$

$$\left.\begin{aligned} M_o\mu A(a) + M_1\mu + P_1 &= 0 \\ -2M_o\mu D(a) + 2M_1\mu + P_1 &= 0 \end{aligned}\right\rangle \text{ solve for } M_1 \text{ and } P_1$$

CONTINUOUS FOUNDATIONS

Practical Example



long corridor in a building

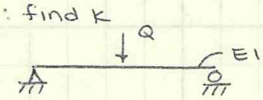
9 roof cross beams
EI, spaced at c

cross beam, $E\hat{I}$

load of weight $W = P$

can this be approximated as a continuous elastic foundation? Exact problem in HW.

Step 1: find k



one of the cross beams

$$\Delta(a/2) = \delta = \frac{QL^3}{48EI}, \quad L = a$$

stiffness of beam $k_b = \frac{Q}{\delta} = \frac{48EI}{a^3}$

stiffness per unit width of corridor:

$$k = \frac{k_b}{c} = \frac{48EI}{ca^3}$$

Step 2: return to beam on an elastic foundation

$$\mu^4 = \frac{k}{4E\hat{I}}, \quad \frac{12I}{a^3 c \hat{I}}$$

define $E\hat{I} = \alpha EI$

$$\mu^4 = \frac{12}{\alpha c a^3}, \quad \mu = \frac{1}{a}\left[\frac{12a}{c\alpha}\right]^{1/4}$$

Step 3: calculate forces

$$V(0) = V_{max} = \frac{P\mu}{2k}$$ ← deflection v = displacement of cross beam in the middle

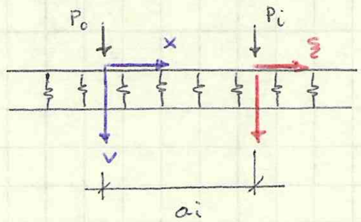
$$V(c) = \frac{P\mu}{2k} A(c)$$

deflection of next beam over

M_{max} of each cross-beam

CONTINUOUS SYSTEMS

Shifting of coordinates



For P_o at $x=0$,

$$v(x) = \frac{P_o \mu}{2k} e^{-\mu x}\left[\cos\mu x + \sin\mu x\right], \quad x>0$$

For P_i at $x = a_i$, $\xi = x - a_i$

$$\cancel{v(x) =}\ v(\xi) = \frac{P_i \mu}{2k} e^{-\mu\xi}\left[\cos\mu\xi + \sin\mu\xi\right],$$

$$\xi > 0$$

For $x > a_i$,

$$v(x) = \frac{P_i\mu}{2k} e^{-(x-a_i)}\left[\cos\mu(x-a_i) + \sin\mu(x-a_i)\right]$$

For P_o at $x=0$,

$$v(x) = \frac{P_o\mu}{2k} e^{\mu x}\left[\cos(-\mu x) + \sin(-\mu x)\right], \quad x<0$$

For P_i at $x = a_i$,

$$v(x) = \frac{P_i\mu}{2k} e^{\mu(x-a_i)}\left[\cos \underbrace{\mu(a_i - x)}_{= -\mu(x-a_i)} + \sin\mu(a_i - x)\right]$$

For $x < a_i$

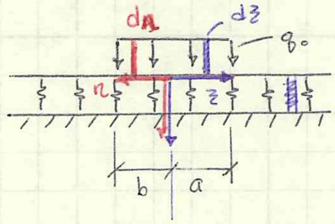
Similarly,

$$\left.\begin{aligned} v'(x) &= \frac{-P_i\mu^2}{2k} B(x-a_i) \\ M(x) &= \frac{P_i}{4k} C(x-a_i) \\ V(x) &= \frac{-P_i}{2} D(x-a_i) \end{aligned}\right\} x > a_i$$

For $x < a_i$, take note of symmetry of function (or anti-symmetry)

CONTINUOUS SUPPORTS

Example

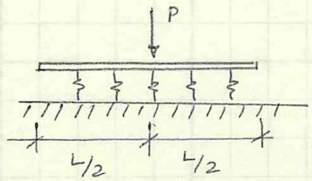


Find deflection at $\xi = \eta = 0$

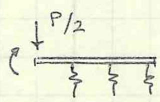
$$dv(0) = \frac{q_0\, d\xi\, \mu}{2k}\left[\cos\mu\xi + \sin\mu\xi\right]e^{-\mu\xi}$$

$$v(0) = \int_0^a \frac{q_0\mu}{2k}e^{-\mu\xi}\left[\cos\mu\xi + \sin\mu\xi\right]d\xi + \int_0^b \frac{q_0\mu}{2k}e^{-\mu\eta}\left[\cos\mu\eta + \sin\mu\eta\right]d\eta$$

Example – finite length beam on elastic foundation



beam of total length L



$$v(x) = e^{\mu x}\left[c_1\sin\mu x + \cos\mu x\, c_2\right] + e^{-\mu x}\left[c_3\sin\mu x + c_4\cos\mu x\right]$$

boundary conditions no longer include $v(\infty) = 0$

$v'(0) = 0$

$V(0) = P/2$

$V(L/2) = 0$

$M(L/2) = 0$

using to calculate constants,

$$c_1 = \frac{P\mu}{4k}\left[\frac{\sin\mu L + \cos\mu L - e^{-\mu L}}{\sin\mu L + \sinh\mu L}\right]$$

$$c_2 = \frac{P\mu}{4k}\left[\frac{2 - \sin\mu L + \cos\mu L - e^{-\mu L}}{\sin\mu L + \sinh\mu L}\right]$$

$$c_3 = \frac{2P\mu}{4k} - c_1$$

$$c_4 = \frac{2P\mu}{4k} + c_2$$

Homework comments (#5)

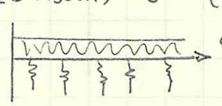
distributed load

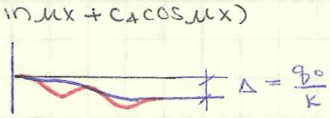
$$kv + EI\frac{d^4v}{dx^4} = q_0$$

$v = v_H + v_P$

$\hookrightarrow v_H(x) = e^{\mu x}(c_1\sin\mu x + c_2\sin\mu x) + e^{-\mu x}(c_3\sin\mu x + c_4\cos\mu x)$

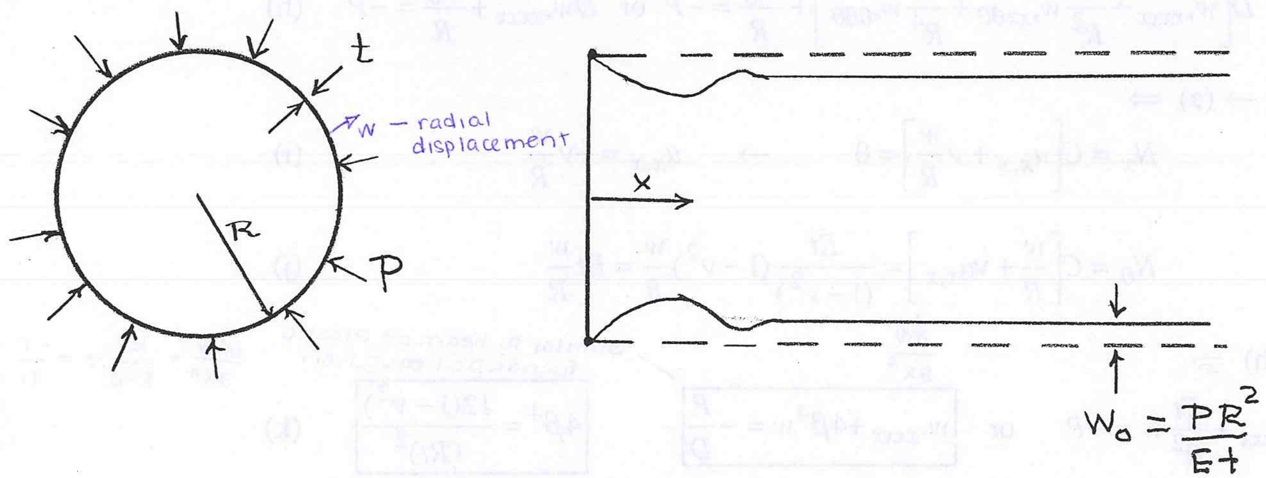
$\hookrightarrow v_P(x) = q_0/k$ for





Simply Supported Circular Cylinder under Uniform External Pressure

using shell and plate theory



EQM Eqns:

$$N_{x,x} + \frac{1}{R} N_{x\theta,\theta} = 0 \qquad N_{x\theta,x} + \frac{1}{R} N_{\theta,\theta} = 0 \tag{a}$$

$$M_{x,xx} + \frac{2}{R} M_{x\theta,x\theta} + \frac{1}{R^2} M_{\theta\theta,\theta\theta} - \frac{1}{R} N_\theta = P \tag{b}$$

$$N_x = 0$$

Problem is axisymmetric

$$\therefore \qquad N_{x\theta} = 0 \qquad v = 0, \quad \frac{\partial}{\partial\theta} = 0 \tag{c}$$

Nothing varies with θ

Const. Eqns:

$$\begin{Bmatrix} N_x \\ N_\theta \\ N_{x\theta} \end{Bmatrix} = C \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \varepsilon_{x\theta} \end{Bmatrix}, \qquad C = \frac{Et}{(1-\nu^2)} \tag{d}$$

$N_x = \frac{Et}{1-\nu^2}\left[\varepsilon_x + \nu\varepsilon_\theta\right]$

force per unit length

moments per unit length

curvatures

$$\begin{Bmatrix} M_x \\ M_\theta \\ M_{x\theta} \end{Bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_\theta \\ \kappa_{x\theta} \end{Bmatrix}, \quad D = \frac{Et^3}{12(1-\nu^2)} \tag{e}$$

Kinematics

$$\varepsilon_x = u_{,x}, \qquad \varepsilon_\theta = \frac{v_{,\theta} + w}{R}, \quad \varepsilon_{x\theta} = \frac{u_{,\theta}}{R} + v_{,x} \tag{f}$$

$= \frac{\partial u}{\partial x}$

$= \left[\frac{\partial v}{\partial \theta} + w\right]\frac{1}{R}$

$= \frac{1}{R}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x}$

EM339: MATERIALS

TEST TWO

Shear Effects

$$V = \alpha A G \frac{dv_s}{dx} \qquad \alpha = \frac{2}{3} \text{ for}$$ 

$$\tau = \frac{VQ}{It}, \quad \gamma = \frac{\tau}{G}, \quad G = \frac{E}{2(1+\nu)}$$

α: shape factor; $\frac{dv_s}{dx}$: γ

consider for short, deep beams ($L/h < 10$)

$$v_s'(0) \neq 0, \quad = \frac{V(x)}{\alpha G A}$$

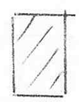
$$\frac{d^2v}{dx^2} = \frac{-1}{EI}\left[M(x) + \frac{EI}{\alpha G A} q(x)\right]$$

Axial Loads

$$EI \frac{d^4v}{dx^4} - N\frac{d^2v}{dx^2} = q(x) \qquad \sigma = \frac{N_0}{A} \pm \frac{M_{max} \cdot c}{I}$$

$$\Delta = \frac{-N_0 L}{EA} + \frac{1}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx$$

Inelastic Behavior

$$M = \int_A \sigma y \, dA, \quad = 2b\int_0^{h/2} \sigma y \, dy \text{ for}$$ 

plot M/M_0 vs K/K_0

M_0: moment of first yield $= \frac{1}{6}\sigma_0 b h^2$

K_0: curvature at first yield $= 2\sigma_0/Eh$

$\varepsilon = Ky$

Energy Methods

$$U_0 = \int_0^{\varepsilon_x} \sigma_x \, d\varepsilon_x, \quad U_0^c = \int_0^{\sigma_x} \varepsilon_x \, d\sigma_x$$ can write in each of 6 directions

if elastic (linearly), $U_0 = U_0^c$

$$U = \int_V U_0 \, dV$$

$$P = \frac{\partial U}{\partial \Delta}, \quad \Delta = \frac{\partial U^c}{\partial P} = \frac{\partial U}{\partial P}$$
if linearly elastic!

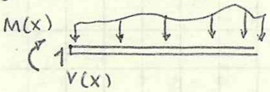
$$U = \int_0^L \frac{1}{2}P\frac{d\delta}{dx}dx = \int_0^L \frac{1}{2}EA\left(\frac{d\delta}{dx}\right)^2 = \int_0^L \frac{1}{2}\frac{P^2}{EA}$$

$$U = \int_0^L \frac{1}{2}M(x)K(x)dx = \int_0^L \frac{1}{2}EI(x)K^2(x)\,dx = \int_0^L \frac{M^2(x)}{2EI}dx$$

$$U = \int \frac{1}{2}T\frac{d\phi}{dx}dx = \int \frac{1}{2}GJ\left(\frac{d\phi}{dx}\right)^2 dx = \int \frac{T^2}{2GJ}dx$$

SHEAR EFFECTS

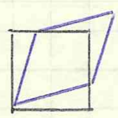
Deformations in beam bending
considering beam theory



fundamental assumption: $\varepsilon = yK$
strain varies linearly
plane sections remain plane

shear stress means shear strain

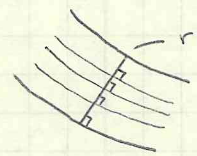
$$\gamma(y) = \frac{\tau(y)}{G}$$



angles change

since we have $V(x)$ and $\tau(y)$,
we must have

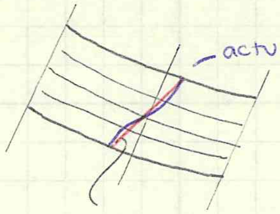
$$\gamma(y) = \frac{\tau(y)}{G}$$



right angles stay at 90°, no shear strain!

so: assumption of plane sections is not appropriate in all situations.

Actual behavior



actual shape, includes γ

Timoshenko correction for shear deformations

- Let planes remain plane, but add a uniform shear strain to the beam cross-section (of angle γ).
- Let γ be equal to τ_{max}/G

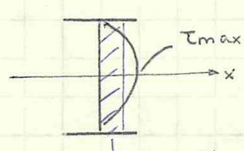
$$\gamma = \frac{\tau_{max}}{G} = \frac{\partial V_s}{dx}$$

change in slope

$$V = \alpha A \tau_{max}$$

correction

area



such that the areas are the same (average)

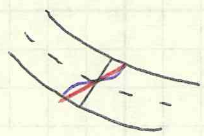
$$V = \alpha A G \frac{dV_s}{dx}$$

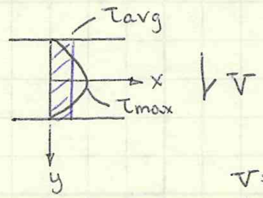
$$M = -EI \frac{d^2V_b}{dx^2}$$

total $V = V_s + V_b$

SHEAR EFFECTS

Beam approximations





$$\underset{\text{reduction factor}}{V = \alpha A \tau_{max}} = \alpha A G \gamma = \alpha G A \frac{dv_s}{dx} \leftarrow \text{displacement due to shear}$$

and

$$-M = EI \frac{d^2 v_b}{dx^2}$$

$$\underline{v = v_s + v_b}$$

$$\frac{dV}{dx} = -q(x)$$

also,

$$\frac{dV}{dx} = \alpha GA \frac{d^2 v_s}{dx^2} = -q(x)$$

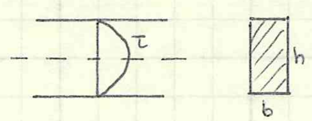
$$\frac{d^2 v_s}{dx^2} = \frac{-q(x)}{\alpha GA} \quad \text{and} \quad \frac{d^2 v_b}{dx^2} = \frac{-M}{EI}$$

$$\frac{d^2}{dx^2}(v_b + v_s) = -\left(\frac{M(x)}{EI} + \frac{q(x)}{\alpha GA}\right)$$

$$\boxed{\frac{d^2 v}{dx^2} = \frac{-1}{EI}\left[M(x) + \frac{EI}{\alpha GA} q(x)\right]}$$

Defining α

- for a rectangular cross-section



$$\tau = \frac{V(x)}{2I}\left[\frac{h^2}{4} - y_1^2\right]$$

$$\tau_{max} = \tau(0) = \frac{V(x) h^2}{8I}$$

$$\alpha \tau_{max} A = \tau_{max} b h \alpha = V(x)$$

$$\frac{bh^3}{8I} \alpha = 1.0 \qquad \underline{\underline{\alpha = \frac{2}{3}}}$$

$\leftarrow \frac{1}{12} bh^3$

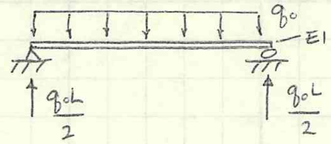
defined as $\tau_{max} \cdot A \cdot$ some value to equal actual area

QUIZ #1: 17 october 07

- likely two problems
- elastic foundation handout provided
- know: 2D stress, strain continuity equations
  beam behavior
  transformation equations
- topics finish before shear

SHEAR EFFECTS

Example problem



$$M(x) = \frac{q_0 L}{2} x - \frac{q_0 x^2}{2}$$

$$V(x) = \frac{q_0 L}{2} - q_0 x$$

$$\frac{d^2 v}{dx^2} = \frac{-1}{EI}\left[\underbrace{\frac{q_0 L}{2} x - \frac{q_0 x^2}{2}}_{M(x)} + \frac{EI}{\alpha G A}\underbrace{q_0}_{q(x)}\right]$$

$$\frac{dv}{dx} = \frac{-1}{EI}\left[\frac{q_0 L}{4} x^2 - \frac{q_0 x^3}{6} + \frac{EI}{\alpha G A} q_0 x + A\right]$$

$$-EIv = x^3 \frac{q_0 L}{12} - \frac{q_0 x^4}{24} + \underbrace{\frac{EI}{\alpha G A}\frac{q_0}{2} x^2}_{\text{only term making things different}} + Ax + B$$

Boundaries:

$v(0) = 0, \quad v(L) = 0$

$B = 0$

$$\frac{q_0 L^4}{24} + \frac{EI}{\alpha G A}\frac{q_0}{2} L^2 = -AL$$

$$A = \frac{-q_0 L^3}{24} - \frac{EI}{\alpha G A}\frac{q_0}{2} \cdot L$$

combining,

$$v(x) = \frac{-q_0 L^4}{EI}\left[\frac{1}{12}\left(\frac{x}{L}\right)^3 - \frac{1}{24}\left(\frac{x}{L}\right)^4 + \frac{EI}{\alpha G A}\frac{1}{2L^2}\left[\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)\right] - \frac{1}{24}\left(\frac{x}{L}\right)\right]$$

max deflection at $x = L/2$

or, $x/L = 1/2$

$$v(x) = \frac{+q_0 L^4}{EI}\left[5/384\right] + \frac{1}{8}\frac{EI}{\alpha G A} \cdot \frac{q_0 L^2}{EI}$$

$$v(x) = \frac{5 q_0 L^4}{384 EI}\left[1 + \underbrace{\frac{48}{5}\frac{EI}{\alpha G A L^2}}_{\text{shear correction factor}}\right]$$

Now consider:

$\frac{I}{A} = r^2$ (radius of gyration)

$$[\quad] = \left[1 + \frac{48}{5} \cdot \frac{E}{G} \cdot \frac{1}{\alpha} \left(\frac{r}{L}\right)^2\right]$$

slenderness ratio $= L/r$

SHEAR EFFECTS

continuing example:

$$v(x) = \frac{5q_oL^4}{384EI}\left[1 + \frac{48}{5}\cdot\frac{E}{G}\cdot\frac{1}{\alpha}\cdot\left(\frac{r}{L}\right)^2\right]$$

for an isotropic solid,

$G = \frac{E}{2(1+\nu)}$, $\frac{E}{G} = 2(1+\nu)$ and is generally around 2.5

$\frac{48}{5} > 1$, $\frac{1}{\alpha} > 1$, $\frac{E}{G} > 1$ ... all mean shear does something

but, for beams, $L \gg h$, or than r

$(r/L)^2 \ll 1$, unless beam is short compared to its depth

For a rectangular beam:

$$r = \frac{I}{A} = \frac{1}{bh}\cdot \tfrac{1}{12}bh^3 = \tfrac{1}{12}h^2$$

$$\frac{48}{5}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{1}{12}\left(\frac{h}{L}\right)^2 = 3.12\frac{h^2}{L^2}$$

using 2.6 for $\frac{E}{G}$
$= 3.0$ using 2.5

$L/h = 20$, $3.12/400$

$= 10$, $3.12/100$

$= 5$, $3.12/25$ — small, but maybe not negligible (12%)

e.g., 5 foot span, 1 ft depth ~~but 1ft span~~

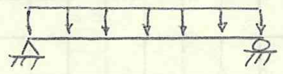
conclusion:

If beam is long and slender ($L/h > 10$), the effect of shear is small and can be neglected.

