CE 381R
The Finite Element Method
Fall 2006
SYLLABUS

UNIQUE NUMBER: 15510

INSTRUCTOR:
John L. Tassoulas
ECJ 4.732
Phone: 471-4553
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OFFICE HOURS: MWF 3:00-4:00

COURSE TIME: MWF 9:00-10:00

PLACE: ECJ 6.406

OBJECTIVES: This course is aimed at introducing conceptual and computational aspects of the finite element method.

PREREQUISITES: This course requires understanding of matrix structural analysis, knowledge of elementary linear algebra and familiarity with computer programming.

TEXT: There is no required textbook. Therefore, students are advised to attend classes regularly and take notes carefully. For reference purposes, the following books are recommended:

"The Finite Element Method: Linear Static and Dynamic Finite Element Analysis," by T.J.R. Hughes
"The Finite Element Method: Its Basis and Fundamentals," Sixth Edition, by O.C. Zienkiewicz, R.L. Taylor and J.Z. Zhu
"The Finite Element Method for Solid and Structural Mechanics," Sixth Edition, by O.C. Zienkiewicz and R.L. Taylor

CLASS FORMAT: Lectures supplemented with homework and examinations.

CLASS OUTLINE: Continuous and Discrete Systems: principle of virtual work, direct stiffness method. Finite Element Analysis of Elastic Structures: two-dimensional finite elements (plane stress, plane strain), three-dimensional and axisymmetric finite elements. Interpolation and Numerical Integration: shape functions, isoparametric finite elements, Gauss-Legendre quadrature. Finite Element Analysis of Inviscid, Incompressible Fluids. Convergence of the Finite Element Method: patch test, error estimation. Linear Dynamic Finite Element Analysis.

GRADING:

Each of two fifty-minute examinations	20%
Final Examination	30%
Homework	30%

HOMEWORK POLICY: Students are encouraged to discuss course topics in groups. However, assignments must be carried out by each student independently and turned in by the beginning of class on the day that they will be due.

DROP POLICY: (College of Engineering) *Graduate Students:* From the 1st through the 4th class day, graduate students can drop a course via the web and receive a refund. During the 5th through 12th class days, graduate students must initiate drops in the department that offers the course and receive a refund. After the 12th class day, no refund is given. No class can be added after the 12th class day. From the 13th through the 20th class day, an automatic Q is assigned with approval from the graduate advisor and the Graduate Dean. From the 21st class day through the last class day, graduate students can drop a class with permission from the instructor, graduate advisor, and the Graduate Dean. Students with 20-hr/week GRA/TA appointment or a fellowship may not drop below 9 hours.

Undergraduate Students: From the 1st through the 12th class day, an undergraduate student can drop a course via the web and receive a refund. From the 13th through the 20th class days, an automatic Q is assigned, no refund; approval from the Dean and departmental advisor is required. From the 21st class day through the mid-semester deadline, approval is required from the Dean, instructor of the course and departmental advisor.

EXAMINATIONS: There will be two fifty-minute examinations and the final examination. Notebooks and textbooks can be open during the examinations. The date of each fifty-minute examination will be set and announced by the instructor. Alternative dates for the two fifty-minute examinations will be arranged only for students with documented medical emergencies. The **Final Examination** is scheduled for **Wednesday, December 13, 9:00-12:00 noon.**

EVALUATION: The University Measurement and Evaluation Center forms will be used to evaluate the course and the instructor.

DISHONESTY: University procedures will be followed in dealing with cases of suspected scholastic dishonesty.

ATTENDANCE: Regular class attendance is expected in accordance with The University's General Information catalog and the College of Engineering policy.

IMPORTANT NOTE: The University of Texas at Austin provides, upon request, appropriate academic adjustments for qualified students with disabilities. Any student with a documented disability (physical or cognitive) who requires academic accommodations should contact the Services for Students with Disabilities area of the Office of the Dean of Students at 471-6259 as soon as possible to request an official letter outlining authorized accommodations. For more information, contact that Office or TTY at 471-4641, or the College of Engineering Director of Students with Disabilities at 471-4321.