A short beam, as compared to the depth, is a "deep beam"
$L/h < 10$

SHEAR EFFECTS

Typical sections



$$\frac{d^2v}{dx^2} = \frac{-1}{EI}\left[M(x) + \frac{EI}{\alpha GA}q_0\right]$$

for a rectangle,

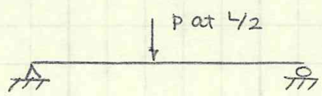
$$v(L/2) = \frac{5}{384}\frac{q_0 L^4}{EI}\left[1 + \frac{48}{5}\frac{1}{\alpha}\frac{E}{G}\frac{I}{AL^2}\right]$$

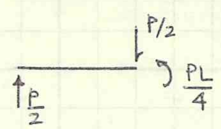
isotropic material $= 2.5$
anisotropic ~ 20

allowable L/d increases by $10\times$ when shear must be considered
$L/d > 8 \rightarrow L/d > 20$ or more

conclusion: nonisotropic, non-solid cross-sections, even semi-long spans must include shear considerations

Example





$$M(x) = -EI\frac{d^2v_b}{dx^2}$$

$$V(x) = \alpha GA\frac{dv_s}{dx}$$

$$\frac{dV}{dx} = -q(x)$$

Bending

$$M(x) = -EI\frac{d^2v_b}{dx^2} = \frac{P}{2}x$$

$$-EI\frac{dv_b}{dx} = \frac{P}{4}x^2 + A$$

$$-EIv_b = \frac{P}{12}x^3 + Ax + B$$

$$v_b(0) = 0, \quad B = 0$$

$$v_b'(L/2) = 0$$

$$\frac{P}{4}\cdot\frac{L^2}{4} + A = 0,$$

$$A = \frac{-PL^2}{16}$$

Shear

$$V(x) = \frac{P}{2} = \alpha GA\frac{dv_s}{dx}$$

$$v_s\,\alpha GA\,\cancel{\frac{dv_s}{dx}} = \frac{P}{2}x + A \quad \text{or, } C$$

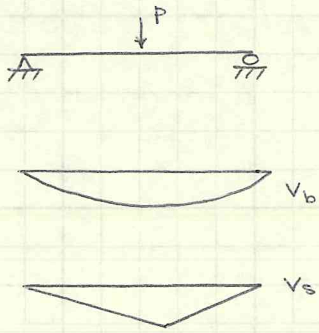
$$\cancel{\alpha GA v_s = \frac{P}{4}x^2 + Ax + B}$$

$$v_s(0) = 0, \quad C = 0$$

$$v_{tot} = \left[\frac{P}{12}x^3 - \frac{PL^2}{16}x\right]\frac{-1}{EI} + \frac{1}{\alpha GA}\cdot\frac{P}{2}x$$

SHEAR EFFECTS

Example (cont'd)



odd, eh? approximation, does not capture true shape of shear distribution

$$v = v_b + v_s = \frac{PL^3}{EI}\left[-\frac{1}{12}\left(\frac{x}{L}\right)^3 + \frac{1}{16}\left(\frac{x}{L}\right)\right] + \frac{Px}{\alpha GA}\cdot\frac{1}{2}$$

v_{max} at $x = L/2$: $$v = \frac{PL^3}{48EI}\left[1 + \frac{12EI}{L^2\alpha GA}\right]$$

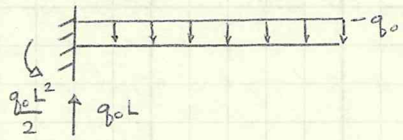
if shear is considered, deflections increase just not always by a lot

Three Structures Problems

- stress checks
- buckling
- vibration

SHEAR EFFECTS

Response of beams



$$q(x) = q_0$$

$$M(x) = q_0 L x - \frac{q_0 L^2}{2} - \frac{q_0 x^2}{2}$$

$$\frac{d^2 v}{dx^2} = \frac{-1}{EI}\left[M(x) + q(x)\frac{EI}{\alpha G A}\right]$$

$$= \frac{-1}{EI}\left[q_0 L x - q_0 L^2 \cdot \frac{1}{2}\right] - \frac{q_0}{\alpha G A}$$

$- q_0 x^2/2$

$$\frac{dv}{dx} = \frac{-1}{EI}\left[\frac{q_0}{2} L x^2 - q_0 L^2 x/2 - \frac{q_0}{6} x^3\right] - \frac{q_0}{\alpha G A} x + A$$

$$v = \frac{-1}{EI}\left[\frac{q_0}{6} L x^3 - \frac{q_0}{4} L^2 x^2 - \frac{q_0}{24} x^4\right] - \frac{q_0 x^2}{2 \alpha G A} + A x + B$$

so,

$$v(x) = \frac{q_0 L^4}{EI}\left[\frac{-1}{6}\left(\frac{x}{L}\right)^3 - \frac{1}{4}\left(\frac{x}{L}\right)^2 + \frac{1}{24}\left(\frac{x}{L}\right)^4\right] - \frac{q_0 L^2}{2 \alpha G A}\left(\frac{x}{L}\right)^2 + Bx + C$$

$v(0) = 0, \quad C = 0$

consider:

case 1 – $v'(0) = 0, \quad B = 0$

completely fixed condition at end

no slope, no change in shear

$$v_b'(0) = v_s'(0) = 0$$

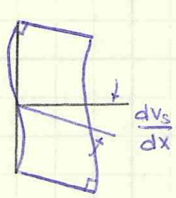
$$v(L) = \frac{q_0 L^4}{8EI} \underbrace{- \frac{q_0 L^2}{2 \alpha G A}}$$

shear effects are <u>decreasing</u> the deflection

this is not possible

incorrect assumption: $v_s'(0) = 0$

needs to be fixed



$V(x) = \alpha G A \frac{dv_s}{dx}$ is not satisfied if $v_s'(0) = 0$

We must allow fixed end to develop shear deformations

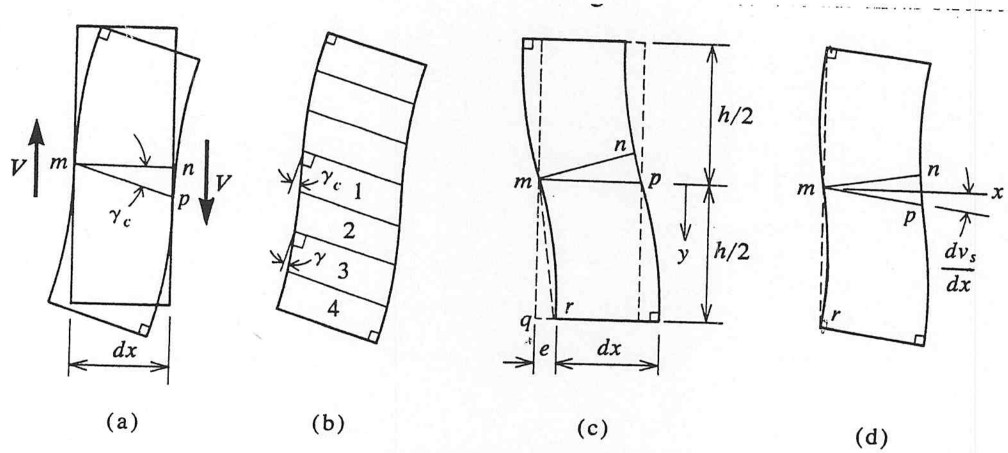


Fig. 6-25. Shear deformations in a beam.

SHEAR EFFECTS

Response of beams

case 2: let $v_s'(0) = \frac{V(x)}{\alpha GA} = \frac{q_0 L}{\alpha GA}$

then $v'(0) = v_s'(0) = B = \frac{q_0 L}{\alpha GA}$

now $v(L) = \frac{q_0 L^4}{8EI} + \underbrace{\frac{q_0 L}{2\alpha GA}}$

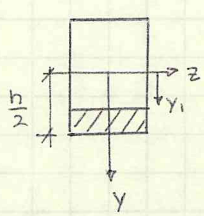
$= \frac{-q_0 L^2}{2\alpha GA}\left(\frac{x}{L}\right)^2 + \frac{q_0 L}{\alpha GA} \cdot x$

at L: $\frac{q_0 L^2}{\alpha GA}\left(-1/2 + 1\right) = \frac{1}{2}\frac{q_0 L^2}{\alpha GA}$

positive value!

Shear Stress Distribution through beam cross-section

example:



remember: $\tau(y_1) = \frac{V}{Ib}\int_{y_1}^{h/2} y\,dA$

let $Q = \int_{y_1}^{h/2} y\,dA$ first moment of area about the centroid

$Q = A'\bar{y}'$, where prime refers to the shaded area

$\tau = \frac{VQ}{Ib}$

for a rectangle,

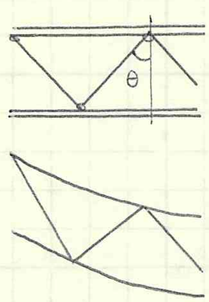
$Q = b\left(\frac{h}{2} - y_1\right)\left(y_1 + \frac{1}{2}\left(\frac{h}{2} - y_1\right)\right)$

$= \frac{b}{2}\left(\frac{h^2}{4} - y_1^2\right)$

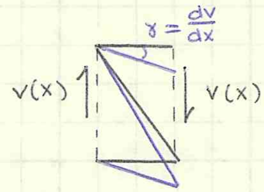
$\tau = \frac{V}{Ib}\frac{b}{2}\left(\frac{h^2}{4} - y_1^2\right) = \frac{V}{2I}\left(\frac{h^2}{4} - y_1^2\right)$

Homework:

Problem 3:



$V(x) = \underbrace{\alpha GA}\frac{dv_s}{dx}$ need a new relationship

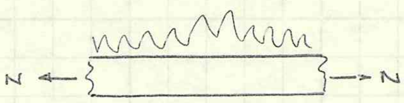


now: relate $V(x)$ to γ

$\gamma = \frac{\tau}{G}$, $\tau = \frac{VQ}{It}$

SHEAR EFFECTS

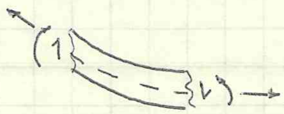
Beams with axial loads



axially, $N = EA\frac{\delta}{L} = EA\frac{du}{dx}$

bending, $M = -EI\frac{d^2v}{dx^2}$

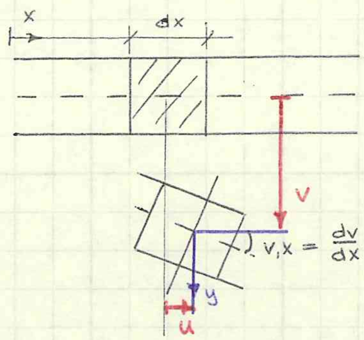
individual response; do not simply add



also produces a moment (in addition to axial)

< PΔ effect >

Kinematics



$$\varepsilon_x = \frac{d\bar{u}}{dx} + \frac{1}{2}\left[\left(\frac{d\bar{u}}{dx}\right)^2 + \left(\frac{d\bar{v}}{dx}\right)^2\right]$$

equation applies to large strains

small strains allow us to simplify:

$$\varepsilon_x = \frac{d\bar{u}}{dx} + \frac{1}{2}\left(\frac{d\bar{v}}{dx}\right)^2$$

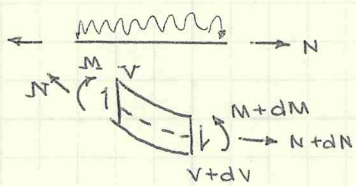
Now adopt displacements referenced to the centroid and the neutral axis

$$\bar{v} \simeq v$$

$$\bar{u} = u - y\frac{dv}{dx}$$

$$\therefore \varepsilon_x = \underbrace{\frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2}_{\substack{\varepsilon_x^\circ \\ \text{membrane strain}}} - \underbrace{y\frac{d^2v}{dx^2}}_{\substack{y\kappa_x \\ \text{curvature}}} = \varepsilon_x^\circ + y\kappa_x$$

Equilibrium



angle of axial forces:

$$\frac{dv}{dx},\ \frac{dv}{dx} + d\left(\frac{dv}{dx}\right)$$

ΣF_x: $N + \frac{dN}{dx}dx - N = 0$, $\frac{dN}{dx} = 0$

$N = \text{constant}$

ΣF_y:

$$-\cancel{V} - N\cancel{\frac{dv}{dx}} + \cancel{V} + \frac{dv}{dx}dx + \left[N + \cancel{\frac{dN}{dx}}^{\,0}dx\right]\left[\cancel{\frac{dv}{dx}} + \frac{d^2v}{dx^2}dx\right] + q(x)\,dx = 0$$

$$\frac{dv}{dx}dx + N\frac{d^2v}{dx^2}dx = -q(x)\,dx\,,\quad \frac{dv}{dx} + N\frac{d^2v}{dx^2} = -q(x)$$

AXIAL LOAD IN BEAMS

Derivation, cont'd

Equilibrium

$$\frac{dM}{dx} = V$$

$$\frac{dN}{dx} = 0$$

$$\frac{dV}{dx} + N\frac{d^2v}{dx^2} = -q(x)$$

important equations to this point in differentiation / equilibrium / derivation

$$\frac{d^2M}{dx^2} + N\frac{d^2v}{dx^2} = -q(x)$$

$$\frac{dN}{dx} = 0$$

Note:

$$N = \int_A \sigma \, dA, \quad N = \int_A \left[E(\varepsilon_x^\circ + yK_x)\right] dA$$

$$N = \int_A E\varepsilon_x^\circ \, dA + \int_A EyK_x \, dA = EA\varepsilon_x^\circ$$

$$M = \int_A \sigma y \, dA = \int_A E \overset{0}{\cancel{y}} \varepsilon_x^\circ \, dA + \int_A Ey^2 K_x \, dA$$

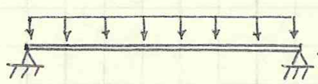
$$= EIK_x = -EI\frac{d^2v}{dx^2}$$

Combining,

$$\boxed{EI\frac{d^4v}{dx^4} - \frac{Nd^2v}{dx^2} = q(x)}$$

$$\frac{dN}{dx} = 0, \quad \varepsilon_x^\circ = \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2$$

Big Example



beam is not free to move axially — adds axial stress in beam

No horizontal displacement means tension is "applied"

constants: EI, EA, L, etc.

$$EI\frac{d^4v}{dx^4} - N_0\frac{d^2v}{dx^2} = q(x)$$

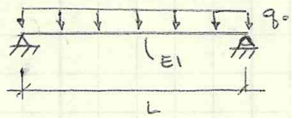
differential equation with constant coefficients — solveable.

However, N_0 is unknown

makes the problem harder

AXIAL EFFECTS

Beam response



$$\frac{dN}{dx} = 0 \,, \quad N = \text{constant}$$

$$EI\frac{d^4v}{dx^4} - N_0\frac{d^2v}{dx^2} = q_0 = q(x)$$

$$\text{or} \quad \frac{d^4v}{dx^4} - \frac{N_0}{EI}\frac{d^2v}{dx^2} = \frac{q(x)}{EI}$$

Now, solve the equations

$$\frac{d^4v_H}{dx^4} - \lambda^2\frac{d^2v_H}{dx^2} = 0 \,, \quad \lambda^2 = \frac{N_0}{EI}$$

$$\frac{d^2}{dx^2}\left[\frac{d^2v_H}{dx^2} - \lambda^2 v_H\right] = 0$$

$$v_H(x) = A\sinh\lambda x + B\cosh\lambda x + Cx + D$$

$$v_p = \frac{-q_0}{N_0}\frac{x^2}{2}$$

$$v = v_H + v_p = A\sinh\lambda x + B\cosh\lambda x + Cx + D - \frac{q_0}{N_0}\frac{x^2}{2}$$

applying boundaries:

$$v(0) = 0 \ : \quad B + D = 0$$

$$v''(x) = \lambda^2(A\sinh\lambda x + B\cosh\lambda x) - \frac{q_0}{N_0}$$

$$v''(0) = 0 = B\lambda^2 - \frac{q_0}{N_0}$$

$$B = \frac{+q_0}{N_0\lambda^2} \,, \quad D = \frac{-q_0}{N_0\lambda^2}$$

$$v(L) = 0, \quad A\sinh\lambda L + \frac{q_0}{N_0\lambda^2}\cosh L\lambda + CL - \frac{q_0}{N_0\lambda^2} - \frac{q_0}{N_0}\frac{L^2}{2}$$

$$v''(L) = 0 \,, \quad \lambda^2\left(A\sinh\lambda L + \frac{q_0}{N_0\lambda^2}\cosh L\lambda\right) - \frac{q_0}{N_0} = 0$$

$$A = \frac{q_0}{N_0\lambda^2}(1 - \cosh\lambda L)/\sinh\lambda L$$

$$C = \frac{q_0}{N_0}\cdot\frac{L}{2}$$

Thus,

$$v = \frac{q_0}{N_0\lambda^2}\left[\frac{1-\cosh\lambda L}{\sinh\lambda L} + \cosh\lambda L + \frac{L\lambda^2}{2}x - 1 - \frac{x^2\lambda^2}{2}\right]$$

using half angle theorems,

$$v(x) = \frac{q_0}{N_0\lambda^2}\left[\frac{\cosh\left(\frac{\lambda L}{2} - \lambda x\right)}{\cosh\frac{\lambda L}{2}} - 1\right] + \frac{q_0}{2N_0}(L - x)x$$

Note that λ is unknown.

AXIAL EFFECTS

Equations: need N_o, or λ

$$N_o = EA\varepsilon_x^{\circ} = EA\left[\frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2\right]$$

↑ stretch of the neutral axis

$$\frac{du}{dx} = \frac{N_o}{EA} - \frac{1}{2}\left(\frac{dv}{dx}\right)^2 \quad \text{now integrate}$$

$$\int_0^L \frac{du}{dx}\cdot dx = \int_0^L \frac{N_o}{EA}dx - \int_0^L \frac{1}{2}\left(\frac{dv}{dx}\right)^2 dx$$

$$\cancel{u(L)}^{\,0} - \cancel{u(0)}^{\,0} = \frac{N_o L}{EA} - \int_0^L \frac{1}{2}\left(\frac{dv}{dx}\right)^2 dx$$

$$\frac{N_o L}{EA} = \frac{1}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx$$

amount of stretch is equal to the difference in length from start to finish. Duh.