CATHERINE HOVELL
03 November 06

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Examination 1

1. (40 points)

Consider a finite element of a solid in one, two or three dimensions. Let $\underset{\sim}{B}$ be the matrix relating strain to nodal displacements:

$$\underset{\sim}{\varepsilon} = \underset{\sim}{B} \cdot \underset{\sim}{U}$$

Assume that the stress, σ, is known in the element (as a function of position). On the basis of the Principle of Virtual Work (and the Galerkin Finite Element Method), obtain an expression for the vector of nodal forces required for equilibrium.

$\varepsilon = B \cdot u$ ✓

$\sigma = E \cdot \varepsilon = E(B \cdot u)$ ✓ D OK

$\delta\varepsilon = B \cdot \delta u$ ✓

$\int_\Omega \delta\varepsilon \cdot \sigma \, d\Omega = \delta u_o \cdot P$

Force = area under σ curve

$F = \int_A \sigma \cdot dA$ OK

B, E matrices are constant
for the 2-node, 1-D element

$\int_\Omega B \cdot \delta u \cdot E \cdot (B \cdot u) \, d\Omega = \delta u \cdot P = \int_\Omega B \cdot \delta u \cdot \sigma \, d\Omega = \delta u \cdot P$

using Galerkin, δu is in terms of $\delta a, \delta b \ldots$
as u is in terms of a, b... $[x \quad x^2 \quad x^3 \ldots]$
✓ related by shape functions, N (N_o = at boundaries)

$\delta u P = N \cdot \delta a \cdot P$ $\quad \int B \cdot \delta u \cdot \sigma \, d\Omega = B \int N \cdot \delta a \cdot \sigma \, d\Omega$
Is this evaluated at $x = 0$?

$\delta a \cdot B \int N^T \sigma \, d\Omega = \delta a \cdot N_o^T P$ $\quad$ for all $\delta a \neq 0$, so δa can cancel

$P = N_o^{T-1} B \int_\Omega N^T \sigma \, d\Omega$

! Is this operation possible in general?

(I think there's an error in my matrix manipulations)

$P = B \int \sigma \, d\Omega$

N^{-1} and N, if multiplied, equal [I]

simple approx ($\sigma = P/A$) says there's an extra B term here, should be
$P = E \int B \cdot u \, d\Omega$

$$\boxed{P = B \cdot E \int_\Omega \underbrace{\overbrace{B \cdot u}^{=\varepsilon}}_{\sigma} \, d\Omega}$$

$\Rightarrow P = \int_\Omega \underset{\sim}{B}^T \underset{\sim}{\sigma} \, d\Omega$

+25

2. (30 points)

Consider a finite element of a linearly-elastic solid in plane stress (or plane strain) undergoing a change in temperature, ΔT. The relationship between the strain and the vector of nodal displacements is

$$\underset{\sim}{\varepsilon} = \underset{\sim}{B} \cdot \underset{\sim}{U}$$

while the stress-strain relationship can be written as:

$$\underset{\sim}{\sigma} = \underset{\sim}{D} \cdot (\underset{\sim}{\varepsilon} - \underset{\sim}{\varepsilon}_T)$$

where $\underset{\sim}{D}$ is the elastic rigidity matrix and $\underset{\sim}{\varepsilon}_T$ is the "thermal" strain:

$$\underset{\sim}{\varepsilon}_T = \begin{bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{bmatrix}$$

α being the coefficient of thermal expansion (assumed to be the same in the x and y directions). On the basis of the Principle of Virtual Work (and the Galerkin Finite Element Method), obtain an expression for the vector of nodal forces due to the temperature change ("thermal" load vector).

$$\varepsilon = B \cdot u$$

$$\sigma = D \cdot (\varepsilon - \varepsilon_T) = D \cdot (B \cdot u - \varepsilon_T)$$

$$\delta\varepsilon = B \cdot \delta u \qquad \delta u = N \, \delta a$$

We are dealing with a 2-D element in this question.

thermal forces

$$\int_\Omega B \cdot \delta u \cdot D \cdot (B \cdot u - \varepsilon_T) \, d\Omega = \underbrace{\delta u_0 P_0 + \delta u_L P_L}_{\underset{\sim}{\delta u_0} \cdot \underset{\sim}{P}} + \int_\Omega \delta u \cdot T \, d\Omega$$

$$\int_\Omega B \cdot N \cdot \delta a \cdot D \; (B \cdot u - \varepsilon_T) \, d\Omega = N_0 \delta a \cdot P + \int_\Omega N \delta a \cdot T \cdot d\Omega$$

- δa's can be removed using matrix manipulation.
- B, D, ε_T known; integrate N matrices over domain Ω
- solve for P, nodal forces

$$d\int_{\Omega^e} \underset{\sim}{B}^T \underset{\sim}{D} \, \underset{\sim}{\varepsilon}_T \, d\Omega^e$$

"thermal" load vector

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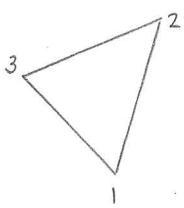
3. (30 points)

Consider a 3-node, triangular, finite element of a homogeneous, linearly-elastic solid in plane stress (or plane strain). Let $\underset{\sim}{K}$ be the element stiffness matrix. Now, consider another element, similar to the first, with nodal coordinates given by

$$x_i' = S \cdot x_i + X \qquad i = 1,2,3$$
$$y_i' = S \cdot y_i + Y$$

where x_i, y_i, $i=1,2,3$, are the nodal coordinates of the first element, and S, X and Y are (arbitrary) constants. What is the stiffness matrix of the second element?



$$K = d\int_\Omega B^T D B \, d\Omega$$

- D matrix is constant, depends on E, ν
- triangle is scaled up or down, moved
- B depends on the derivatives of N functions

$$N_1 = \frac{1}{2A}\left[x(y_2-y_3) - y(x_2-x_3) + x_2y_3 - x_3y_2\right]$$

$$N_1' = \frac{1}{2AS^2}\left[x(Sy_2-Sy_3) - y(Sx_2-Sx_3) + (Sx_2+X)(Sy_3+Y) - (Sx_3+X)(Sy_2+Y)\right]$$

S S

S^2, S

$S^2x_2y_3 - S^2x_3y_2$

$$\underline{\underline{K' = SK}}$$

X

+24

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04 Dec 06

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Examination 2

1. (70 points)

Consider a hybrid (displacement – pressure) formulation of an axisymmetric 4-node rectangular finite element (of an isotropic, elastic solid) with a single pressure node (constant pressure over the element).

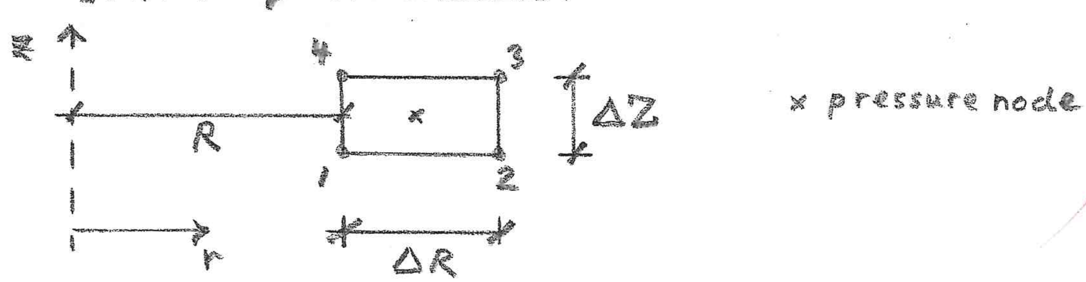
a) Write an equation for the augmented principle of virtual work, incorporating the volumetric constraint, suitable for the axisymmetric formulation.

b) Obtain expressions for $\underset{\sim}{G}$ and $\underset{\sim}{E}$ (the parts of the element augmented stiffness matrix associated with pressure). Keep in mind that, in cylindrical coordinates, the volumetric strain, ε, is given by:

$$\varepsilon = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}$$

(assuming axisymmetric conditions; u and w are the radial and axial displacement components; r and z are the radial and axial coordinates). Do not evaluate the integrals involved in the expressions.

c) Assuming that the element has been mapped onto the standard square in ξ–η space, write expressions for the first two entries in $\underset{\sim}{G}$ as integrals with respect to ξ and η. (Show all functions in terms of ξ and η.) If the integrals were to be evaluated numerically, what would be the minimum number of integration points (Gauss-Legendre) necessary for ~~exact~~ integration? Use the following dimensions:



2. (30 points)