(problem is no longer linear)

Goal: find $\frac{dv}{dx}$

$$\frac{dv}{dx} = \frac{q_o}{N_o\lambda^2}\left[\frac{-\lambda\sinh\left(\frac{\lambda L}{2} - \lambda x\right)}{\cosh\frac{\lambda L}{2}}\right] + \frac{q_o}{2N_o}(L - 2x)$$

from equation last page

$$\frac{N_o L}{EA} = \frac{1}{2}\left(\frac{q_o}{N_o\lambda^2}\right)^2 \int_0^L \left[\left[\frac{-\lambda\sinh\left(\frac{\lambda L}{2} - \lambda x\right)}{\cosh\frac{\lambda L}{2}}\right] + \frac{\lambda^2}{2}(L - 2x)\right]^2 dx$$

$$\text{let } \xi = \frac{\lambda L}{2}\left(1 - \frac{2x}{L}\right)$$

$$d\xi = -\lambda dx$$

$$\frac{N_o L}{EA} = \frac{1}{2}\left(\frac{q_o}{N_o\lambda^2}\right)^2 \int_{\frac{\lambda L}{2}}^{-\frac{\lambda L}{2}} \left[\frac{-\lambda\sinh(\xi)}{\cosh(\lambda L/2)} + \lambda\xi\right]^2 \left(\frac{-1}{\lambda}\right) d\xi$$

$$= \frac{-1}{2}\left(\frac{q_o}{N_o\lambda^2}\right)^2\cdot\lambda \int_{\frac{\lambda L}{2}}^{-\frac{\lambda L}{2}} \left[\xi^2 - \frac{2\xi\sinh\xi}{\cosh\lambda L/2} + \frac{\sinh^2\xi}{\cosh^2\frac{\lambda L}{2}}\right] d\xi$$

$$\frac{-2N_o^3\lambda^3 L}{q_o^2 EA} = \left[\frac{\xi^3}{3} - \frac{2\xi\cosh\xi}{\cosh\frac{\lambda L}{2}} + \frac{2\sinh\xi}{\cosh\frac{\lambda L}{2}} + \frac{1}{2}\,\frac{\sinh\xi\cosh\xi - \xi}{\cosh^2\frac{\lambda L}{2}}\right]_{\frac{\lambda L}{2}}^{-\frac{\lambda L}{2}}$$

$$= -\frac{2}{3}\left(\frac{\lambda L}{2}\right)^3 + 2\lambda L - 5\tanh\frac{\lambda L}{2} + \frac{\lambda L/2}{\cosh^2\frac{\lambda L}{2}}$$

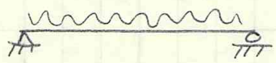
$$\text{Let } \mu = \frac{\lambda L}{2}$$

$$\frac{16 N_o\mu^3}{q_o^2 EAL^2} = \frac{64\mu^5 N_o^2 I}{q_o^2 L^4 A} = \frac{2}{3}\mu^3 - 4\mu + 5\tanh\mu - \frac{\mu}{\cosh^2\mu}$$

AXIAL EFFECTS

more derivation

consider the corresponding linear problem

 ← wheels

$$\frac{d^4 v}{dx^4} = \frac{q_0}{EI}, \quad v(L/2) = \frac{5L^4 q_0}{384EI}$$

let $\hat{q}_0$ be the load when $v(L/2) = h$ (the depth of the beam)

$$h = \frac{5\hat{q}_0 L^4}{384EI}$$

define $$\Lambda = \frac{q_0}{\hat{q}_0} = \frac{v(L/2)}{h}$$

Return to the nonlinear problem:

$$\lambda^2 = \frac{N_0}{EI}$$

$$\frac{64\mu^5 N_0^2 I}{q_0 L^4 A} = \frac{64\mu^5 N_0^2}{\frac{q_0^2}{\hat{q}_0}\cdot \hat{q}_0 \cdot L^4} \cdot \frac{bh^3/12}{bh} = \frac{64\times 25}{(384)^2\times 12}\cdot\frac{\mu^5}{\Lambda^2}\cdot\frac{N_0^2 L^4}{(EI)^2} = \frac{1}{3}\left(\frac{5}{24}\right)^2 \frac{\mu^9}{\Lambda^2}$$

rectangular section, use definition above

$$h = \frac{5}{384}\frac{\hat{q}_0 L^4}{EI}$$

Now,

$$\left(\frac{5}{24}\right)^2 \frac{\mu^9}{3\Lambda^2} + \frac{\mu}{\cosh^2\mu} - \frac{2}{3}\mu^3 + 4\mu - 5\tanh\mu = 0$$

Back to $v(x)$, non-linear form

$$v(L/2) = \frac{q_0 L^2}{4N_0\mu^2}\left[\frac{1}{\cosh\mu} - 1\right] + \frac{q_0 L^2}{8N_0} = \frac{q_0 L^2}{8N_0\mu^2}\left[\frac{2(1-\cosh\mu) - \mu^2\cosh\mu}{\cosh\mu}\right]$$

AXIAL EFFECTS

Equations a GoGo

$$v(L/2) = \frac{q_0 L^2}{4N_0\mu^2}\left[\frac{1}{\cosh\mu} - 1\right] + \frac{q_0 L^2}{8N_0} = \frac{q_0 L^2}{8N_0\mu^2}\left[\frac{2(1-\cosh\mu)+\mu^2\cosh\mu}{\cosh\mu}\right]$$

$$\left(\frac{5}{24}\right)^2\frac{\mu^4}{3\Lambda^2} + \frac{\mu}{\cosh^2\mu} - \frac{2}{3}\mu^3 + 4\mu - 5\tanh\mu = 0$$

from $\frac{N_0 L}{EA} = -\int_0^L \frac{1}{2}\left(\frac{dv}{dx}\right)^2 dx$

where:

$$\Lambda = \frac{q_0}{\hat{q}_0} = \frac{v(L/2)}{h} \quad \text{for linear beam}$$

$$\left.\begin{aligned} \mu &= \frac{\lambda L}{2} \\ \lambda^2 &= \frac{N_0}{EI} \end{aligned}\right\} \mu = \frac{L}{2}\sqrt{\frac{N_0}{EI}}$$

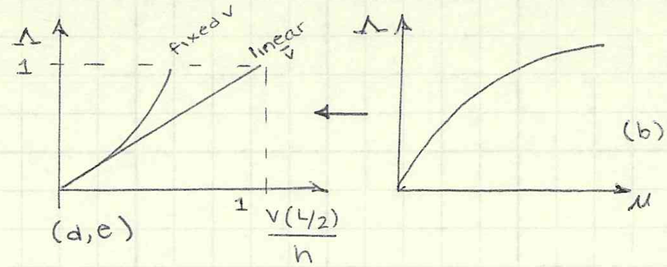
Now, non-dimensionalize:

$$\frac{v(L/2)}{h} = \frac{\Lambda\, 384 EI L^2}{5\times 8 \times N_0 \times L^4 \mu^2}\left[\frac{2(1-\cosh\mu)+\mu^2\cosh\mu}{\cosh\mu}\right]$$

$$= \frac{\Lambda}{\mu^2}\cdot\frac{12}{5}\left[\frac{2(1-\cosh\mu)+\mu^2\cosh\mu}{\cosh\mu}\right]$$

4

Now, do all this ourselves in the Homework (#6)



TEST COMMENTS

Number One

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}\sigma_x$$

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}\sigma_y$$

$$\gamma_{xy} = \frac{\sigma_{xy}}{G} = \frac{2(1+\nu)}{E}\sigma_{xy}$$

Given: $\sigma_x = 10\,\text{ksi}$, $\sigma_{xy} = 5\,\text{ksi}$, $\varepsilon_y = 0.0002$, $\sigma_z = 0$, $E = 10000\,\text{ksi}$, $\nu = 0.3$

von mises criterion:

$$\sigma_o^2 = \sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\sigma_{xy}^2 \quad , \; \sigma_o = \text{yield stress (uniaxial)}$$

Things to remember:

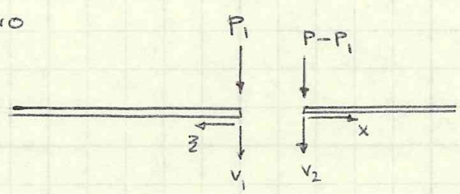
$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_y + \sigma_x)\right]$$

can always remove terms (2D, etc.). can't make strains zero correctly.

Number Two



$$\mu_1 = \left[\frac{k}{8EI}\right]^{1/4}, \quad \mu_2 = \left[\frac{k}{4EI}\right]^{1/4}$$

$$v_1(\xi) = e^{-\mu_1\xi}\left[C_3\sin\mu_1\xi + C_4\cos\mu_1\xi\right] \qquad v_2(x) = e^{-\mu_2 x}\left[D_3\sin\mu_2 x + D_4\cos\mu_2 x\right]$$

boundaries:

$$v_1(0) = v_2(0)$$

$$v_1'(0) = -v_2'(0)$$

$$M_1(0) = M_2(0) \longrightarrow -2EIv_1''(0) = -EIv_2''(0)$$

$$-2EIv_1'''(0) = -P_1$$

$$-EIv_2'''(0) = -P + P_1$$

Answer:

$$\alpha = \frac{\mu_1}{\mu_2} = \left(\frac{1}{2}\right)^{1/4}$$

$$C_4 = D_4 = \frac{\alpha(2\alpha+1)}{\alpha+1}C_3$$

$$2\alpha^2 C_3 = D_3$$

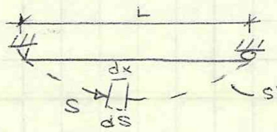
$$\left[\frac{4\alpha^3 + 4\alpha^2 + 3\alpha + 2}{\alpha+1}\right]C_3 = \frac{2\mu_2 P}{k}$$

AXIAL EFFECTS

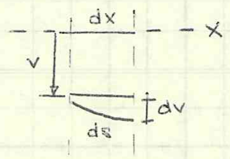
continuing...

$$\varepsilon_x = \varepsilon_x^\circ + y k_x \qquad k_x = -\frac{d^2 v}{dx^2}$$

$$\varepsilon_x^\circ = \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2$$



should have a $\Delta \leftarrow$, too



$$ds = (dx^2 + dv^2)^{1/2}$$

$$= \left[1 + \left(\frac{dv}{dx}\right)^2\right]^{\frac{1}{2}} dx$$

$$\simeq \left[1 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2\right] dx$$

$$\text{length} = \int ds = \int_0^{L-\Delta} \left(1 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2\right) dx$$

$$= (L-\Delta) + \int_0^{L-\Delta} \frac{1}{2}\left(\frac{dv}{dx}\right)^2 dx$$

in this integral, Δ matters very little, change limits to $[0, L]$

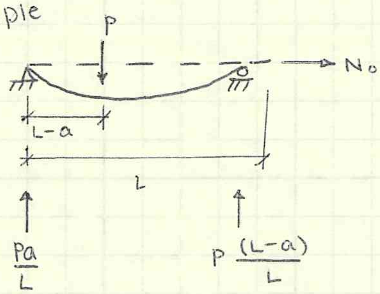
$$= L + \frac{N_0 L}{EA}$$

stretch from axial force

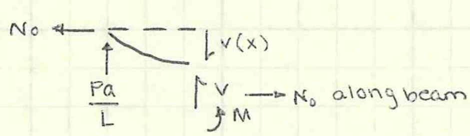
$$L + \frac{N_0 L}{EA} = L - \Delta + \frac{1}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx$$

$$\Delta = \underbrace{\frac{-N_0 L}{EA}}_{\text{stretch along the axis}} + \underbrace{\frac{1}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx}_{\text{pull from axial load (?)}}$$

Example



For $x < L-a$,



$$M(x) = R_1 x - N_0 v_1(x) = -EI v_1''$$

$$v_1'' = \frac{N_0}{EI} v_1 - \frac{R_1}{EI} x$$

For $x > L-a$,

$$M(x) = R_1 x - N_0 v_1 - P(x - L + a)$$

AXIAL EFFECTS

Example, cont'd

Two equations:

$$v_1'' - \frac{N_0}{EI} v_1 = \frac{-R_1 x}{EI}$$

$$v_2'' - \frac{N_0}{EI} v_2 = \frac{-R_1 x}{EI} + \frac{Px}{EI} - \frac{P}{EI}(L-a)$$

again, define $K^2 = \frac{N_0}{EI}$

$$v_1 = A\sinh Kx + B\cosh Kx + \frac{R_1}{N_0}x$$

$$v_2 = C\sinh x + D\cosh Kx + \frac{R_1}{N_0}x - \frac{P}{N_0}x + \frac{P}{N_0}(L-a)$$

Boundaries:

$v_1(0) = 0$, $v_2(L) = 0$

$B = 0$ $\quad C\sinh KL + D\cosh KL + \frac{R_1}{N_0}L - \frac{P}{N_0}a = C\sinh KL + D\cosh KL$

$$R_1 = \frac{Pa}{L}$$

$$D = \cancel{-}\tanh KL \cdot C$$

$v_1(a) = v_2(a)$

$$A\sinh Ka + \cancel{\frac{R_1}{N_0}a} = C\sinh a + D\cosh Ka + \cancel{\frac{R_1}{N_0}a} - \frac{2P}{N_0}a + \frac{P}{N_0}L$$

$$A = C + D\tan^{h} Ka + \frac{P}{N_0}\left(\frac{L-2a}{\sinh Ka}\right), \ \sinh Ka \neq 0$$

... except it's actually

$$v_1(L-a) = v_2(L-a)$$

$$A\sinh K(L-a) = C\sinh K(L-a) + D\cosh K(L-a)$$

$$A = \left[1 - \tanh KL \coth K(L-a)\right] C$$

$v_1'(L-a) = v_2'(L-a)$

$$AK\cosh K(L-a) = CK\left[\cosh K(L-a) - \tanh KL \sinh(L-a)\right] - \frac{P}{N_0}$$

$$C = \frac{P}{N_0 K} \cdot \frac{\sinh K(L-a)}{\tan KL}$$

$$D = \frac{-P}{N_0 K}\sinh K(L-a)$$

$$B = 0$$

$$A = \frac{P}{N_0 K}\left[\frac{\sinh K(L-a)}{\tan KL} - \cosh K(L-a)\right]$$

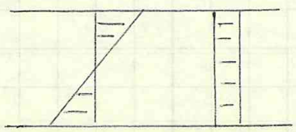
$$= \frac{-P}{N_0 K} \cdot \frac{\sinh Ka}{\sinh KL}$$

$$v_1(x) = \frac{P}{N_0 K}\left[\frac{\sinh Ka}{\sinh KL} \cdot \sinh Kx + aKx\right] \quad x < L-a$$

AXIAL EFFECTS

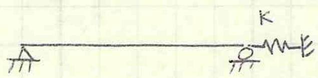
Stress calculations

$$\sigma_{max} = \frac{M_{max} \cdot c}{I} + \frac{N_o}{A}$$



If symmetric, σ_{max} is on side where stresses add. If not symmetric, both sides must be checked

Homework:



N_o in beam, spring

Δ must match between beam, spring

"take it as far as you can" [solution-wise]

Summary:

$$EI\frac{d^4v}{dx^4} - N_o\frac{d^2v}{dx^2} = q(x)$$

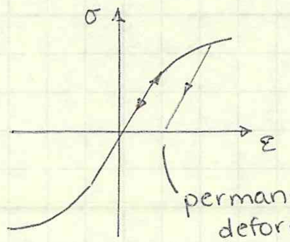
previously, second term was $+kv$
(continuous foundation)

both:

$$EI\frac{d^4v}{dx^4} - N_o\frac{d^2v}{dx^2} + kv = q(x)$$

INELASTIC BEHAVIOR

Beam bending

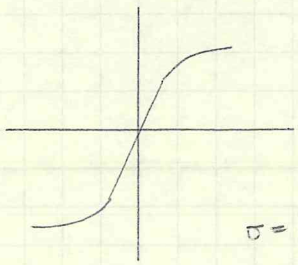


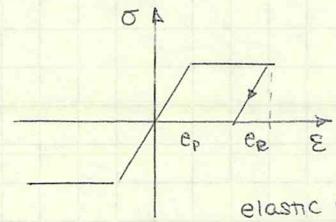
permanent plastic deformation

we don't wish to engage the inelastic behavior of a structure. only in extreme situations - earthquake, collapse, blast, etc.
or, car accidents

INELASTIC BEHAVIOR

Beam behavior





$\sigma = \sigma(\varepsilon) = \sigma(yK)$

$\int_A \sigma \, dA = 0, \quad M = \int_A y \sigma \, dA$

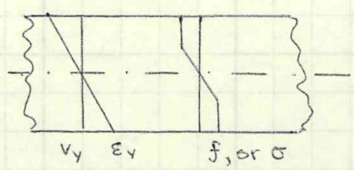
$\frac{M}{EI}$ no longer defines curvature

elastic perfectly plastic response

ε_e: elastic strain

ε_p: plastic strain

Example



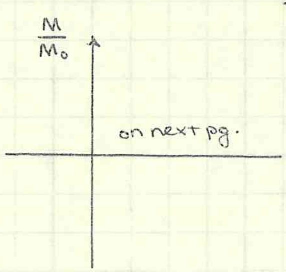
f cannot exceed σ_0

$$M = 2b\left[\int_0^{\bar{h}/2} E\varepsilon y\,dy + \int_{\frac{\bar{h}}{2}}^{h/2} \sigma_0 y\,dy\right]$$, where $\frac{\bar{h}}{2}$ = distance to limit of elastic section

$$= 2b\left[EK\frac{y^3}{3}\Big|_0^{\bar{h}/2} + \sigma_0 \frac{y^2}{2}\Big|_{\bar{h}/2}^{h/2}\right]$$

if $\bar{h}/2 = h/2$, $M = \frac{1}{12}bh^3 \cdot EK = EI \cdot K$, or $M/EI = K$

$$= 2b\left[\frac{1}{3}EK\left(\frac{\bar{h}}{2}\right)^3 + \frac{\sigma_0}{2}\left(\frac{h^2}{4} - \frac{\bar{h}^2}{4}\right)\right]$$



, K_0 = curvature at which material first yields, or $\frac{\sigma_0}{Eh/2}$

$$K = \frac{2\sigma_0}{E\bar{h}} \quad ; \quad K/K_0 = \bar{h}/h$$

$$\frac{E\bar{h}^3}{12}K = \frac{h^2}{6}\left(\frac{\bar{h}}{h}\right)^2 \frac{\cancelto{\sigma_0}{E\bar{h}K}}{2} = \frac{\sigma_0 h^2}{2}\left(\frac{K_0}{K}\right)^2$$

$$\frac{\sigma_0 h^2}{4}\left[1 - \left(\frac{\bar{h}}{h}\right)^2\right] = \frac{\sigma_0 h^2}{4}\left[1 - \left(\frac{K_0}{K}\right)^2\right]$$

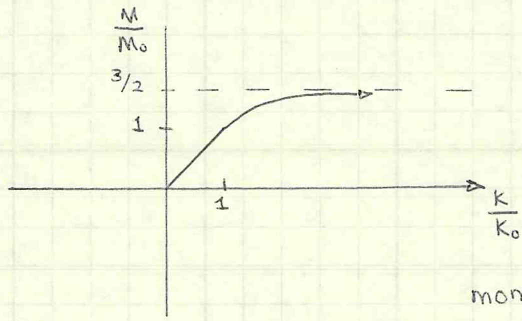
$$M = \frac{\sigma_0 b h^2}{6}\left[\left(\frac{K_0}{K}\right)^2 + \frac{3}{2}\left[1 - \left(\frac{K_0}{K}\right)^2\right]\right], \quad M_0 = \frac{bh^3}{12}\frac{\sigma_0}{h/2} = \frac{bh^2}{6}\sigma_0$$

moment at first yield, $K = K_0$

INELASTIC BEHAVIOR

Beam Response

Moment-curvature



$$\frac{M}{M_o} = \frac{1}{2}\left[3 - \left(\frac{K_o}{K}\right)^2\right], \quad K \geq K_o$$

$$\frac{M}{M_o} = \frac{K}{K_o}, \quad K < K_o$$

moment asymptotes to $3/2$

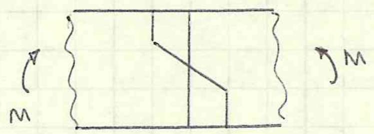
note: equation has K_o/K, graph is K/K_o

when the slope $= 0$, further bending can occur without adding any moment

hinge forms ("plastic hinge")

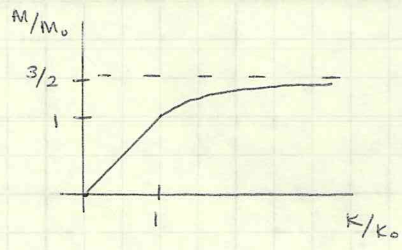
AXIAL RESPONSE

or, inelastic behavior

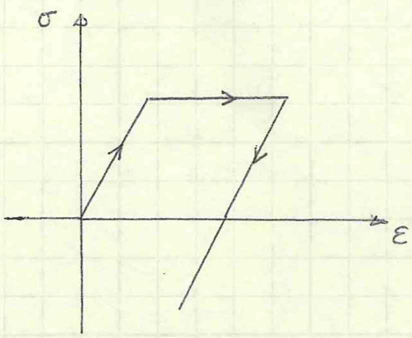


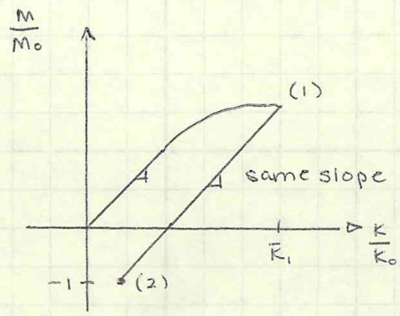
$$M_o = \frac{\sigma_o b h^2}{2} \qquad \frac{M}{M_o} = \frac{1}{2}\left[3 - \left(\frac{K_o}{K}\right)^2\right] \quad K > K_o$$

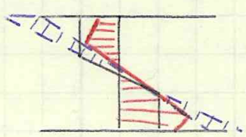
$$K_o = \frac{2\sigma_o}{Eh} \qquad \text{for } K \le K_o,\ \frac{M}{M_o} = \frac{K}{K_o}$$



unloading from a plastic state







$\Delta\sigma = E\Delta\varepsilon$

$= Ey\Delta K$, K varies linearly when unloading

— new σ profile
(should be symmetric)

Let $K_1 = \frac{2\sigma_o}{E\bar{h}_1}$, $\frac{K_o}{K_1} = \frac{\bar{h}_1}{h}$

for $|y| \le \frac{\bar{h}_1}{2}$, $\sigma = E\varepsilon = EyK$

for $|y| > \frac{\bar{h}_1}{2}$, $\sigma = \sigma_o - E\Delta\varepsilon = \sigma_o - E(\varepsilon_1 - \varepsilon_o)$

$= \sigma_o - Ey(K_1 - K)$

$$M = 2b\left[\int_0^{\bar{h}_1/2} Ey^2 K\,dy + \int_{\frac{\bar{h}_1}{2}}^{h/2} \sigma_o y + Ey^2K - Ey^2K_1\,dy\right]$$

$$= 2b\left[E\left(\frac{\bar{h}_1}{2}\right)^3\frac{K}{3} + \frac{\sigma_o}{2}\left(\frac{h^2}{4} - \frac{\bar{h}_1^2}{4}\right) + E\frac{K}{3}\left(\frac{h^3}{8} - \frac{\bar{h}_1^3}{8}\right) - \frac{EK_1}{24}\left(h^3 - \bar{h}_1^3\right)\right]$$

$$\frac{M}{M_o} = \frac{1}{2}\left[3 - \left(\frac{K_o}{K_1}\right)^2\right] + (\bar{K} - \bar{K}_1) \qquad = \frac{M_1}{M_o} - \frac{\Delta K}{K_o}$$