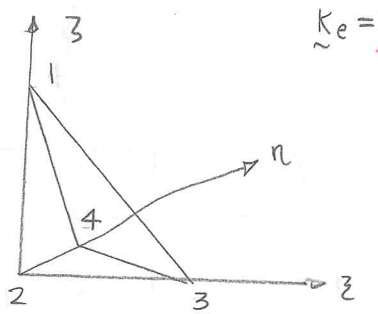
Consider a 4-node, tetrahedral, finite element of a homogeneous, linearly-elastic solid. Let $\underset{\sim}{K}$ be its stiffness matrix. Now, consider another element, similar to the first, with nodal coordinates given by

$$x_i' = S \cdot x_i + X$$

$$y_i' = S \cdot y_i + Y \qquad i = 1, 2, 3, 4$$

$$z_i' = S \cdot z_i + Z$$

where x_i, y_i, z_i, $i = 1, 2, 3, 4$, are the nodal coordinates of the first element, and S, X, Y and Z are (arbitrary) constants. What is the stiffness matrix of the second element?



$$\underset{\sim}{K}_e = \iiint B^T D B \det(J)\, d\xi\, d\eta\, d\zeta$$

$$\det(J) = \frac{\partial x}{\partial \xi} \times \frac{\partial x}{\partial \eta} \cdot \frac{\partial x}{\partial \zeta} \qquad (x = [x\ y\ z]^T)$$

— $\det(J)' = S^3 \det(J)$ ✓

as volume in real space increased by S in each dimension, vol. in $\xi \cdot \eta \cdot \zeta$ = same.

$N_1 = \zeta$

$N_2 = 1 - \zeta - \xi - \eta$

$N_3 = \xi$

$N_4 = \eta$

— D is constant, $D' = D$ ✓

Non-mapped:

$$K = \int_\Omega B^T D B \, d\Omega$$

+15

B relates to $\frac{\partial N}{\partial \_}$

$$N_i = \frac{A \times B \cdot C}{\text{vol.}}$$

$$A = \begin{bmatrix} x_3 - x_2 \\ y_3 - y_2 \\ z_3 - z_2 \end{bmatrix}, \quad B = \begin{bmatrix} x_4 - x_2 \\ y_4 - y_2 \\ z_4 - z_2 \end{bmatrix}$$

S

$\underline{\underline{K' = K}}$?

$$C = \begin{bmatrix} x - x_2 \\ y - y_2 \\ z - z_2 \end{bmatrix}$$

Ni changes by $\frac{S^3}{S^3} = 1$ ✓

derivatives in $\underset{\sim}{B}$ same fro...

... also doesn't change ✗ $B' = \ldots \frac{1}{\ } B$

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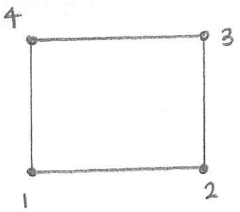
04 December 06

EXAM #2

1. a.



$$\begin{bmatrix} \underset{\sim}{K_s} & \underset{\sim}{G} \\ \underset{\sim}{G}^T & \underset{\sim}{F} \end{bmatrix} \begin{bmatrix} \underset{\sim}{u} \\ p \end{bmatrix} = 2\pi \int N^T T \cdot r\,dr + 2\pi \int N^T b \cdot r\,dr$$

∂B_{2D} $\quad B_{2D}$ $\quad d\Omega_{2D}$ $\quad \Omega_{2D}$

b. $\varepsilon = \dfrac{\partial u}{\partial r} + \dfrac{u}{r} + \dfrac{\partial w}{\partial z}$

in non-axisymmetric,

$$G = -\int_\Omega B_b^T N_p \, d\Omega$$

$$F = -\int_\Omega N_p^T \frac{1}{K} N_p \, d\Omega$$

$$G = -2\pi \int B_b^T N_p \cdot r \, d\Omega$$

$$B_b = \left[\frac{\partial N_1}{\partial r} + \frac{N_1}{r} \quad \frac{\partial N_1}{\partial z} \;\middle|\; \text{repeat for other N functions} \; \dots \right]$$

$N_p = [1]$, as there is only one pressure node

$$F = -2\pi \int N_p^T \frac{1}{K} N_p \cdot r \, d\Omega$$

K = bulk modulus = $\lambda + 2/3\, G$

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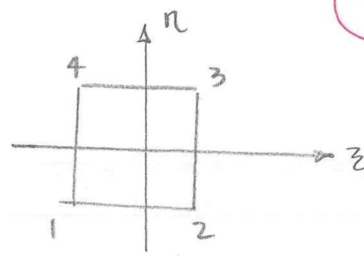
EXAM #2

+55

1. c. $G = -2\pi \int B_b^T N_p \cdot r \, dr$

when mapped,

$$G = -2\pi \int B_b^T N_p \, r \det(J) \, d\Omega$$



$$\det(J) = \frac{\Delta Z \cdot \Delta R}{4}$$

$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$

$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$

$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$

$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$

$$G = \frac{-\pi}{2} \Delta Z \Delta r \int_{-1}^{1}\int_{-1}^{1} B_b^T N_p \, r \, d\xi \, d\eta$$

For rows 1, 2, 7, 8,

$r = R$

For rows 3–6,

$r = R + \Delta R$

X

$r = R \cdot \left(\frac{1}{2} \cdot (1-\xi)\right) + (R+\Delta R)\left(\frac{1}{2} \cdot (1+\xi)\right)$

1: $\frac{-\pi}{2} \Delta Z \Delta R \, R \cdot N_p \iint r B_{b1} \, d\xi \, d\eta$

considering use of det(J), use

$$B_{b1} = \frac{\partial N_i}{\partial z} + \frac{N_i}{\cancel{R}\, r}$$

$\frac{\partial N_i}{\partial r} = \frac{\partial N_i}{\partial \xi} \cdot \frac{2}{\Delta R}$ (for this element)

$\longrightarrow \frac{-\pi}{2} \Delta Z \cdot \Delta R \cdot R \cdot N_p \int_{-1}^{1}\int_{-1}^{1} \left(-\frac{1}{4}(1-\eta) + \frac{N_1}{\cancel{R}\, r}\right) d\xi \, d\eta$

2: $-\frac{\pi}{2} \Delta Z \Delta R \cdot R \cdot N_p \iint \frac{\partial N_1}{\partial \eta} \, d\xi \, d\eta$

$\quad \rightarrow -\frac{1}{4}(1-\xi) \, d\xi \, d\eta$

$\frac{\partial N_i}{\partial z} = \frac{\partial N_i}{\partial \eta} \frac{2}{\Delta Z}$ (for this element)

$\longrightarrow \frac{\pi}{8} \Delta Z \Delta R \cdot R \cdot N_p \int_{-1}^{1}\int_{-1}^{1} (1-\xi) \, d\xi \, d\eta)$

Minimum number of Gauss points

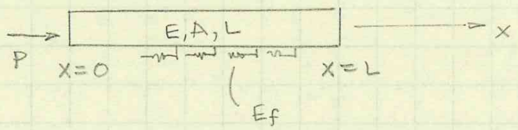
$= 2 \times 1$

with respect to ξ

with respect to η

ASSIGNMENT #1

100



$$\frac{d\sigma}{dx} - \frac{E_f}{A}u = 0 \;,\; \sigma(0) = \frac{-P}{A} \quad \sigma(L) = 0$$

1.

$$u(x) = \frac{P}{\lambda EA} \; \frac{e^{\lambda(x-L)} + e^{-\lambda(x-L)}}{e^{\lambda L} - e^{-\lambda L}} \;,\; \lambda = \sqrt{\frac{E_f}{EA}}$$ ← a constant

$$\varepsilon = \frac{du}{dx}$$

$$\varepsilon = \frac{P}{\lambda EA\left(e^{\lambda L} - e^{-\lambda L}\right)}\left[\lambda e^{\lambda(x-L)} - \lambda e^{-\lambda(x-L)}\right]$$

$$\sigma = E \cdot \varepsilon = \frac{P}{A} \; \frac{e^{\lambda(x-L)} - e^{-\lambda(x-L)}}{e^{\lambda L} - e^{-\lambda L}}$$

$$\sigma(0) = \frac{P}{A} \; \underbrace{\frac{e^{-\lambda L} - e^{\lambda L}}{e^{\lambda L} - e^{-\lambda L}}}_{-1}$$

$$\underline{\sigma(0) = \frac{-P}{A}}$$ ✓

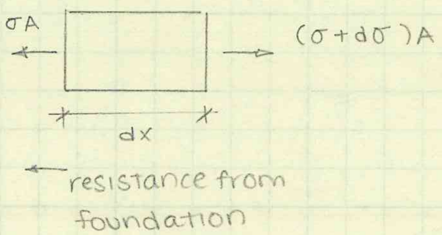
✓

$$\sigma(L) = \frac{P}{A} \; \frac{e^{0} - e^{0}}{e^{\lambda L} - e^{-\lambda L}} = 0$$

$$\underline{\sigma(L) = 0}$$ ✓

ASSIGNMENT #1

2.