$\bar{K} = \frac{K}{K_o}$, $\bar{K}_1 = \frac{K_1}{K_o}$

INELASTIC BENDING

Loading and unloading

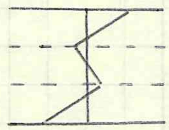
Let $\frac{M}{M_o} = 0$ on unloading path

$$\frac{M}{M_o} = \frac{M_1}{M_o} - \frac{\Delta K}{K_o} = 0$$

$$\frac{M_1}{M_o} = \frac{K}{K_o} - \frac{K_1}{K_o} = \bar{K} - \bar{K}_1$$

$$\bar{K} = \bar{K}_1 - \frac{1}{2}\left[3 - \left(\frac{1}{\bar{K}_1^2}\right)\right]$$

residual curvature



remembers plastic deformation up to $y = \bar{h}_1/2$

$M = 0$
$N = 0$

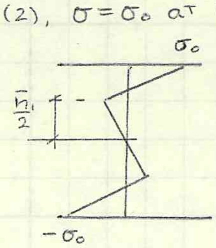
residual stresses across cross-section

aircrafts use shot pinning to remove tensile stresses

Unload max, or opposite curvature yield

K_2: the curvature at which the outer fibers yield again because of reverse loading

At (2), $\sigma = \sigma_o$ at $y = h/2$, $y = -h/2$

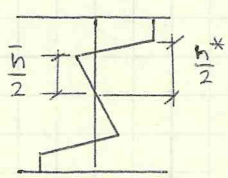


$$-\sigma_o = \sigma_o + \frac{h}{2}K_2 E - \frac{h}{2}K_1 E$$

$$K_2 = \frac{2}{Eh}\left[-2\sigma_o + \frac{h}{2}EK_1\right]$$

or $\frac{K_2}{K_o} = -2 + \frac{K_1}{K_o}$, $\frac{M_2}{M_o} = \frac{-1}{2}\left[1 + \left(\frac{K_o}{K_1}\right)^2\right]$

loading past point (2) – more yielding at extremes



at each yield occurance, a discontinuity forms in the stress distribution

At $\frac{h^*}{2}$, $\sigma = -\sigma_o = \sigma_o - \frac{h^*}{2}EK_1 + \frac{h^*}{2}EK$

$$\frac{h^*}{h} = \frac{2}{\bar{K}_1 - \bar{K}}$$

$$|y| \le \frac{\bar{h}_1}{2}, \quad \sigma = E\varepsilon = EyK$$

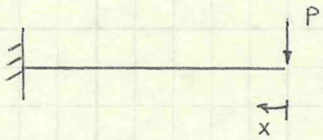
$$\frac{\bar{h}_1}{2} < |y| \le \frac{h^*}{2}, \quad \sigma = \sigma_o - \frac{h^*}{2}EK_1 + \frac{h^*}{2}EK$$

$$\frac{h^*}{2} < |y| \le \frac{h}{2}, \quad \sigma = -\sigma_o$$

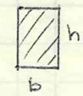
M now has 3 integrals, $\frac{M}{M_o} = -\frac{1}{2\bar{K}_1^2} + \frac{3}{2} + \frac{4}{(\bar{K}_1 - \bar{K})^2}$

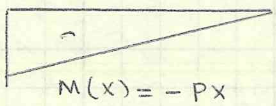
INELASTIC BEHAVIOR

Beam example



rectangular cross-section





$M(x) = -Px$

$$M_0 = \frac{\sigma_0 bh^2}{6} \qquad P_0 = \frac{\sigma_0 bh^2}{6L} \text{ to yield}$$

$$M_u = \frac{\sigma_0 bh^2}{4} \qquad P_u = \frac{\sigma_0 bh^2}{4L} \text{ to fall (hinge)}$$

consider $P_0 < P < P_u$

$$M(x_1) = M_0$$
$$M(L) = M$$

$x < x_1$, beam is elastic

$$M(x) = -EIv_1'' = -Px$$

$$v_1(x) = \frac{Px^3}{6EI} + Ax + B$$

$x_1 < x \leq L$, beam is elastic-plastic

$$K = \frac{K_0}{\left[3 - \frac{1}{2}\frac{M}{M_0}\right]^{1/2}}$$

$$-v_2'' = K, \quad M = -Px$$
$$M_0 = -Px_1$$

$$v_2'' = \frac{-K_0}{\left[3 - \frac{1}{2}\frac{x}{x_1}\right]^{1/2}}$$

in integrating, $v_2(L) = 0$, $v_2'(L) = 0$

$$v_1(x_1) = v_2(x_1)$$
$$v_1'(x_1) = v_2'(x_1)$$

Get to this in homework — $\frac{v(0)}{v_0(0)} = f\left(\frac{P}{P_0}\right)$

displacement when $P = P_0$ (yield)

$$K = \frac{-v''}{\left[1 + (v')^2\right]^{3/2}}$$

previously simplified by assuming v' was small — no longer true.

ENERGY METHODS

introduction

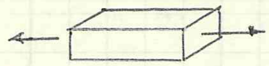
load causes deformation, stores energy

think of an elastic band or a spring

$$U = \frac{1}{2} k x^2$$

Definitions:

Strain Energy – the energy stored in a body due to the deformations induced by external loads.

σ_x  σ_x, causes ε_x

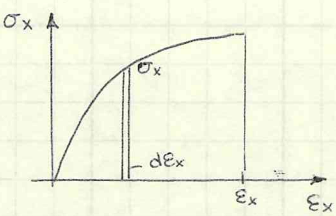
Force $= \sigma_x \, dy \, dz$

change in length $= d\varepsilon_x \, dx$

$$\text{Energy} = \int_0^{\varepsilon_x} \sigma_x \, dy \, dz \, d\varepsilon_x \, dx = U$$

strain energy density, $U_0 = \dfrac{\text{strain energy}}{\text{unit volume}}$

$$U_0 = \int_0^{\varepsilon_x} \sigma_x \, d\varepsilon_x$$



complimentary strain energy

area over the $\sigma_x \cdot \varepsilon_x$ curve

$$U_0^c = \int_0^{\sigma_x} \varepsilon_x \, d\sigma_x$$

mathematical value; has no physical meaning

More generally, for a six-dimensional stress state,

$$U_0 = \int_0^{\varepsilon_x} \sigma_x \, d\varepsilon_x + \int_0^{\varepsilon_y} \sigma_y \, d\varepsilon_y + \int_0^{\varepsilon_z} \sigma_z \, d\varepsilon_z + \int_0^{\gamma_{xy}} \sigma_{xy} \, d\gamma_{xy} + \int_0^{\gamma_{xz}} \sigma_{xz} \, d\gamma_{xz} + \int_0^{\gamma_{yz}} \sigma_{yz} \, d\gamma_{yz}$$

complementary strain energy is similar ($\varepsilon d\sigma \ldots$)

Known increments,

$$dU_0 = \sigma_x \, d\varepsilon_x + \sigma_y \, d\varepsilon_y + \sigma_z \, d\varepsilon_z + \sigma_{xy} \, d\gamma_{xy} + \sigma_{yz} \, d\gamma_{yz} + \sigma_{xz} \, d\gamma_{xz}$$

ENERGY METHODS

Strain energy

If the material is potential then

$$\sigma_x = \frac{\partial U_0}{\partial \varepsilon_x}, \quad \sigma_y = \frac{\partial U_0}{\partial \varepsilon_y}, \quad \ldots$$

$$dU_0 = \frac{\partial U_0}{\partial \varepsilon_x} d\varepsilon_x + \frac{\partial U_0}{\partial \varepsilon_y} d\varepsilon_y + \ldots \frac{\partial U_0}{\partial \gamma_{xz}} d\gamma_{xz}$$

$$U_0^c = \int_0^{\sigma_x} \varepsilon_x \, d\sigma_x + \ldots \int^{\sigma_{xz}} \varepsilon_{xz} \, d\sigma_{xz}$$

er, γ_{xz}

$$dU_0^c = \varepsilon_x \, d\sigma_x + \varepsilon_y \, d\sigma_y \ldots + \gamma_{xz} \, d\sigma_{xz}$$

again, if potential (elastic (?)),

$$\varepsilon_x = \frac{\partial U^c}{\partial \sigma_x}, \quad \varepsilon_y = \frac{\partial U_0^c}{\partial \sigma_y}, \quad \ldots \quad \gamma_{xz} = \frac{\partial U_0^c}{\partial \sigma_{xz}}$$

the total strain energy ~~[illegible]~~ in a body

$$U = \int_V U_0 \, dx\,dy\,dz$$

function of position

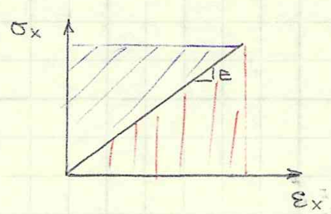
the total complementary strain energy is

$$U_c = \int_V U_0^c(x) \, dx\,dy\,dz$$

Now, what does being potential mean?

- elastic material range
- linear or non-linear

Linearly Elastic Material



$\sigma_x = E\varepsilon_x$

$$U_0 = \int_0^{\varepsilon_x} \sigma_x \, d\varepsilon_x = \int_0^{\varepsilon_x} E\varepsilon_x \, d\varepsilon_x = \frac{1}{2}E\varepsilon_x^2 = \frac{1}{2}\sigma_x\varepsilon_x = \frac{1}{2}\frac{\sigma_x^2}{E}$$

$$U_0^c = \int_0^{\sigma_x} \varepsilon_x \, d\sigma_x = \int_0^{\sigma_x} \frac{\sigma_x}{E} d\sigma_x = \frac{1}{2}\frac{\sigma_x^2}{E} = \frac{1}{2}\sigma_x\varepsilon_x = \frac{1}{2}E\varepsilon_x^2$$

for linearly elastic, $U_0 = U_0^c$

More generally,

$$U = \int_V U_0 \, dV = \int_V \frac{1}{2}\left[\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \sigma_{xy}\gamma_{xy} + \sigma_{yz}\gamma_{yz} + \sigma_{xz}\gamma_{xz}\right] dV$$

$dV = dx\,dy\,dz$

$$= \int_V \frac{1}{2E}\left[(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - 2\nu(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + 2(1+\nu)(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{xz}^2)\right] dV$$

$$= \int_V \frac{E}{2}\left[\frac{\nu}{(1+\nu)(1+2\nu)}(\varepsilon_x + \varepsilon_y + \varepsilon_z)^2 + \frac{1}{1+\nu}(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) + \frac{1}{2(1+\nu)}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{xz}^2)\right] dV$$

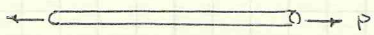
ENERGY METHODS

Linearly elastic material

von Mises – related to energy due to distortion from shear (as opposed to volumetric change)

Structural problems in one-D

Axial rod



length L

elongation δ

area A

stiffness E

$$\sigma = \frac{P(x)}{A}$$

$$\varepsilon_x = \frac{\delta}{L} = \frac{d\delta}{dx}$$

$$U_0 = \int_V \frac{1}{2}\sigma_x \varepsilon_x = \int_0^L \frac{1}{2}\sigma_x \varepsilon_x A\,dx$$

$$= \int_V \frac{1}{2}\frac{P}{A}\frac{d\delta}{dx}\,dV, \quad dV = dA\,dx$$

$$= \int_A \int_0^L \frac{1}{2}\frac{P}{A}\frac{d\delta}{dx}\,dA\,dx$$

$$= \int_0^L \frac{1}{2} P \frac{d\delta}{dx}\cdot dx, \quad \text{but } P = EA\frac{d\delta}{dx}$$

$$= \int_0^L \frac{1}{2} EA\left(\frac{d\delta}{dx}\right)^2 dx$$

$$= \int_0^L \frac{1}{2}\frac{P^2}{EA}\,dx$$

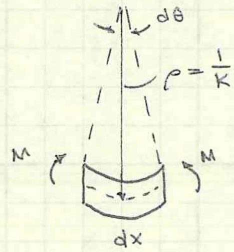
Important to know:

$$\boxed{U = \int_0^L \frac{1}{2} P \frac{d\delta}{dx}\,dx = \int_0^L \frac{1}{2} EA\left(\frac{d\delta}{dx}\right)^2 dx = \int_0^L \frac{1}{2}\frac{P^2}{EA}\,dx}$$

ENERGY METHODS

1D Structural Problems, Cont'd

Beam bending



$$u = \int_0 \frac{1}{2} \sigma_x \varepsilon_x \, dV$$

$\uparrow$ V, rather

$$\varepsilon_x = yK, \quad u = \int_V \frac{1}{2} \sigma_x y K \, dV$$

$$= \int_0^L \int_A \frac{1}{2} \sigma_x y K \, dA \, dx$$

Rearrange to get:

$$u = \int_0^L \frac{K}{2} \underbrace{\int_A \sigma_x y \, dA}_{= M(x) = \int_A \sigma_x y \, dA} \, dx$$

$$u = \int_0^L M(x) K(x) \cdot \frac{1}{2} \, dx$$

Or, alternatively,

$$\sigma_x = EyK, \quad u = \int_V \frac{1}{2} E y K \varepsilon_x \, dV$$

$$= \int_0^L \int_A \frac{1}{2} (yK)^2 \, dA \, dx$$

$$= \int_0^L \frac{EK^2}{2} \underbrace{\int_A y^2 \, dA}_{= I(x), \text{ moment of inertia}} \, dx$$

$$= \int_0^L \frac{1}{2} EI K^2 \, dx, \quad M = EIK$$

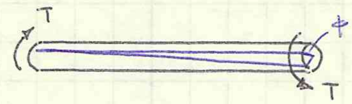
Summary:

$$\boxed{u = \int_0^L \frac{1}{2} M(x) K(x) \, dx = \int_0^L \frac{1}{2} EI(x) K^2(x) \, dx = \int_0^L \frac{M^2(x)}{2EI} \, dx}$$

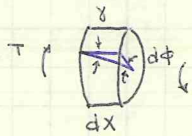
ENERGY METHODS

1D structural problems (cont'd)

Rod under torsion



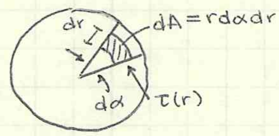
length L, material G
full radius R, variable r



shear stresses control / exist

$$\sigma_{xy} = G\gamma, \quad \gamma = r\frac{d\phi}{dx}$$
$$= Gr\frac{d\phi}{dx}$$

the stress and strain vary linearly through the thickness R



$$U = \int_0^L \int_A \frac{1}{2}\tau\gamma\, dA\, dx$$
$$= \int_0^L \int_A \frac{1}{2} G r \frac{d\phi}{dx}\cdot\frac{d\phi}{dx}\cdot r\, dA\, dx$$
$$= \int_0^L \int_A \frac{1}{2}\tau r \frac{d\phi}{dx}\, dA \cdot dx$$

$$U = \int_0^L \frac{1}{2}\int_A \tau r\cdot r\cdot dr\cdot d\alpha \qquad \text{insert } \frac{d\phi}{dx}dx$$
$$= \int_0^L \frac{1}{2}\int_A^{2\pi}\int_0^R \tau r^2 dr\,\frac{d\phi}{dx}\cdot d\alpha\, dx$$

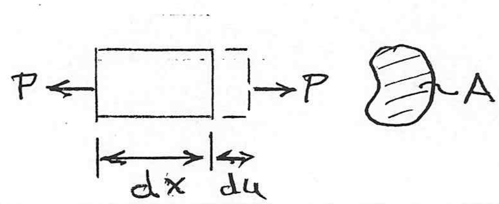
$$U = \int_0^L \frac{1}{2}\frac{d\phi}{dx}\underbrace{\int_0^R 2\pi\tau r^2\, dr}_{\text{equals torque } T}\, dx, \quad U = \int_0^L \frac{1}{2}\frac{d\phi}{dx}\cdot T\, dx$$

$$U = \int_0^L \int_A \frac{1}{2}G\gamma^2\, dA\, dx$$
$$= \int_0^L \frac{G}{2}\int_A r^2\left(\frac{d\phi}{dx}\right)^2 dA\, dx$$
$$= \int_0^L \frac{G}{2}\left(\frac{d\phi}{dx}\right)^2 dx\, \underbrace{2\pi\int_0^R r^3 dr}_{J}$$

$$\boxed{U = \int_0^L \frac{1}{2}T\frac{d\phi}{dx}\cdot dx = \int_0^L \frac{1}{2}GJ\left(\frac{d\phi}{dx}\right)^2 dx = \int_0^L \frac{T^2}{2GJ}\, dx}$$

Axially Loaded Bars

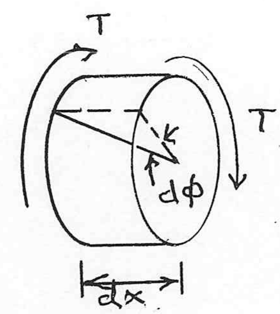
$$\frac{P}{A} = \sigma = E\frac{\delta}{L}$$



$$U = \int_0^L \frac{1}{2} P \frac{du}{dx} dx = \int_0^L \frac{1}{2} EA\left(\frac{du}{dx}\right)^2 dx = \int_0^L \frac{1}{2} \frac{P^2}{EA} dx$$

Torsion of Circular Bars

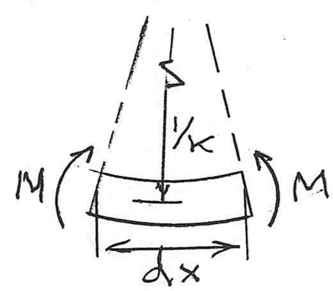
$$\frac{T}{I_p} = \frac{\tau}{r} = G\frac{\phi}{L}$$



$$U = \int_0^L \frac{1}{2} T \frac{d\phi}{dx} dx = \int_0^L \frac{1}{2} GJ\left(\frac{d\phi}{dx}\right)^2 dx = \int_0^L \frac{1}{2} \frac{T^2}{GJ} dx$$

Bending of Beams

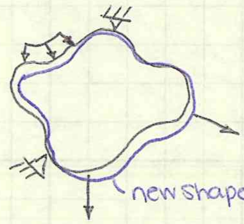
$$\frac{M}{I} = \frac{\sigma}{y} = E\kappa$$



$$U = \int_0^L \frac{1}{2} M\kappa dx = \int_0^L \frac{1}{2} EI\kappa^2 dx = \int_0^L \frac{1}{2} \frac{M^2}{EI} dx$$

ENERGY METHODS

The Principle of virtual work



body in equilibrium, loaded by external forces

the body deforms (displacements $\underline{u}(x)$), and develops internal stresses $\underline{\sigma}(x)$.

Now apply a new set of forces that produce a kinematically admissible displacement ~~xxx~~ field $\delta\underline{u}$ ($\delta\underline{u}$ satisfies all displacement boundary conditions). we will call $\delta\underline{u}$ a "virtual" displacement because it is an assumed process.

Because of $\delta\underline{u}$, the original external forces move their point of application and consequently do work, given by:

$$\underbrace{\int_{S_T}}_{\text{boundary}} \underbrace{\underline{T}}_{\substack{\text{traction vector} \\ F/in^2}} \cdot \delta\underline{u}\, dA + \int_V \underbrace{\underline{F}}_{\substack{\text{body forces} \\ (F/in^3)}} \cdot \delta\underline{u}\, dV = \delta W_{ext}$$

At the same time, $\delta\underline{u}(x)$ causes internal strains and these result in internal work by the original internal stress field. the change in internal work is:

$$\delta U = \int_V \underline{\sigma} \cdot \delta\underline{\varepsilon}\, dV$$

The principle of virtual work states that the work done by the external forces equals the change in strain energy:

$$\delta W_{ext} = \delta U$$

(True for all materials)
small and large deformations, too

[Sidebar: a principle cannot be proved, unlike a theorem. we just accept it as truth]

ENERGY METHODS

Castigliano's Theorems

First Theorem

- consider a body loaded by point loads P_i. The body is in equilibrium and the resultant displacements are Δ_i.
- Apply virtual displacements $\delta\Delta_i$. They are arbitrary, but satisfy the displacement boundary conditions.
- From the P.V.W., $\delta U = \delta W_{ext}$

$$\delta W_{ext} = \sum_{i=1}^{n} P_i \delta\Delta_i$$

summation rather than integral as integral is not needed

$$\delta U = \delta U(\delta\Delta_1, \delta\Delta_2, \delta\Delta_3 \ldots \delta\Delta_N)$$

conservative system, or material is potential (elastic range)

$$= \frac{\partial U}{\partial \Delta_1}\delta\Delta_1 + \frac{\partial U}{\partial \Delta_2}\delta\Delta_2 \ldots + \frac{\partial U}{\partial \Delta_N}\delta\Delta_N$$

Because $\delta\Delta_i$ are arbitrary, anything can be used (as long as the BCs are met). We can choose to apply only $\delta\Delta_I$.

Therefore, $\delta U = \frac{\partial U}{\partial \Delta_I}\delta\Delta_I$

$$\delta W = P_I\,\delta\Delta_I = \frac{\partial U}{\partial \Delta_I}\delta\Delta_I = \delta U$$

as long as $\delta\Delta_I \neq 0$ (trivial),

$$\boxed{P_I = \frac{\partial U}{\partial \Delta_I}}$$

This can be done for all i, thus

$$P_i = \frac{\partial U}{\partial \Delta_i}$$

Thus, if you can write the potential energy of the body in terms of $\Delta_1, \Delta_2 \ldots \Delta_N$, the force can be found at that point.