$F = E_f u\,dx$ so: $\Sigma F = 0$:

$$(\sigma + d\sigma)A - \sigma A - E_f u\,dx = 0$$

$$d\sigma A - E_f u\,dx = 0$$

$$\frac{d\sigma}{dx} - \frac{E_f u}{A} = 0$$

Multiply by δu

$$\delta u \left(\frac{d\sigma}{dx} - \frac{E_f u}{A} \right) = 0$$

integrate over volume

$$A \int_0^L \delta u \left(\frac{d\sigma}{dx} - \frac{E_f u}{A} \right) dx = 0$$

$$= A \int_0^L \delta u \frac{d\sigma}{dx}\, dx - \int_0^L \delta u E_f u\, dx$$

using integration by parts,

$$= A \int_0^L \frac{d}{dx}(\delta u \cdot \sigma)\, dx - A \int_0^L \frac{d\delta u}{dx} \sigma\, dx - E_f \int_0^L \delta u\, u\, dx$$

$$= A \cdot \delta u \cdot \sigma \Big|_0^L - A \int_0^L \frac{d\delta u}{dx} \cdot \sigma\, dx - E_f \int_0^L \delta u \cdot u\, dx$$

define $\delta \varepsilon = \frac{d\delta u}{dx}$, as $\varepsilon = \frac{du}{dx}$

$$A\delta u(L)\sigma(L) - A\delta u(0)\sigma(0)$$

$$= \delta u_L P_L + \delta u_0 P_0$$

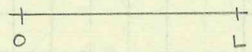
$$= \delta u_L P_L + \delta u_0 P_0 - A \int_0^L \delta\varepsilon\, \sigma\, dx - E_f \int_0^L \delta u \cdot u\, dx$$

$$A \int_0^L \delta \varepsilon\, \sigma\, dx = \delta u_L P_L + \delta u_0 P_0 - E_f \int_0^L \delta u \cdot u\, dx$$

✓

ASSIGNMENT #1

3.



– one element
– linear approximation

$$u = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{matrix} \leftarrow \text{disp. of node 1, at } x=0 \\ \leftarrow \text{disp. of node 2, at } x=L \end{matrix}$$

$$N_1 = \frac{x_2 - x}{x_2 - x_1}, \quad N_2 = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{du}{dx} = \varepsilon = \underline{\begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \frac{dN_1}{dx} = \frac{-1}{x_2 - x_1}, \quad \frac{dN_2}{dx} = \frac{1}{x_2 - x_1}$$

$$\sigma = \varepsilon E = \begin{bmatrix} E \end{bmatrix}$$

$$\delta u = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta u_2 \end{bmatrix}$$

$$\delta \varepsilon = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta u_2 \end{bmatrix}$$

From Part 2, governing equation is

$$A \int_0^L \delta\varepsilon \, \sigma \, dx = \delta u_L P_L - \delta u_0 P_0 - E_f \int_0^L \delta u \cdot u \, dx$$

$$A \int_0^L \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta u_2 \end{bmatrix} \begin{bmatrix} E \end{bmatrix} \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} dx =$$

$$\begin{bmatrix} \delta u_0 & \delta u_2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} - E_f \int_0^L \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta u_2 \end{bmatrix} \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} dx$$

Pull δu out from each part of equation

$$A \int_0^L \begin{bmatrix} \frac{dN_1}{dx} \\ \frac{dN_2}{dx} \end{bmatrix} \begin{bmatrix} E \end{bmatrix} \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} dx \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} - E_f \int_0^L \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \begin{bmatrix} N_1 & N_2 \end{bmatrix} dx \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Integrating,

$$\underbrace{\begin{bmatrix} \frac{EA}{L} & \frac{-EA}{L} \\ \frac{-EA}{L} & \frac{EA}{L} \end{bmatrix}}_{K} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} - \underbrace{E_f \int_0^L N^T N \, dx}_{\Gamma} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