Note that the forces P_i and the displacements Δ_i should be interpreted as generalized forces and displacements ($F \rightarrow M \rightarrow T$, etc.)

$$\frac{\partial U}{\partial \phi_i} = T_i, \quad \frac{\partial U}{\partial \theta_i} = M_i$$

ENERGY METHODS

Principle of complementary virtual work

consider a body in equilibrium.

Disturb the body by applying a statically admissible stress field $\delta \underline{\sigma}$, which satisfies equilibrium and stress boundary conditions

The change in complementary external work equals the change in complementary internal energy (strain energy).

$$\delta U^c = \int_V \underline{\varepsilon} \cdot \delta \underline{\sigma} \, dV$$

$$\delta W^c = \int_{S_{\underline{u}}} \underline{u} \cdot \delta \underline{T} \, dA$$

remember — mathematical entities, not real

$$\delta U^c = \delta W^c, \quad \int_V \underline{\varepsilon} \cdot \delta \sigma \, dV = \int_{S_{\underline{u}}} \underline{u} \cdot \delta \underline{T} \, dA$$

Castigliano's Theorems

Second Theorem

— consider a body in equilibrium under the action of point forces P_i and corresponding displacements Δ_i

— Disturb the structure by applying δP_i

— The change in complementary strain energy is

$$\delta U^c = \delta U^c(\delta P_1, \delta P_2 \ldots \delta P_N)$$

— The change in complementary virtual work is

$$\delta W^c = \sum_{i=1}^{N} \Delta_i \delta P_i$$

— Finally,

$$\delta U^c = \delta W^c, \quad \delta U^c(\delta P_1, \delta P_2 \ldots \delta P_N) = \sum_{i=1}^{N} \Delta_i \delta P_i$$

— since δP_i are ~~arbitrar~~ arbitrary,

$$\frac{\partial U^c}{\partial P_I} \delta P_I = \Delta_I \delta P_I \quad \text{or} \quad \boxed{\frac{\partial U^c}{\partial P_I} = \Delta_I} \quad \text{for all } \delta P_i$$

or, $\dfrac{\partial U^c}{\partial M_I} = \theta_I, \quad \dfrac{\partial U^c}{\partial T_I} = \phi_I$

ENERGY METHODS

Castigliano's Theorems

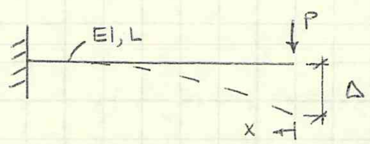
Note: if the material is linearly elastic, $U^c = U$

$$\therefore \frac{\partial U}{\partial P_i} = \Delta_i$$

Two theorems:

$$\frac{\partial U}{\partial \Delta_i} = P_i \,, \quad \frac{\partial U^c}{\partial P_i} = \Delta_i$$

Examples



Find Δ.

linearly elastic, $U^c = U$: need to write the strain energy U in terms of P

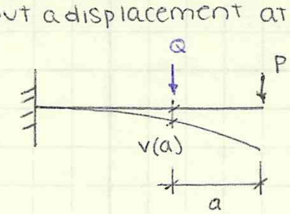
$$M(x) = -Px$$

$$U = \int_0^L \frac{M^2(x)}{2EI}\,dx = \int_0^L \frac{P^2 x^2}{2EI}\,dx$$

$$\frac{\partial U}{\partial P} = \int_0^L \frac{M(x)}{EI}\frac{\partial M}{\partial P}\,dx \,, \quad \frac{\partial M}{\partial P} = -x$$

$$\Delta = \int_0^L \frac{(-Px)(-x)}{EI}\,dx = \frac{P}{EI}\frac{x^3}{3}\Big|_0^L = \underline{\frac{PL^3}{3EI}}$$ expected result

What about a displacement at a position i?



what is $v(a)$? No force applied there... add a dummy force Q at $x = a$

$$M(x) = -Px - Q\langle x-a \rangle$$ (remember half-range functions)

$$U = \int_0^L \frac{M^2(x)}{2EI}\,dx \,, \quad \frac{dU}{dQ} = \int_0^L \frac{M(x)}{EI}\cdot\frac{\partial M}{\partial Q}\,dx$$

$$\frac{\partial M}{\partial Q} = -\langle x-a \rangle$$

$$v(a) = \frac{dU}{dQ} = \int_0^L \frac{Px + Q\langle x-a \rangle}{EI}\langle x-a \rangle\,dx$$

Now, we can drop Q ($Q = 0$) to simplify algebra

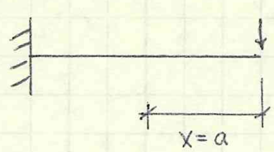
$$= \int_0^L \frac{Px}{EI}\langle x-a \rangle\,dx$$, which equals 0 until $x = a$

$$= \int_a^L \frac{Px}{EI}(x-a)\,dx$$

ENERGY METHODS

Examples (cont'd)

Dummy force method



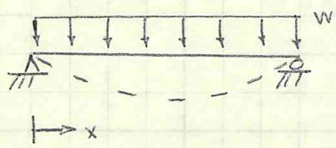
$$v(a) = \int_a^L \frac{P}{EI}x^2 - \frac{P}{EI}xa\,dx$$

$$= \frac{P}{EI}\left[\frac{x^3}{3} - \frac{x^2}{2}\cdot a \Big|_a^L\right]$$

$$= \frac{P}{EI}\left[\frac{L^3}{3} - \frac{L^2 a}{2} - \frac{a^3}{3} + \frac{a^3}{2}\right]$$

$$= \frac{PL^3}{EI}\left[\frac{1}{3} - \frac{1}{2}\left(\frac{a}{L}\right) + \frac{1}{6}\left(\frac{a}{L}\right)^3\right]$$

New problem



EI, L

Find the rotation θ at supports

$$\theta_i = \frac{\partial u}{\partial M_i}$$

$$M(x) = \frac{wL}{2}x - \frac{wx^2}{2} + M_0 \quad + \frac{M_0}{L}x$$

M_0 = dummy moment at $x = 0$

additional term in support reaction

$$u = \int_0^L \frac{M^2(x)}{2EI}dx, \quad \frac{\partial u}{\partial M} = \int_0^L \frac{M(x)}{EI}dx$$

$$\theta = \int_0^L \frac{M(x)}{EI}\frac{\partial M}{\partial M_0}dx$$

$$\frac{\partial M}{\partial M_0} = 1 - \frac{x}{L}$$

dummy forces added just to get this multiplier

$$= \int_0^L \frac{1}{EI}\left[\frac{wL}{2}x - \frac{wx^2}{2} - \frac{M_0 x}{L} + M_0\right]\left(1 - \frac{x}{L}\right)dx$$

Now, $M_0 = 0$

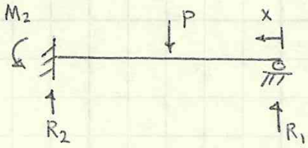
$$\theta = \frac{1}{EI}\int_0^L \left[\frac{wL}{2}x - \frac{wx^2}{2}\right]\left(1 - \frac{x}{L}\right)dx$$

$$= \frac{1}{EI}\left[\frac{wL}{4}x^2 - \frac{wx^3}{6} - \frac{w}{6}x^3 + \frac{wx^4}{8L}\Big|_0^L\right] = \frac{wL^3}{EI}\left[\frac{1}{4} - \frac{1}{6} - \frac{1}{6} + \frac{1}{8}\right]$$

$$\theta = \frac{wL^3}{24EI}$$

ENERGY METHODS

Examples (cont'd)



$$R_1 + R_2 = P$$
$$R_1 L + M_2 = \frac{PL}{2}$$

statically indeterminate

$$M(x) = R_1 x - P\langle x - L/2\rangle$$

$$U = \int_0^L \frac{M^2(x)}{2EI}\,dx$$

declare R_1 to be the redundant

at $x=0$, $\Delta = 0$

$$\therefore \frac{\partial U}{\partial R_1} = 0$$

$$\frac{\partial U}{\partial R_1} = \int_0^L \frac{M(x)}{EI}\cdot\frac{\partial M}{\partial R_1}\,dx = 0$$

$$\frac{\partial M}{\partial R_1} = x$$

$$\int_0^L \frac{1}{EI}\left[R_1 x - P\langle x - L/2\rangle\right] x\,dx = 0$$

$$= \int_0^L \frac{R_1 x^2}{EI} - \int_{L/2}^L \frac{P}{EI}\left(x^2 - Lx/2\right)dx$$

$$= \frac{1}{EI}\left[\frac{R_1}{3}x^3\Big|_0^L - P\left(\frac{x^3}{3} - \frac{Lx^2}{4}\right)\Big|_{L/2}^L\right] = 0$$

$$= \frac{1}{EI}\left[\frac{R_1 L^3}{3} - P\left(\frac{L^3}{3} - \frac{L^3}{4} - \frac{L^3}{24} + \frac{L^3}{16}\right)\right] = 0$$

$$R_1 = 3P\left(1/3 - 1/4 - 1/24 + 1/16\right) = P\frac{5}{16}$$

$$R_2 = \frac{11}{16}P$$

$$M_2 = \frac{3PL}{16}$$

ENERGY METHODS

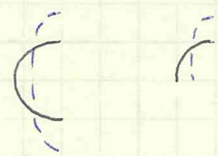
Example

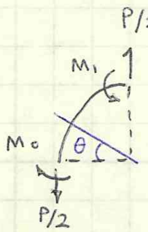
exaggerated deformed shape

thin ring of radius R, EI, etc. being pulled — consider a link on a chain

what happens? deflections?

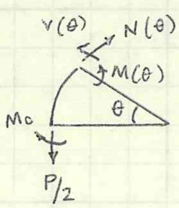
using equilibrium, the problem is very difficult.
using energy method, it is much easier
plus, consider symmetry





$$M_1 = M_0 - \frac{P}{2}R$$

no shear, as shear is antisymmetric



treat M_0 as the redundant

$$\Sigma M_{cut}: \quad M(\theta) = M_0 - \frac{P}{2}(R - R\cos\theta)$$

$$U = \int_0^{\pi/2} \frac{M^2(\theta)}{2EI} R \cdot d\theta$$

neglect energy due to the axial force

$$\frac{\partial U}{\partial M_0} = \int_0^{\pi/2} \frac{M(\theta)}{EI} \cdot \frac{\partial M(\theta)}{\partial M_0} R\,d\theta = \Theta_0 = 0$$

$$\frac{\partial M(\theta)}{\partial M_0} = 1$$

$$0 = \int_0^{\pi/2} \frac{M_0 - P/2\,(R - R\cos\theta)}{EI} R\,d\theta$$

$$M_0\theta - \frac{PR}{2}\theta + \frac{PR}{2}\sin\theta \Big|_0^{\pi/2} = 0$$

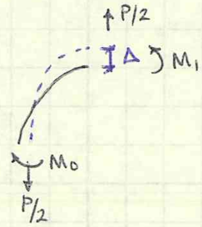
$$M_0 = \frac{PR}{2}(1 - 2/\pi)$$

$$\underline{\underline{M(\theta) = \frac{PR}{2}(\cos\theta - 2/\pi)}}$$

ENERGY METHODS

Example (cont'd)

Now, M_0 is known. Find Δ



$$M_0 = \frac{PR}{2}\left(1 - {}^{2}/_{\pi}\right)$$

$$M_1 = \frac{-PR}{\pi}$$

effect from each side

$$\Delta = \frac{\partial u}{\partial P} = 2\int_0^{\pi/2} \frac{M}{EI}\frac{\partial M}{\partial P} R\,d\theta$$

$$\frac{\partial M}{\partial P} = \frac{R}{2}\left(\cos\theta - {}^{2}/_{\pi}\right)$$

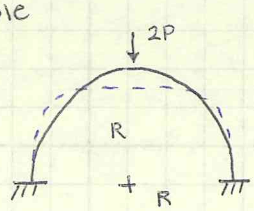
$$\Delta = \frac{2}{EI}\int_0^{\pi/2} \frac{PR}{2}\left(\cos\theta - {}^{2}/_{\pi}\right)\frac{R}{2}\left(\cos\theta - {}^{2}/_{\pi}\right)R\,d\theta$$

$$= \frac{PR^3}{2EI}\int_0^{\pi/2} \cos^2\theta - {}^{4}/_{\pi}\cos\theta + {}^{4}/_{\pi^2}\,d\theta$$

$$= \frac{1}{2}\frac{PR^3}{EI}\left[{}^{\pi}/_{4} - {}^{2}/_{\pi}\right]$$

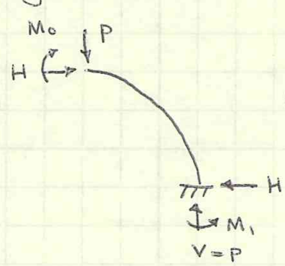
- no kinematics
- no consideration for curve (R instead of x, $R\,d\theta$ not dx)
- use symmetry
- find redundant

Example

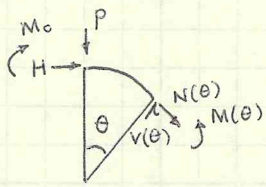


Find the deflection

use symmetry:



$M_1 = M_0 + HR - PR$
- 3 unknowns
- 2 redundants



$$M(\theta) = M_0 + H(R - R\cos\theta) - PR\sin\theta$$

M_0, H are redundants

$\theta_0 = 0$, $H_0 = 0$

$$u = \int_0^{\pi/2} \frac{M^2(\theta)}{EI\cdot 2} R\,d\theta, \quad \frac{\partial u}{\partial H} = 0 = \int_0^{\pi/2} 2\,\frac{M(\theta)}{EI}\frac{\partial M}{\partial H} R\,d\theta$$

$$\frac{\partial u}{\partial M_0} = 0 = 2\int_0^{\pi/2} \frac{M(\theta)}{EI}\frac{\partial M}{\partial M_0} R\,d\theta$$

ENERGY METHODS

Example (cont'd)

$$M(\theta) = M_0 + HR(1-\cos\theta) + PR\sin\theta$$

$$\frac{\partial M}{\partial H} = R(1-\cos\theta) \qquad \int_0^{\pi/2} (M_0 + HR(1-\cos\theta) + PR\sin\theta)(R - R\cos\theta)\,R\,d\theta = 0$$

$$\frac{\partial M}{\partial M_0} = 1$$

Skipping the integrations,

$$H = -2P\frac{(\pi-4)}{\pi^2-8}$$

$$\frac{M_0}{R} = \frac{4P(\pi-3)}{\pi^2-8}$$

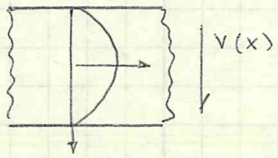
$$M(\theta) = \frac{PR}{\pi^2-8}\left[4(\pi-3) + 2(4-\pi)(1-\cos\theta)\right] + PR\sin\theta$$

$$\delta = \frac{\partial U}{\partial P} = \frac{1}{EI}\int_0^{\pi/2} M(\theta)\frac{\partial M}{\partial P}\,R\,d\theta$$

ENERGY METHODS

Shear deformation

strain energy due to shear in ~~beam~~ bending



$$\tau = \frac{VQ}{Ib}, \quad Q = \int_{y_1}^{h/2} y\,dA$$

$$\gamma = \frac{VQ}{GIb}$$

$$U = \int_V \frac{1}{2}\tau\gamma\,dV$$

$$= \int_0^L \int_A \frac{1}{2}\left(\frac{VQ}{Ib}\right)\left(\frac{VQ}{GIb}\right) dA\,dx$$

$$= \int_0^L \frac{V^2(x)}{2GA}\,dx \underbrace{\frac{A}{I^2}\int_A \frac{Q^2}{b^2}\,dA}_{\beta}$$

β: the form shear factor – constant that depends on the cross section

$$\boxed{= \int_0^L \beta \frac{V^2(x)}{2GA}\,dx}, \quad V = \alpha G A \gamma = \alpha G A \frac{dv_s}{dx}$$

Now,

$$\boxed{\beta = \frac{A}{I^2}\int_A \frac{Q^2}{b^2}\,dA}$$

example:

$A = bh, \quad I = \frac{1}{12}bh^3$

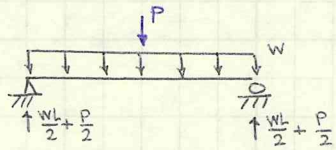
$$Q = \int_{y_1}^{h/2} y\,dA = \frac{b}{2}\left[\frac{h^2}{4} - y_1^2\right]$$

$$\beta = \frac{bh \times 144}{(bh^3)^2}\int_{-\frac{h}{2}}^{h/2} \frac{1}{b^2}\cdot\frac{b^2}{4}\left[\frac{h^2}{4} - y_1^2\right]^2 b\,dy$$

$= \frac{6}{5}$ – considered more correct (than $\alpha = 2/3$?)

ENERGY METHODS

Example



Find maximum deflection including the effects of shear

add P at $x = L/2$ to find deflections at that point

$$U = 2\int_0^{L/2} \frac{M^2(x)}{2EI}\,dx + 2\int_0^{L/2} \beta \frac{V^2(x)}{2GA}\,dx$$

$$\left.\begin{aligned} M(x) &= \frac{wL}{2}x + \frac{P}{2}x - \frac{wx^2}{2} \\ V(x) &= \frac{wL}{2} + \frac{P}{2} - wx \end{aligned}\right\} \text{up to } x = L/2$$

$$v(L/2) = \left.\frac{\partial U}{\partial P}\right|_{P=0} = \left[\int_0^{L/2} \frac{M(x)}{EI}\frac{\partial M}{\partial P}\,dx + \int_0^{L/2} \beta \frac{V(x)}{GA}\frac{\partial V}{\partial P}\,dx\right](2)$$

$$\frac{\partial M}{\partial P} = \frac{x}{2}, \quad \frac{\partial V}{\partial P} = \frac{1}{2}$$

$$= \int_0^{L/2} \frac{2}{EI}\left[\frac{wL}{2}x + \frac{P}{2}x - \frac{wx^2}{2}\right]\frac{x}{2}\,dx + 2\int_0^{L/2} \beta \frac{\frac{wL}{2} + \frac{P}{2} - wx}{GA}\cdot\frac{1}{2}\,dx$$

$$= \frac{2}{EI}\left[\frac{wL}{12}x^3 + \cancel{\frac{P}{12}x^3} - \frac{w}{16}x^4\right]\Bigg|_0^{L/2} + \frac{2\beta}{2GA}\left[\frac{wL}{2}x + \cancel{\frac{Px}{2}} - \frac{w}{2}x^2\right]\Bigg|_0^{L/2}$$

Now, $P = 0$ (and I read correctly)

$$v_{max} = \frac{5}{384}\frac{wL^4}{EI} + \frac{\beta L^2}{8GA}$$

$$\text{or, } v_{max} = \frac{5}{384}\frac{wL^4}{EI}\left[1 + \underbrace{\frac{48}{5}\beta\frac{EI}{L^2GA}}_{\text{increase in } \Delta \text{ due to shear}}\right]$$

from equilibrium, we had

$$\frac{48}{5}\frac{1}{\alpha}\frac{EI}{L^2GA}$$

$\beta \sim \frac{1}{\alpha}$ both are approximations

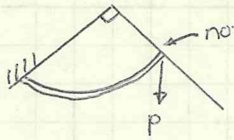
β is better!

for a rectangle,

$$\frac{1}{\alpha} = \frac{3}{2}, \quad \beta = \frac{6}{5}$$

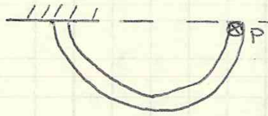
ENERGY METHODS

Example / HW



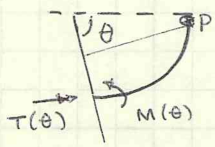
not connected, but showing 90° of beam

combined bending and torsion



find deflection down at P

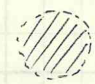
look at a section



$M(\theta) = -PR\sin\theta$

$T(\theta) = RP(1-\cos\theta)$

beam x-section



$$U = \int_0^{\pi} \frac{M^2(\theta)}{2EI} R\,d\theta + \int_0^{\pi} \frac{T^2(\theta)}{2JG} R\,d\theta$$

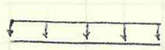
$$\Delta = \frac{\partial U}{\partial P}, \quad \frac{\partial M}{\partial P} = -R\sin\theta, \quad \frac{\partial T}{\partial P} = R(1-\cos\theta)$$

$$\Delta = \int_0^{\pi} \frac{PR\sin\theta}{EI} R^2 \sin\theta\,d\theta + \int_0^{\pi} \frac{PR(1-\cos\theta)}{JG} R^2(1-\cos\theta)\,d\theta$$

$$\Delta = \frac{PR^3}{EI}\int_0^{\pi} \sin^2\theta\,d\theta + \frac{PR^3}{GJ}\int_0^{\pi}(1-\cos\theta)^2\,d\theta$$

$$= \frac{PR^3}{EI}\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi} + \frac{PR^3}{GJ}\left[\theta - 2\sin\theta + \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi}$$

$$= \frac{\pi}{2}\frac{PR^3}{EI} + \frac{3\pi}{2}\frac{PR^3}{GJ}$$



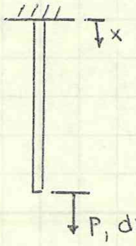
$dM = -(x-\xi)w\,d\xi$

$$M = -\int_0^x (x-\xi)w\,d\xi = -w\left[x\xi - \xi^2/2\right]_0^x = \frac{-wx^2}{2}$$

need to use this in the HW problem

ENERGY METHODS

Example

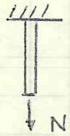


Heavy bar of density ρ

Find the tip deflection

$$\varepsilon = c\sigma^n$$

P, dummy force



$N(x) = \rho g A(L-x) + P$

$\frac{\partial U^c}{\partial P} = \Delta$, but non-linear stress-strain relationship means $\underline{\partial U} \neq \partial U^c$

$$U_0^c = \int_0^\sigma \varepsilon\, d\sigma = \int_0^\sigma c\sigma^n\, d\sigma = c\frac{\sigma^{n+1}}{n+1}$$

$$U^c = \int_0^L \int_A U_0^c\, dA\, dx = \int_0^L c\frac{\sigma^{n+1}}{n+1} A\, dx \quad , \quad \sigma(x) = \frac{N(x)}{A}$$

$$= \int_0^L Ac\frac{N(x)^{n+1}}{A^{n+1}} \cdot \frac{1}{n+1}\, dx$$

$$= \int_0^L \frac{cA}{n+1}\left(\frac{N(x)}{A}\right)^{n+1} dx$$

$$= \frac{c}{(n+1)A^n} \int_0^L \left[P + \rho g A(L-x)\right]^{n+1} dx$$

derivative loses +1

$$\Delta = \frac{\partial U^c}{\partial P} = \frac{c}{A^n}\int_0^L \left[P + \rho g A(L-x)\right]^n dx \cdot \frac{\partial N}{\partial P} = 1$$

$$= \frac{c}{A^n}\left(\frac{-1}{\rho g A}\right)\left[\frac{P + \rho g A(L-x)}{n+1}\right]^{n+1}\Bigg|_0^L$$

$$\underline{\Delta = \frac{-c}{A^n(n+1)\rho g A}\left[P^{n+1} - (P + \rho g A L)^{n+1}\right]}$$

Remember, $P = 0$

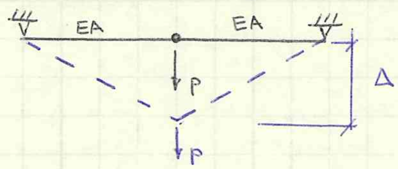
Now, check:

$n=1$, $\varepsilon = c\sigma$, $\Delta = \frac{c\rho g L^2}{2}$

$n=2$, $\varepsilon = c\sigma^2$, $\Delta = \cancel{\frac{cL^3}{3A^2}} = \frac{c(\rho g)^2 L^3}{3}$

ENERGY METHODS

Example



deformed length

$$L' = \left[L^2 + \Delta^2\right]^{1/2}$$

strain

$$\varepsilon = \frac{\left[L^2+\Delta^2\right]^{1/2} - L^2}{L^2} = \left[1 + (\Delta/L)^2\right]^{1/2} - 1$$

if $\Delta/L \ll 1$, approximate:

$$\varepsilon = 1 + \tfrac{1}{2}\left(\tfrac{\Delta}{L}\right)^2 - \tfrac{1}{2}\cdot\tfrac{-1}{2}\cdot\tfrac{1}{2}\left(\tfrac{\Delta}{L}\right)^4 - 1$$

$$= \tfrac{1}{2}\left(\tfrac{\Delta}{L}\right)^2 + \tfrac{-1}{8}\left(\tfrac{\Delta}{L}\right)^4$$

two bars

$$U = 2\left[\int_0^L \int_A \tfrac{1}{2}\sigma\varepsilon\, dx\, dA\right]$$

$$= \int_0^L \underbrace{\int_A \sigma\, dA}_{\text{axial force, constant in this problem}}\, \varepsilon\, dx$$

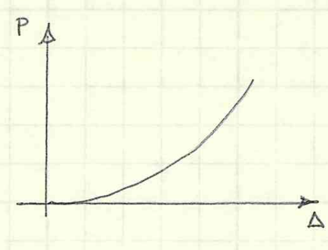
$$= N\int_0^L \varepsilon\, dx = \frac{N}{2}\int_0^L \left(\tfrac{\Delta}{L}\right)^2 - \tfrac{1}{4}\left(\tfrac{\Delta}{L}\right)^4 dx$$

$$= N\varepsilon L, \quad N = EA\varepsilon$$

$$U = EAL\left[\tfrac{1}{2}(\Delta/L)^2\right]^2 \quad \text{drop second term}$$

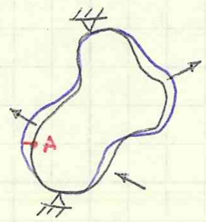
$$U = \tfrac{1}{4}EAL\left(\tfrac{\Delta}{L}\right)^4$$

$$\underline{P = \frac{\partial U}{\partial \Delta} = EAL\left(\tfrac{\Delta}{L}\right)^3}$$



ENERGY METHODS

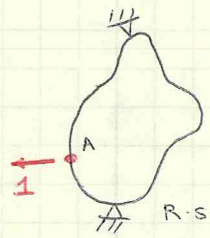
Unit Load Method



V·system

What is the displacement at point A?

Consider a body loaded by a set of loads $\{M_V, N_V, T_V, V_V\}$. We seek the displacement at point A in a particular direction. Set up the same body and boundaries in a second configuration. Add a unit force at A (moment, torque, shear...) in the direction of the required displacement.



R·system

The point force produces internal loads $\{M_R, N_R, T_R, V_R\}$. Now, take the deformation in the V·system and apply it as a virtual displacement to the R·system. In this process, the unit force does the following work:

$$\delta W = 1 \times \Delta$$

The change in internal energy is given by

$$\delta U = \int \underbrace{M_R}_{\text{existing force}} \underbrace{\delta\theta}_{\text{disturbance}} + \int N_R \delta u + \int T_R \delta\phi + \int V_R \delta\Delta^{\lambda}$$

From the P.V.W.,

$$\delta U = \delta W, \quad \Delta = \int M_R \delta\theta + \int N_R \delta u + \int T_R \delta\phi + \int V_R \delta\Delta^{\lambda}$$

If we limit our attention to linear structural components, then

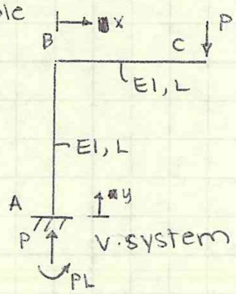
$$\delta\theta = \frac{M_V}{EI} dx, \quad \delta\phi = \frac{T_V}{JG} dx, \quad \delta u = \frac{N_V}{EA} dx, \quad \delta\lambda = \beta \frac{V_V}{GA} dx$$

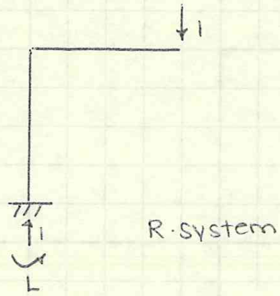
and

$$\boxed{\Delta = \int \frac{M_R M_V}{EI} dx + \int \frac{N_R N_V}{EA} dx + \int \frac{T_V T_R}{JG} dx + \int \beta \frac{V_R V_V}{GA} dx}$$

ENERGY METHODS

Example





Find Δ_{cy}

$$M_v(x) = Px - PL \qquad M_R(x) = x - L$$

$$M_v(y) = -PL \qquad M_R(y) = -L$$

$$EI\Delta = \int_0^L (-PL)(-L)\,dy + \int_0^L (Px - PL)(x - L)\,dx$$

$$= PL^3 + \int_0^L (Px^2 - 2PLx + PL^2)\,dx$$

$$= PL^3 + \frac{P}{3}L^3 - PL^3 + PL^3 = \frac{4}{3}PL^3$$

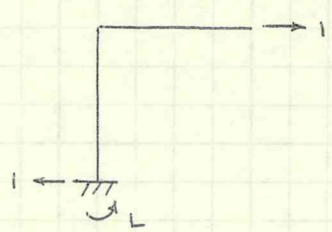
$$\Delta_y = \frac{4PL^3}{3EI}$$

aka, virtual work method

Same as Castigliano's – $\frac{\partial M}{\partial P} = M_R$, in this case

$$\frac{\partial U}{\partial P} = \Delta = \int \frac{M}{EI}\frac{\partial M}{\partial P}\,dx$$

Now, horizontal deflection



$$M_R(y) = -L + \overset{y}{x}$$

$$M_R(x) = L - L = 0$$

$$EI\Delta = \int_0^L (-PL)(\overset{y}{x} - L)\,dy$$

$$= \int_0^L -PLx + PL^2\,dy$$

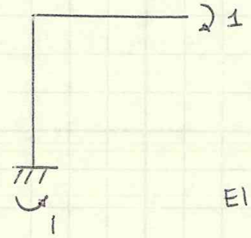
$$= \cancel{\ } -\frac{1}{2}PL^3 + PL^3 = \frac{1}{2}PL^3$$

$$\Delta_x = \frac{PL^3}{2EI}$$

ENERGY METHODS

Example (cont'd)

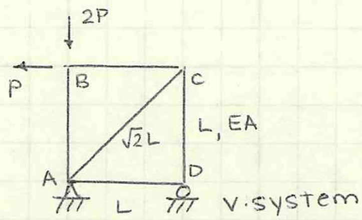
Calculate θ_c:

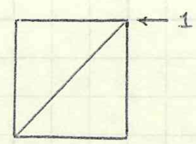


$M_R(y) = -M$
$M_R(x) = -M$ 1

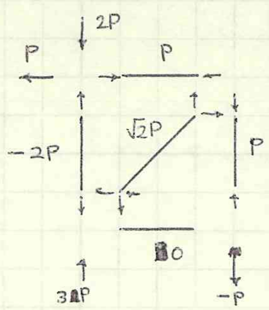
$$EI\theta = -\int_0^L Px - PL\,dx + \int_0^L (-1)(-PL)\,dy$$
$$= PL^2 - \frac{1}{2}PL^2 + PL^2 = \frac{3}{2}PL^2$$
$$\underline{\underline{\theta = \frac{3PL^2}{2EI}}}$$

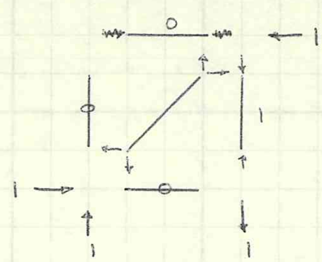
Consider a truss





Find Δ_{xC}





	N_V	N_R	L	$N_V N_R L$
AB	2P	0	L	0
BC	-P	0	L	0
CD	-P	-1	L	PL
DA	0	0	L	0
AC	$\sqrt{2}P$	$\sqrt{2}$	$\sqrt{2}L$	$2P\sqrt{2}L$

$$\boxed{\Delta = \sum \frac{N_V N_R L}{EA}}$$

$$\Delta = PL + 2\sqrt{2}\,PL$$

I think my (+)/(-) are incorrect.

$$\Delta = \int_0^L \frac{N_V N_R}{EA}\,dx$$ for each member, general form

however, it works out through consistency.

85a

THE RAYLEIGH-RITZ METHOD

In many structural mechanics problems the solution of the equilibrium equations can be cumbersome. The R-R method is a direct energy method that can simplify the solution procedure and yield an approximate solution relatively easily.

We approximate the displacements as follows:

$$u(x) = \sum_{i=1}^{L} a_i \phi_i(x),$$

$$v(x) = \sum_{j=1}^{M} b_j \psi_j(x), \qquad (1)$$

$$w(x) = \sum_{k=1}^{N} c_k \chi_k(x)$$

where

ϕ_i , ψ_j and χ_k are kinematically admissible displacement functions of our choice and a_i , b_j and c_k are unknown coefficients.

The minimum potential energy theorem requires that for the exact problem to be in equilibrium

$$\delta V = \delta U - \delta W = 0. \qquad (2)$$

Motivated by (2) let us choose the unknown coefficients $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ such that the approximate value of V is also minimized. That is require that

$$\begin{aligned}\delta V = 0 \approx & \frac{\partial V}{\partial a_1}\delta a_1 + \frac{\partial V}{\partial a_2}\delta a_2 + \ldots + \frac{\partial V}{\partial a_L}\delta a_L \\ & + \frac{\partial V}{\partial b_1}\delta b_1 + \frac{\partial V}{\partial b_2}\delta b_2 + \ldots + \frac{\partial V}{\partial b_M}\delta b_M \\ & + \frac{\partial V}{\partial c_1}\delta c_1 + \frac{\partial V}{\partial c_2}\delta c_2 + \ldots + \frac{\partial V}{\partial c_N}\delta c_N \\ & = 0\end{aligned} \qquad (3)$$

Since $\delta a_i \quad (i = 1, L)$ $\delta b_j \quad (j = 1, M)$ and $\delta c_k \quad (k = 1, N)$ are arbitrary

ENERGY METHODS

Potential Energy

The principle of virtual work states that $\delta U = \delta W$

$$\delta U = \int_V \delta U_0 \, dV = \underbrace{\int_V \frac{\partial U_0}{\partial \varepsilon} \cdot \delta \varepsilon \, dV}_{\text{if the material is potential}}$$

$$\delta W = \int_S \underline{T} \cdot \delta \underline{u} \, dS + \int_V \underline{f} \cdot \delta \underline{u} \, dV$$

if $\underline{T}$ and $\underline{f}$ are conservative,

$$\underline{T} = \frac{\partial g}{\partial \underline{u}}, \quad \underline{f} = \frac{\partial g}{\partial \underline{u}} \leftarrow a\text{, rather}$$

(g and a are potential)

then

$$\delta W = \int_S \frac{\partial g}{\partial \underline{u}} \cdot \delta \underline{u} \, dS + \int_V \frac{\partial a}{\partial \underline{u}} \cdot \delta \underline{u} \, dV$$

$$= \delta \left\{ \int_S g \, dS + \int_V a \, dV \right\}$$

also, $\delta U - \delta W = 0$

$$\delta\{U - W\} = \delta \left\{ \int_V U_0 \, dV - \int_S \underline{T}\,\underline{u} \, dS - \int_V \underline{f} \cdot \underline{u} \, dV \right\} = 0$$

we define $V = U - W$ as the potential energy

and equilibrium requires that $\delta V = \delta(U \overset{(-)}{+} W) = 0$

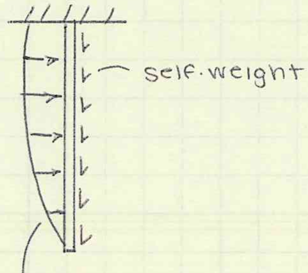
If we make assumptions about the material and forces, we'll return to the principle of virtual work.

For beams:

$$\boxed{V = \int_0^L \frac{EI}{2} \left(\frac{d^2 v}{dx^2} \right)^2 dx + \underbrace{\int_0^L \frac{K}{2} (v)^2 dx}_{\text{elastic foundation}} + \int_0^L \frac{N}{2} \left(\frac{dv}{dx} \right)^2 dx - \underbrace{\int_0^L q(x) \, v \, dx}_{\text{negative because of work}}}$$

ENERGY METHODS

Example



$q(x) = q_0\left[1 - \left(\frac{x}{L}\right)^2\right]$

$$V = \int_0^L \frac{EI}{2}(v'')^2 dx + \int_0^L N(x)\frac{1}{2}(v')^2 dx + \cancel{\int_0^L \frac{N^2(x)}{2EA} dx} - \int_0^L q(x)\, v\, dx$$

neglect; axial stretch is small

$N(x) = \text{weight} = \underbrace{\rho g}_{\gamma} A(L-x)$

Now, approximate shape:

$$v(x) \simeq a_1 \underbrace{\left(\frac{x}{L}\right)^2}_{\text{define as } \phi_1(x)} + a_2 \underbrace{\left(\frac{x}{L}\right)^3}_{\phi_2(x)}$$

must match the kinematic boundary conditions $v(0) = 0$, $v'(0) = 0$, etc.

Nondimensionalize:

$$\frac{d}{dx} = \frac{d}{d\xi}\cdot\frac{d\xi}{dx} = \frac{1}{L}\frac{d}{d\xi}, \quad \text{as } \xi = \frac{x}{L}, \quad d\xi = \frac{dx}{L}$$

$$V = \int_0^1 \frac{EI}{2L^4}(v'')^2 L\, d\xi + \int_0^1 \frac{N(\xi)}{2}\cdot\frac{1}{L^2}\left(\frac{dv}{d\xi}\right)^2 L\, d\xi - \int_0^1 q(\xi)\, v(\xi)\, L\, d\xi$$

$v(\xi)$

$$\phi_1(\xi) = \xi^2, \quad \phi_1'(\xi) = 2\xi, \quad \phi_1''(\xi) = 2$$

$$\phi_2(\xi) = \xi^3, \quad \phi_2'(\xi) = 3\xi^2, \quad \phi_2''(\xi) = 6\xi$$

Rayleigh–Ritz says

$$\frac{\partial V}{\partial a_1} = 0, \quad \frac{\partial V}{\partial a_2} = 0 \ldots \frac{\partial V}{\partial a_n} = 0$$

$$v(\xi) = a_1\xi^2 + a_2\xi^3 = a_1\phi_1 + a_2\phi_2$$

$$v' = a_1\phi_1' + a_2\phi_2'$$

$$v'' = a_1\phi_1'' + a_2\phi_2''$$

ENERGY METHODS

Example math

$$\frac{\partial V}{\partial a_1} = \frac{EI}{L^3}\int_0^1 (a_1\phi_1'' + a_2\phi_2'')\phi_1''\,d\xi + \gamma\int_0^L (1-\xi)(a_1\phi_1' + a_2\phi_2')(\phi_1')\,d\xi$$

$$- q_0 L\int_0^1 (1-\xi^2)\phi_1\,d\xi = 0$$

For $\frac{\partial V}{\partial a_2}$, change ϕ_1'', ϕ_1', ϕ_1 to ϕ_2'', ϕ_2', ϕ_2

Use matrices to solve:

$$\frac{EI}{L^3}\begin{bmatrix} \int_0^L \phi_1''\phi_1''\,d\xi & \int_0^L \phi_1''\phi_2''\,d\xi \\ \int_0^L \phi_1''\phi_2''\,d\xi & \int_0^L \phi_2''\phi_2''\,d\xi \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \gamma\begin{bmatrix} \int_0^1 (1-\xi)\phi_1'\phi_1'\,d\xi & \int_0^L (1-\xi)\phi_1'\phi_2'\,d\xi \\ \int_0^L (1-\xi)\phi_1'\phi_2'\,d\xi & \int_0^L (1-\xi)\phi_2'\phi_2'\,d\xi \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$= q_0 L\begin{bmatrix} \int_0^L (1-\xi^2)\phi_1\,d\xi \\ \int_0^L (1-\xi^2)\phi_2\,d\xi \end{bmatrix}$$

Solving,

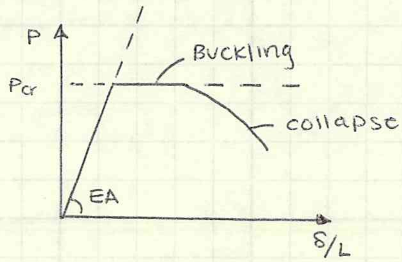
$$\frac{EI}{L^3}\begin{bmatrix} 4 & 6 \\ 6 & 12 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \gamma\begin{bmatrix} 1/3 & 3/10 \\ 3/10 & 3/10 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = q_0 L\begin{bmatrix} 2/15 \\ 1/15 \end{bmatrix}$$

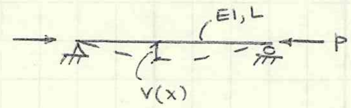
— Acceptable form of solution without values for constants.

— Note symmetric matrices

BUCKLING

Introduction





From earlier,

$$EI\frac{d^4v}{dx^4} - N_o\frac{d^2v}{dx^2} = q(x)$$ beam with an axial load

remember also $\frac{dN}{dx} = 0$, $N(x) = \text{constant} = N_o$

If $N = P$ (compressive), $q(x) = 0$

$$\boxed{EI\frac{d^4v}{dx^4} + P\frac{d^2v}{dx^2} = 0}$$

Now find P for which this equilibrium equation is satisfied.

consider a simply supported column

$$\frac{d^2}{dx^2}\left[\frac{d^2v}{dx^2} + \underbrace{\frac{P}{EI}}_{k^2}v\right] = 0$$

solution to this diff. equation:

$$v(x) = A\sin kx + B\cos kx + Cx + D$$

boundaries:

$v(0) = 0$	$v''(0) = 0$	$B + D = 0$	$B = 0, D = 0$
$v(L) = 0$	$v''(L) = 0$	$C = \frac{-1}{L}\sin kx \cdot A$	$C = 0$

final eq: $A\sin kL = 0$

solution:

either $A = 0$, $v(x) = 0$, trivial solution (no buckling)

or $\sin kL = 0$, $kL = n\pi$, $n = 1, 2, 3 \ldots$

now, $P = \frac{n^2 EI\pi^2}{L^2}$ or $\boxed{P_{cr} = \frac{\pi^2 EI}{L^2}}$

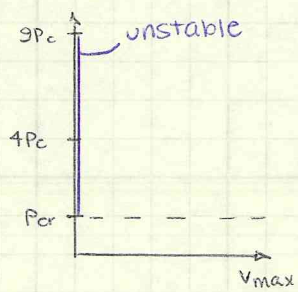
$$v_n(x) = A\sin\frac{n\pi}{L}x, \text{ for } n = 1, 2, 3 \ldots$$

There are many solutions to this problem for different values of P:

n=1 n=2 n=3

BUCKLING EQS.

Example, cont'd



$$P_c = P_{cr} = \frac{n^2 \pi^2 EI}{L^2}$$

load needed ~~[illegible]~~ relates by square of n, not direct

all P_c are solutions

EIGENVALUE PROBLEM

accending sequence of distinct eigenvalues

Anything above first buckling load is unstable

lowest value is critical buckling load,

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

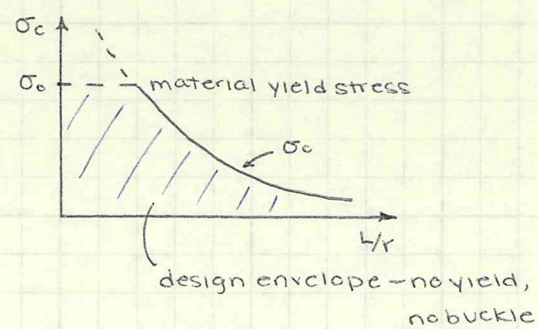
E: material property

I: geometric property

L: dependent on bracing

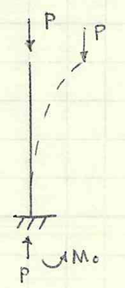
$$\sigma_c = \frac{P_c}{A} = \frac{\pi^2 EI}{AL^2}, \quad \frac{I}{A} = r^2, \text{ radius of gyration}$$

$$= \frac{\pi^2 E r^2}{L^2} = \pi^2 E \left(\frac{r}{L}\right)^2$$

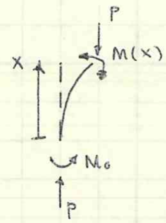


BUCKLING EQS.

Example: new boundary conditions



$M_0 = P \cdot \Delta_0$, where $\Delta_0 = v(L)$



$M(x) = -EIv''(x) = Pv(x) - M_0$

$Pv(x) + EIv''(x) = M_0 = P\Delta_0$

$$\frac{d^2v}{dx^2} + \frac{P}{EI}v(x) = P\Delta_0 \frac{1}{EI}, \quad k^2 = \frac{P}{EI}$$

$$\frac{d^2v}{dx^2} + k^2 v(x) = k^2\Delta_0$$

$$v(x) = A\sin kx + B\cos kx + \underbrace{\frac{M_0}{P}}_{=\Delta_0}$$

boundaries:

$v(0) = 0, \quad B = -\Delta_0$

$v'(0) = 0, \quad A = 0$

$$v(x) = \frac{M_0}{P}\left[1 - \cos kx\right]$$

$$v(L) = \frac{M_0}{P} = \frac{M_0}{P}(1 - \cos kL), \quad \cos kL = 0$$

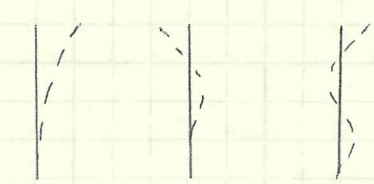
$\frac{M_0}{P} = 0$ – arbitrary, no deflection

$\cos kL = 0, \quad kL = n\frac{\pi}{2}, \; n = 1, 3, 5 \ldots$

$$P = EIk^2, \quad \boxed{P_c = \frac{\pi^2 EI}{4L^2} \cdot n^2,} \quad n = 1, 3, 5 \ldots$$

$$P_{c_{\text{flagpole}}} = \frac{1}{4} P_{c_{\text{simply supported}}}$$

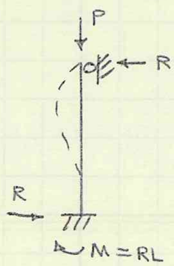
Buckled shape

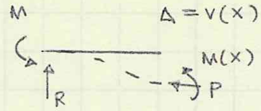


n=1 n=3 n=5

STABILITY

Another Example





$$+Pv(x)$$
$$M(x) = Rx - M = -EIv''(x)$$

$$-EIv''(x) - Pv(x) = Rx - M = Rx - RL$$

$$v''(x) + \frac{P}{EI}v(x) = R(L-x)\frac{1}{EI}$$

$$v(x) = A\sin kx + B\cos kx + \underbrace{Cx + D}_{= \frac{R}{P}(L-x)}$$

$$v(0) = 0, \quad B + \frac{RL}{P} = 0, \quad B = \frac{-RL}{P}$$

$$v'(x): \quad Ak\cos kx - Bk\sin kx - \frac{R}{P}$$

$$v'(0) = 0, \quad Ak = \frac{R}{P}, \quad A = \frac{R}{Pk}$$

Combining,

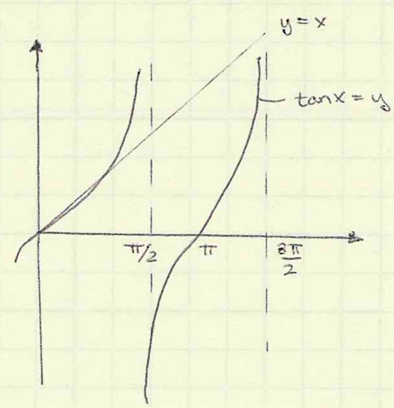
$$v(x) = \frac{R}{Pk}\sin kx + \frac{-RL}{P}\cos kx + \frac{R}{P}(L-x)$$

but R is still unknown.

$$v(L) = 0, \quad \frac{R}{P}\left[\frac{1}{k}\sin kL - L\cos kL\right] = 0$$

$$\cos kL = \frac{\sin kL}{kL}$$

$$\underline{\tan kL = kL}$$



first location: $kL = 4.49$

$$k^2 = \frac{P}{EI}, \quad P = \frac{(4.49)^2 EI}{L^2}$$

$$P = \frac{2.045\pi^2 EI}{L^2}$$

STABILITY

one more ...

$$EI\frac{d^4v}{dx^4} + P\frac{d^2v}{dx^2} = 0$$

$$v(x) = A\sin kx + B\cos kx + Cx + D$$

$v(0)=0 \qquad v(L)=0$

$v'(0)=0 \qquad v'(L)=0$

$D=0$

$Ak + C = 0$

$C = -Ak$

$$\begin{array}{c} v \\ \\ v' \\ \\ \end{array}\begin{bmatrix} 0 & 1 & 0 & 1 \\ \sin kL & \cos kL & L & 1 \\ k & 0 & 1 & 0 \\ k\cos kL & -k\sin kL & 1 & 0 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0$$

$|\det| = 0: \quad 2\cos kL - 2 + kL\sin kL = 0$

$$\sin\frac{kL}{2}\left(\frac{kL}{2}\cos\frac{kL}{2} - \sin\frac{kL}{2}\right) = 0$$

Now, either:

$$\sin\frac{kL}{2} = 0, \quad \frac{kL}{2} = n\pi$$

OR:

$$\tan\frac{kL}{2} = \frac{kL}{2}, \quad kL = 2(4.49)$$

Lowest mode uses smallest value,

$$2\pi < 2(4.49)$$

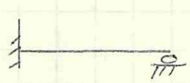
$$kL = 2\pi$$

$$\boxed{P = \frac{4\pi^2 EI}{L^2}}$$

Summary:

$\frac{\pi^2 EI}{L^2} \qquad v = A\sin\frac{\pi x}{L}$

$\frac{\pi^2 EI}{4L^2} \qquad v = A\left(1 - \cos\frac{\pi x}{2L}\right)$

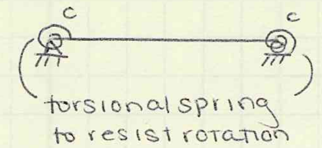


$\frac{2.04\pi^2 EI}{L^2}$

$\frac{4\pi^2 EI}{L^2}$

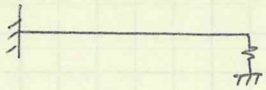
STABILITY

consider the boundaries more



$M = c\theta$

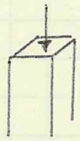
bounded by conditions with no springs and two springs with $k=\infty$



bounded by flagpole and propped cantilever

Load effects

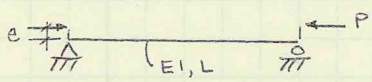
To transition directly from elastic to buckled, load must be perfect

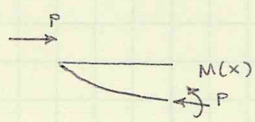


what if load is off-center?

bending, inelasticity in some parts but not necessarily all

Eccentric Loading





$$M(x) = P\,(v(x) + e) = -EIv''$$

$$EIv'' + Pv = -Pe$$

$$v'' + \frac{P}{EI}v = -\frac{P}{EI}e \quad , \quad k^2 = \frac{P}{EI}$$

$$v'' + k^2 v = \underbrace{-k^2 e}_{\text{non-zero term}}$$

Solution:

$$v(x) = A\sin kx + B\cos kx - \underbrace{e}_{\text{particular solution}}$$

boundaries: $v(0) = 0$

$$B = e$$

$$v(L) = 0$$

$$A\sin kL + e\cos kL - e = 0$$

$$A = \frac{e(1-\cos kL)}{\sin kL} = e\tan\frac{kL}{2}$$

$$v(x) = e\tan\frac{kL}{2}\sin kx + e\cos kx - e$$

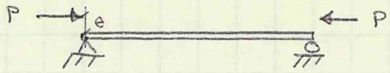
$$v_{max} = e\left[\frac{1}{\cos kL/2} - 1\right]$$

(takes some algebra)

displacement occurs for any and all values of P, as opposed to just at the buckling load (eigenvalues)

STABILITY

Eccentric Loading



$v'' + k^2 v = -k^2 e$ — if no eccentricity, $=0$

solution:

$$v(x) = e\left[\tan\frac{kL}{2}\sin kx + \cos kx\right] - e$$

v_{max} at $x = L/2$,

$$v_{max} = e\left[\tan\frac{kL}{2}\sin\frac{kL}{2} + \cos\frac{kL}{2} - 1\right]$$

$$= e\left[\frac{1}{\cos\frac{kL}{2}} - 1\right]$$

$$kL = \sqrt{\frac{PL^2}{EI}}, \text{ as } P = EIk^2$$

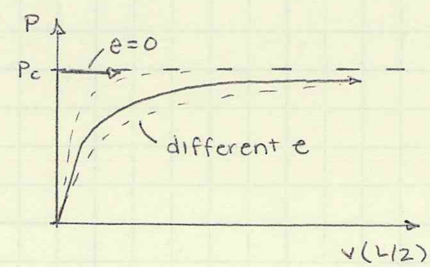
$$= \left[\frac{P}{P_c}\cdot\frac{P_c L^2}{EI}\right]^{\frac{1}{2}} = \left[\frac{P}{P_c}\pi^2\frac{EI\,L^2}{L^2 EI}\right]^{\frac{1}{2}} = \pi\sqrt{\frac{P}{P_c}}$$

thus,

$$v_{max} = e\left[\frac{1}{\cos\frac{\pi}{2}\sqrt{\frac{P}{P_c}}} - 1\right], \quad P \to P_c$$

$v_{max} \to \infty$ ($1/0$ term)

max P cannot go above P_c ($e=0$)



SOLUTION VALID WHEN COLUMN IS ELASTIC

$EI =$ constant

when does this fail to be accurate?

need to know P to cause first yield (bending)

$$M_{max} = Pe + Pv(L/2) = P\frac{e}{\cos\frac{kL}{2}}$$

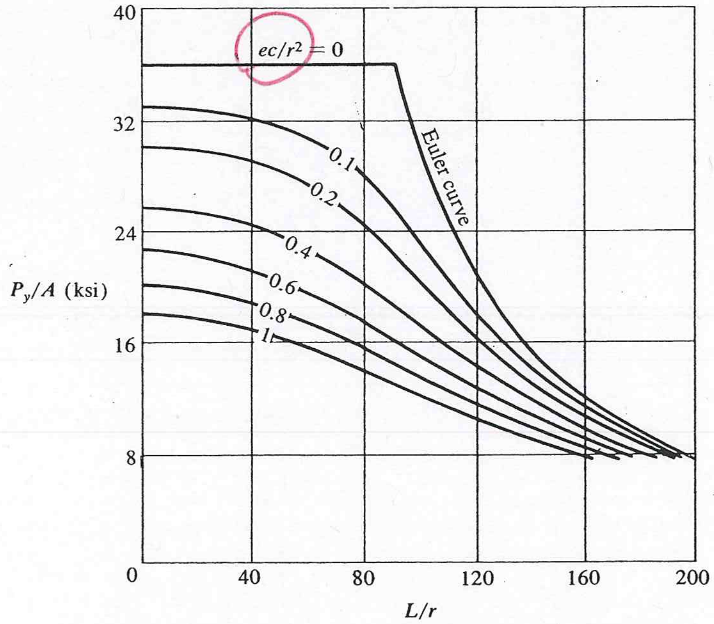
$$\sigma_{max} = \frac{P}{A} + \frac{M_{max}c}{I} = \frac{P}{A} + \frac{Pec}{I\cos\frac{kL}{2}}$$

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{A}{I}\cdot\frac{ec}{\cos\frac{kL}{2}}\right], \quad \frac{I}{A} = r^2$$

$$\boxed{\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\cdot\frac{1}{\cos\left(\frac{L}{2r}\sqrt{\frac{P}{AE}}\right)}\right]} = \sigma_o ?$$

30 November 07

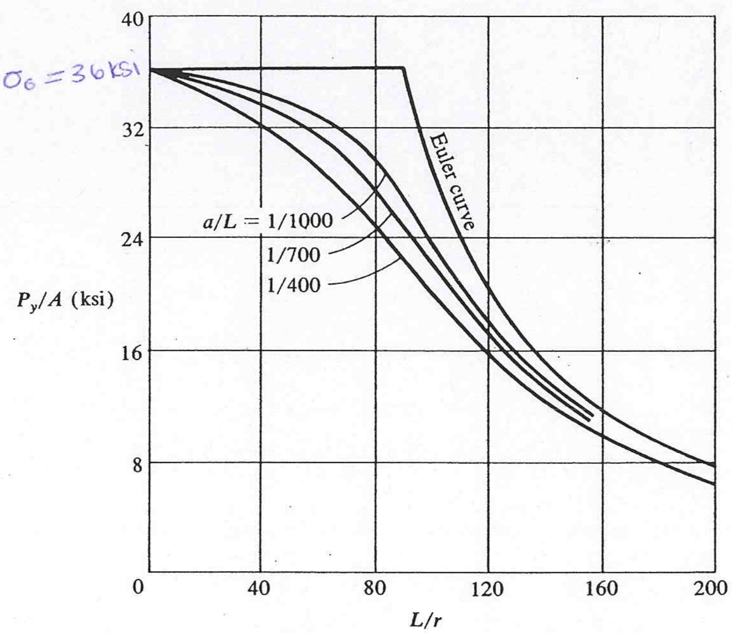
Eccentric Loading



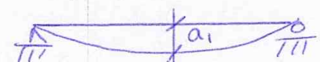
Graph of Eq. ( ) for $\sigma_y = 36{,}000$ psi and $E = 30 \times 10^6$ psi.

Imperfect columns

$\sigma_0 = 36\text{ksi}$



Graph of Eq. for an idealized wide-flange section with $\sigma_y = 36{,}000$ psi and $E = 30 \times 10^6$ psi.

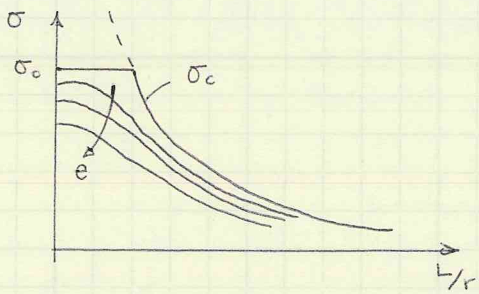


$$\frac{a}{L} = \frac{a_1}{L}$$

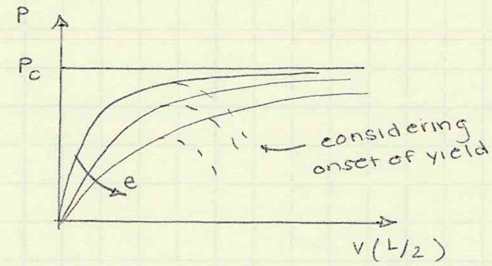
STABILITY

Eccentric loading

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\cdot\frac{1}{\cos\left(\frac{L}{2r}\sqrt{\frac{P}{EA}}\right)}\right]$$



$$\sigma_c = \frac{P_c}{A} = \pi^2 E\left(\frac{r}{L}\right)^2$$

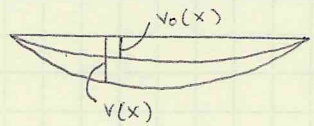


imperfect columns

tolerances exist in all construction

$\frac{1}{4}$ in out-of-straight per __ ft, etc.

initial imperfection $= v_0(x)$



$$\varepsilon = yK = -y(v'' - v_0'')$$

→ when $v = v_0$, the strain is zero

– when $v_0 = 0$, $\varepsilon = -yv'' = yK$

$$M(x) = +EI\chi^* = Pv(x)$$

$$= -EI(v'' - v_0'') - Pv = 0$$

$$v'' + \frac{P}{EI}v = v_0''$$

$$\underline{v'' + k^2 v = v_0''}$$

Generalization:

$$\text{Let } v_0(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}$$

$$v_0'' = -\sum_{n=1}^{\infty} a_n\left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L} = v'' + k^2 v$$

$$\text{Let } v(x) = \sum_{N=1}^{\infty} A_N \sin\frac{N\pi x}{L}$$

Boundaries:

$v(0) = v(L) = 0$

Check:

$$\sum_{N=1}^{\infty} A_N\left[-\left(\frac{N\pi}{L}\right)^2 + k^2\right]\sin\frac{N\pi x}{L} = \sum_{n=1}^{\infty} -a_n\left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L}$$

$n = N$

$$A_N = \frac{a_n\left(\frac{n\pi}{L}\right)^2}{\left(\frac{n\pi}{L}\right)^2 - k^2}$$

STABILITY

Imperfect columns, cont'd

$$A_N = \frac{EI\left(\frac{n\pi}{L}\right)^2 a_n}{EI\left(\frac{n\pi}{L}\right)^2 - P} = \frac{P_n}{P_n - P} a_n \quad , \quad P_n = \frac{n^2\pi^2 EI}{L^2}$$

$$\therefore \; v(x) = \sum_{n=1}^{\infty} \frac{a_n}{1 - P/P_n} \sin\frac{n\pi x}{L}$$

first mode, second mode, third mode ...

$$v(x) = \frac{a_1}{1 - P/P_c}\sin\frac{\pi x}{L} + \frac{a_2}{1 - \frac{P}{4P_c}}\sin\frac{2\pi x}{L} + \frac{a_3}{1 - P/9P_c}\sin\frac{3\pi x}{L} + \ldots$$

as $P \to P_c, 4P_c, 9P_c \cdots$,
singularities occur.

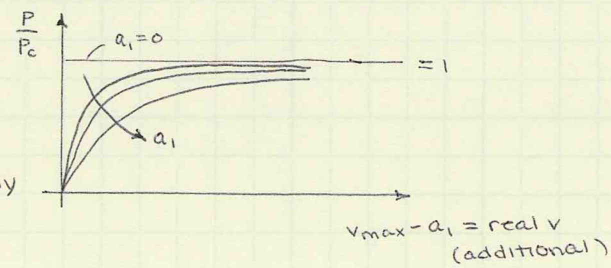
Example

Let $v_0(x) = a_1 \sin\frac{\pi x}{L}$

Then $v(x) = \dfrac{a_1}{1 - P/P_c}\sin\dfrac{\pi x}{L}$, $v_{max} = v(L/2) = \dfrac{a_1}{1 - P/P_c}$

v at $P = 0 = a_1$

$$v_{max} - a_1 = a_1 \frac{P/P_c}{1 - P/P_c}$$



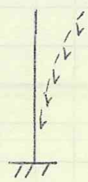
deflection occurs immediately upon applying load
as $P \to P_c$, $v \to \infty$

Now, when does the section yield?

BUCKLING

Consider a column
- only load is self-weight
- answer is approximate

when does the column buckle?



$P(x) = \underbrace{\rho g A}_{\gamma}(L-x)$

P is not constant, k^2 is not constant
non-constant coefficients

Use Rayleigh-Ritz

$$V(x) = \int_0^L \frac{EI}{2}(v'')^2 dx - \int_0^L \frac{P(x)}{2}(v')^2 dx$$

Approximate:

$$v(x) = a_1\phi_1(x) + a_2\phi_2(x)$$

$$\left.\begin{aligned}\phi_1 &= 1-\cos\frac{\pi x}{2L}\\ \phi_2 &= 1-\cos\frac{3\pi x}{2L}\end{aligned}\right]$$ – first eigenmodes for a fixed-free column loaded at the top.

$\frac{\partial V}{\partial a_1}=0, \ \frac{\partial V}{\partial a_2}=0$, results in:

$$0 = \frac{EI}{L^3}\begin{bmatrix}\int_0^1 \phi_1''\phi_1'' d\xi & \int_0^1 \phi_1''\phi_2'' d\xi\\ \int_0^1 \phi_1''\phi_2'' d\xi & \int_0^1 \phi_2''\phi_2'' d\xi\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix} - \gamma\begin{bmatrix}\int_0^1(1-\xi)\phi_1'\phi_1' d\xi & \cdots\\ \int_0^1(1-\xi)\phi_1'\phi_2' d\xi & \end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}$$

$$\phi_1' = \frac{\pi}{2}\sin\frac{\pi}{2}\xi \qquad \phi_1'' = \left(\frac{\pi}{2}\right)^2\cos\frac{\pi}{2}\xi$$

$$\phi_2' = \frac{3\pi}{2}\sin\frac{3\pi}{2}\xi \qquad \phi_2'' = \left(\frac{3\pi}{2}\right)^2\cos\frac{3\pi}{2}\xi$$

$$[\underset{\sim}{k} - \lambda \underset{\sim}{B}]\underset{\sim}{a} = 0$$

$$\lambda = \frac{\gamma L^3}{EI} = \frac{\rho g A L^3}{EI}$$ Eigenvalue problem

Now, either $\underset{\sim}{a}=0$ (no buckling),
or $|\underset{\sim}{k} - \lambda\underset{\sim}{B}| = 0$ (determinant)
forms a quadratic equation for λ
solve for two roots, two eigenvalues.
Number is determined by number of terms (a, ϕ) in approximation.

Exact solution:

$$(\rho g A)_c = 0.794\left(\frac{\pi}{L}\right)^2 EI$$
$\times L$?

- In HW, I varies with height
- Exact solution may not be right

STABILITY

Imperfect columns example, cont'd

$$v_0(x) = a_1 \sin\frac{\pi x}{L}$$

$$v(x) = \frac{a_1}{1-P/P_c}\sin\frac{\pi x}{L} \quad , \quad v(L/2) - a_1 = \frac{a_1 P/P_c}{1-P/P_c}$$

Find P at first yield

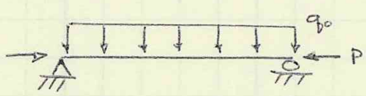
$$M_{max} = M(L/2) = Pv(L/2) = \frac{a_1 P}{1-P/P_c}$$

$$\sigma_{max} = \frac{P}{A} + \frac{M_{max}c}{I} = \frac{P}{A}\left[1 + \frac{a_1 c}{I} A \frac{1}{1-P/P_c}\right] , \quad \frac{I}{A} = r^2$$

$$\boxed{\sigma_{max} = \frac{P}{A}\left[1 + \frac{a_1 c}{r^2}\frac{1}{\left[1 - \frac{P}{A}\left(\frac{L}{r}\right)^2 \frac{1}{E\pi^2}\right]}\right] = \sigma_0}$$

a_1 comes from supplier or from building specifications

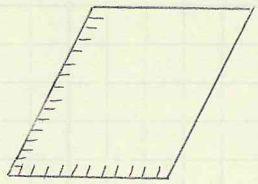
Transverse loads



q_0 causes bending, load P causes more
consider $v_0(x)$ = function of q_0

PLATE BENDING

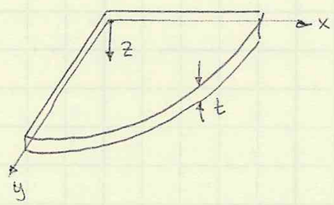
Introduction



like two sets of beams

$$M_x = -EI\frac{d^2v}{dx^2}$$

$$M_y = -EI\frac{d^2v}{dy^2}$$



$t \ll$ other dimensions

Assumptions:

- plane sections remain plane

$$\gamma_{xz} = \gamma_{yz} = 0$$

- small deflections and rotations

Kinematics:

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2} \equiv z K_x$$

$$\varepsilon_y = -z\frac{\partial^2 w}{\partial y^2} \equiv z K_y$$

$$\gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y} = 2zK_{xy}$$

Constitutive Equations:

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_{xy}\end{bmatrix} = \begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}\cdot\frac{1}{E}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{bmatrix}, \quad \begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{bmatrix} = \frac{E}{1-\nu^2}\underbrace{\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}-\frac{\nu}{2}\end{bmatrix}}_{\underline{A}}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_{xy}\end{bmatrix}$$

$$\begin{bmatrix}M_x\\ M_y\\ M_{xy}\end{bmatrix} = \int_{-t/2}^{t/2}\underline{\sigma}\cdot z\,dz = \frac{E}{1-\nu^2}\underline{A}\int_{-t/2}^{t/2} z\,\underline{\varepsilon}\,dz = \frac{E\underline{A}}{1-\nu^2}\int_{-t/2}^{t/2} z^2\,\underline{K}\,dz$$

$$\underline{K} = \begin{bmatrix}K_x\\ K_y\\ 2K_{xy}\end{bmatrix}$$

$$\boxed{\underline{M} = \underbrace{\frac{E}{1-\nu^2}\cdot\frac{t^3}{12}}_{D}\cdot\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}\begin{bmatrix}K_x\\ K_y\\ 2K_{xy}\end{bmatrix}}$$

no b term

moments per unit length (no b)

Also, M_x/M_y, K_x/K_y are coupled through ν.
$M_y \neq 0$ even if $K_y = 0$
$M_y = \nu K_x \cdot D$

PLATE BENDING

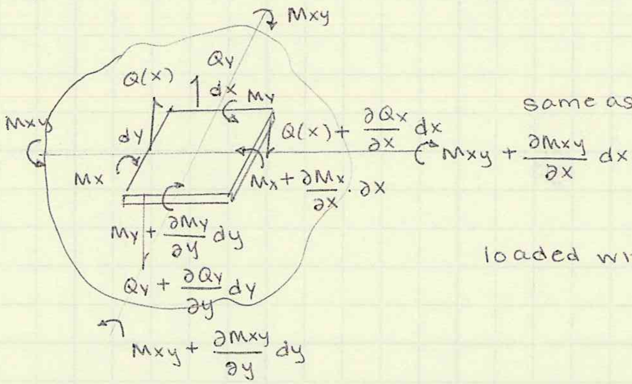
Introduction

Shears in plate

$$Q_x = \int_{-t/2}^{t/2} \sigma_{xz}\, dz \;, \quad Q_y = \int_{-t/2}^{t/2} \sigma_{yz}\, dz$$

both are shear stresses per unit length (or, forces per unit length = intensities)

Equilibrium



same as in beam, but in 2D, not 1

now add M_{xy}

loaded with $q(x,y)$

05 December 07

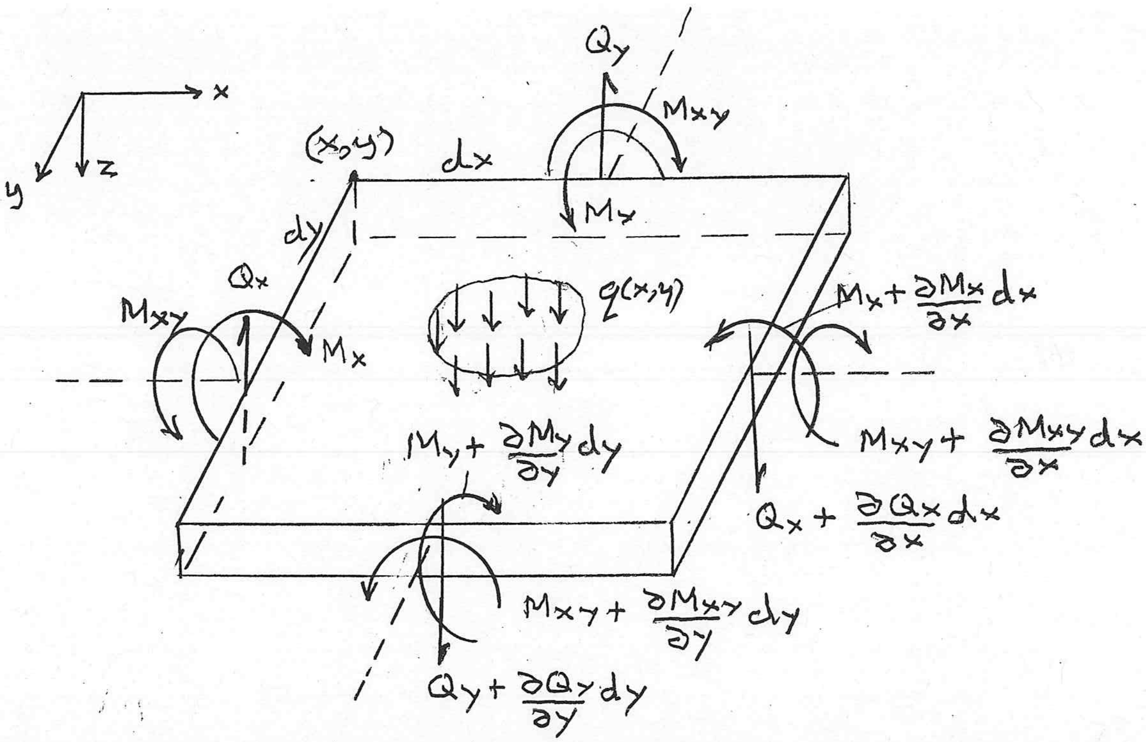


PLATE BENDING

Introduction (cont'd)

considering element of plate (see handout)

$$\Sigma F_z: \quad \left(\cancel{Q_x} + \frac{\partial Q_x}{\partial x}dx\right)dy + \left(\cancel{Q_y} + \frac{\partial Q_y}{\partial y}dy\right)dx - \cancel{Q_x}dy - \cancel{Q_y}dx + q(x,y)\,dx\,dy$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q(x,y)$$

$$\Sigma M_y: \quad \left(M_x + \frac{\partial M_x}{\partial x}dx\right)dy + \left(M_{xy} + \frac{\partial M_{xy}}{\partial y}dy\right)dx - Q_x\,dy\,dx - M_x\,dy - M_{xy}\,dx = 0$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x$$

$$\frac{\partial M_{xy}}{\partial y} + \frac{\partial M_y}{\partial y} = Q_y$$

similar to beam moment equations, with extra terms from twisting.

Now eliminate the shear (Q_x, Q_y)

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} = q(x,y)\,(-1)$$

Note negative sign!

Substitute for M_x, M_y, M_{xy} from the constitutive equations

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{bmatrix} K_x \\ K_y \\ K_{xy} \end{bmatrix}, \quad D = \frac{Et^3}{12(1-\nu^2)}$$

$$D\left\{ \frac{\partial^2}{\partial x^2}(K_x + \nu K_y) + \frac{2\partial^2}{\partial x\partial y}(K_{xy})(1-\nu) + \frac{\partial^2}{\partial y^2}(K_y + \nu K_x) \right\} = -q(x)$$

Now substitute for Ks:

now positive

$$D\left\{ \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right\} = q(x,y)$$

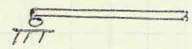
or $\quad D\nabla^2\nabla^2 w = q(x,y) \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \frac{q(x,y)}{D}$

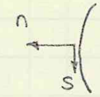
or $\quad D\nabla^4 w = q(x,y)$

PLATEBENDING

Boundary conditions

Simple support





$w = 0$

$M_n = 0$

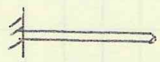
$w = 0$

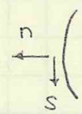
$M_x = 0 = D\left[K_x + \nu K_y\right]$

or, $w_{,xx} + \nu w_{yy} = 0$

curvatures are coupled resulting in two terms in each moment

Fixed support





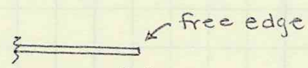
$w = 0$

$w_{,n} = 0$

$w = 0$

$w_{,x} = 0$

Free edge

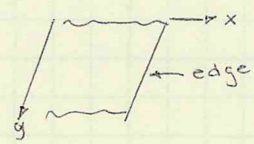


(beam, or plate, continues)

$Q_x = 0$

$M_x = 0$

$M_{xy} = 0$



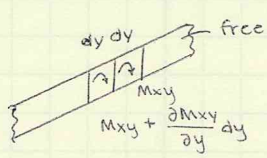
4th order equation (e.g. $EIv'''' = q(x)$)

only needs 2 boundary conditions.

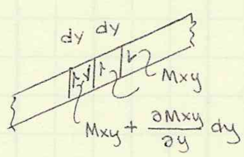
We have 3 here. Too many!

- equations were original sixth order; assumption of plane sections reduced it to fourth. No problem, except in this situation.

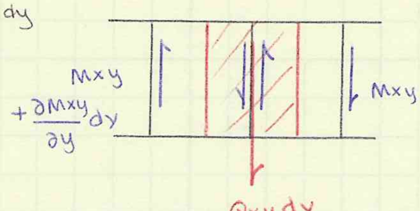
Kerchoff's Remedy



$\equiv$



Now look at center



Equilibrium of central element (width dy)

$$M_{xy} + \frac{\partial M_{xy}}{\partial y}dy + Q_x dy - M_{xy} = 0$$

$$\frac{\partial M_{xy}}{\partial y} + Q_x = 0$$

PLATE BENDING

Boundary conditions

Free edge

From Kerchoff,

$$\boxed{\frac{\partial M_{xy}}{\partial y} + Q_x = 0}$$

works great until you reach a boundary

New boundaries:

$M_x = 0$

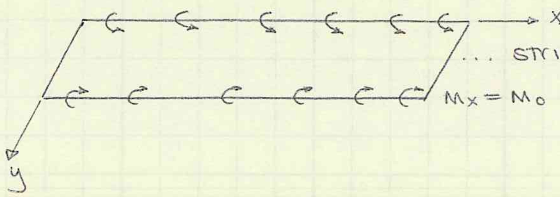
$$\frac{\partial M_{xy}}{\partial y} + Q_x = 0 \quad \text{or} \quad \boxed{\begin{array}{l} M_n = 0 \\ \dfrac{\partial M_{ns}}{\partial s} + Q_n = 0 \end{array}}$$

In displacement form,

$$\boxed{\begin{array}{l} \dfrac{\partial^2 w}{\partial x^2} + \nu \dfrac{\partial^2 w}{\partial y^2} = 0 \\[2ex] \dfrac{\partial^3 w}{\partial x^3} + (2-\nu)\dfrac{\partial^3 w}{\partial x \partial y^2} = 0 \end{array}}$$

EXAMPLE PROBLEMS

cylindrical bending of a long strip



... strip continues (long)

long strip imposes $\frac{\partial}{\partial x} = 0$

$$\therefore \quad M_y = -D\frac{d^2w}{\partial y^2} = M_o$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{-M_o}{D}$$

$$\frac{dw}{dy} = -\frac{M_o}{D}y + A$$

$$w = \frac{-M_o}{2D}y^2 + Ay + B$$

From boundaries,

$$w(0) = 0, \quad B = 0$$

$$w(x,b) = 0,$$

$$-\frac{M_o}{D}\cdot\frac{b^2}{2} + Ab = 0$$

$$A = \frac{M_o b}{2D}$$

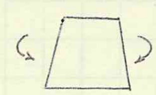
$$\therefore \quad w(x,y) = \frac{M_o b^2}{2D}\left[\frac{y}{b} - \left(\frac{y}{b}\right)^2\right]$$

We observe that

$$M_x = D\left[K_x + \nu K_y\right], \quad K_x = 0$$

$$= -\nu D w_{,yy} = \nu M_o$$

- very small additional curvature (anti-plastic)
- doesn't effect solution (stresses)



Mx is a geometric constraint; prevents this curvature because the strip ~~≠~~ is long.

stress calculations

$$\sigma_y = \frac{-Ez}{1-\nu^2}\left[K_y + \nu \cancel{K_x}\right] = \frac{-Ez}{1-\nu^2}\cdot\frac{M_o}{D}$$

$$\sigma_x = \frac{-Ez}{1-\nu^2}\left[\cancel{K_x} + \nu K_y\right] = \frac{-\nu Ez}{1-\nu^2}\cdot\frac{M_o}{D}$$

maximum stresses occur at extreme fibers

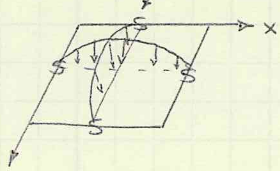
Now, $M_{o_{max}} = f$ (von Mises / Tresca yield criterion)

$$\sigma_o = \sigma_x^2 + \sigma_y^2 - 2\sigma_x\sigma_y + 3\sigma_{xy}^2$$

t (?)

PLATE BENDING

Examples



length a, width b

$$q(x,y) = q_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$D\left[\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] = q(x,y) = q_0 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

From boundaries,

$w(0,y) = 0$, $w(a,y) = 0$

$M_x(0,y) = 0 = -D\left[w_{,xx} + \nu w_{,yy}\right] = 0$

along $x=0$, deflection is 0, curvature (κ_y) is 0 (straight)

$M_x(0,y) = -D w_{,xx} = 0$

or $w_{,xx}(0,y) = 0$, $w_{,xx}(a,y) = 0$

$w(x,0) = 0$, $w(x,b) = 0$

$M_y = w_{,yy}(x,0) = 0$, $w_{,yy}(x,b) = 0$

// 8 boundaries

Solution:

$$w(x,y) = A\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

follows boundaries, $w=0$ at $x=0,a$
$y=0,b$

moments $=0$ at same locations

substitute into equilibrium equation

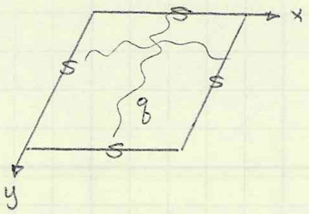
$$D\left[\left(\frac{\pi}{a}\right)^4 + \underbrace{\left(\frac{\pi}{a}\right)^2\left(\frac{\pi}{b}\right)^2}_{\times 2} + \left(\frac{\pi}{b}\right)^4\right] A\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} = q_0 \sin\frac{\pi x}{a}\sin\frac{\pi x}{b}$$

$$\therefore\ A = \frac{q_0}{\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right]^2 D}$$

✓ done.

PLATE BENDING

Example



Use Fourier series to find

$$q(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn} \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi x}{b}$$

same response as last example, with M and N coefficients in original load scheme

$$q(x,y) = q_0 \sin\frac{M\pi x}{a} \sin\frac{N\pi x}{b}$$

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi x}{b}$$

substitute $w(x,y)$ into $D\nabla^4 w = q(x,y)$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D\left[a_{mn}\left[\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{b}\right)^4\right] - q_{mn}\right] \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = 0$$

$$\therefore a_{mn} = \frac{q_{mn}}{[\quad]} = \frac{q_{mn}}{\pi^4 D\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}$$

Example

$$q(x,y) = q_0$$

Fourier series: $\displaystyle\int_0^a\int_0^b q_0 \sin\frac{M\pi x}{a} \sin\frac{N\pi y}{b}\, dx\,dy = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^a\int_0^b q_{mn} \ldots$

$$\ldots \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b} \cdot \sin\frac{M\pi x}{a} \cdot \sin\frac{N\pi y}{b}\, dx\,dy$$

$$\int_0^a \sin\frac{m\pi x}{a} \sin\frac{M\pi x}{a} = \begin{cases} 0 & m \neq M \\ \frac{a}{2} & m = M \end{cases}$$

therefore,

$$q_{mn} = \frac{4q_0}{ab}\int_0^a\int_0^b \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\,dy$$

$$\int_0^a \sin\frac{m\pi x}{a}\, dx = \frac{a}{\pi m}\left[1 - (-1)^m\right]$$

similarly, $\displaystyle\int_0^b \sin\frac{n\pi y}{b}\, dy$

$$q_{mn} = \frac{4q_0}{ab} \cdot \frac{a}{\pi m} \cdot \frac{b}{\pi n}\left[1-(-1)^m\right]\left[1-(-1)^n\right]$$

$$= \frac{16 q_0}{\pi^2 mn} \qquad m,n = 1,3,5\ldots$$

$$w(x,y) = \frac{16 q_0}{\pi^2 mn}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{mn\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2}$$

PLATE BENDING

conclusions

$$D\left[\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] = q(x,y)$$

→ $+ kw$, for plates on elastic foundation

Solution depends highly on:
- the shape of the plate
- the loading
- the boundaries

- it gets very complicated very quickly!
- often use approximations (i.e. finite element)
- use special equations for axial loading

... all beam topics have extensions into plates

Final Exam!

- comprehensive
- open class notes
- open homework (yours and yours only)
  no xeroxes, only hand-copying
- anything on plates will be simple

Tips to Study:

- review notes thoroughly
- notes should only be there to help, not as a basis for work
- put a summary at the front of each section

4-5 problems / Review session Friday at 5pm