ASSIGNMENT #1

3. (cont'd)

$$(K+\Gamma)\underline{u} = \underline{P}$$

$$P_1 = P, \quad P_2 = 0$$

$$(K+\Gamma)u = P$$

$$(K+\Gamma)^{-1}P = u$$

(from MathCAD), $\boxed{u = \frac{P}{EL}\begin{bmatrix} \frac{1600}{13} \\ \frac{1000}{13} \end{bmatrix} \text{in} \quad \text{single element linear approx.}}$

Exact solution:

$$u(x) = \frac{P}{\lambda EA}\,\frac{e^{\lambda(x-L)} + e^{-\lambda(x-L)}}{e^{\lambda L} - e^{-\lambda L}}, \quad \lambda = \sqrt{\frac{E_f}{EA}} = \left[\frac{E/100}{EL^2/100}\right]^{1/2} = \frac{1}{L}$$

at $x=0$,

$$u = \frac{PL}{EA}\,\frac{e^{-1}+e^{1}}{e^{1}-e^{-1}} = 1.313\frac{PL}{EA} = \underline{131.3\frac{P}{EL}}$$

at $x=L$,

$$u = \frac{PL}{EA}\,\frac{e^{0}+e^{0}}{e^{1}-e^{-1}} = 0.8509\frac{PL}{EA} = \underline{85.09\frac{P}{EL}}$$

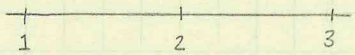
Error for single linear element

$$\frac{131.3 - 1600/13}{131.3} = 6.26\%$$

$$\frac{85.09 - 1000/13}{85.09} = 9.60\%$$

ASSIGNMENT #1

3. (cont'd) Two linear element approximation



Each individual matrix is the same:

$$K_e = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \frac{LE_f}{3} & \frac{LE_f}{6} \\ \frac{LE_f}{6} & \frac{LE_f}{3} \end{bmatrix}, \quad P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

need to be assembled

Also, note: $P_1 = P$

$P_2 = P_3 = 0$

$$K_{tot} = \begin{bmatrix} \frac{2EA}{L} & -\frac{2EA}{L} & 0 \\ -\frac{2EA}{L} & \frac{2EA}{L}(2) & -\frac{2EA}{L} \\ 0 & -\frac{2EA}{L} & \frac{2EA}{L} \end{bmatrix}$$

$$\Gamma_{tot} = LE_f \begin{bmatrix} 1/6 & 1/12 & 0 \\ 1/12 & 2(1/6) & 1/12 \\ 0 & 1/12 & 1/6 \end{bmatrix}, \quad P = \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{u = \frac{P}{EL}\begin{bmatrix} 129.20 \\ 93.88 \\ 83.05 \end{bmatrix} \text{in}}$$

129.20 ⟶ Error = 1.6%

two element linear approx.

83.05 ⟶ Error = 2.4%

ASSIGNMENT #1

3. (cont'd) Four linear elements

$$K = \frac{EA}{L}\begin{bmatrix} 4 & -4 & 0 & 0 & 0 \\ -4 & 8 & -4 & 0 & 0 \\ 0 & -4 & 8 & -4 & 0 \\ 0 & 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & -4 & 4 \end{bmatrix}, \quad \Gamma = \frac{L}{4}E_f\begin{bmatrix} 1/3 & 1/6 & 0 & 0 & 0 \\ 1/6 & 2/3 & 1/6 & 0 & 0 \\ 0 & 1/6 & 2/3 & 1/6 & 0 \\ 0 & 0 & 1/6 & 2/3 & 0 \\ 0 & 0 & 0 & 0 & 1/6 \end{bmatrix}$$

$$P = \begin{bmatrix} P \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{u = \frac{P}{EL}\begin{bmatrix} 130.77 \\ 109.64 \\ 95.43 \\ 87.25 \\ 84.58 \end{bmatrix} \text{in}}$$

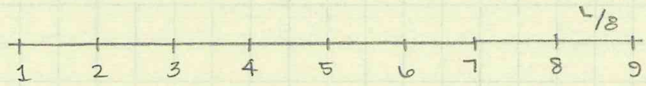
→ Error = 0.4% (130.77)

four element linear approximation

→ Error = 0.6% (84.58)

ASSIGNMENT #1

3. (cont'd) Eight linear elements



Same individual K, Γ matrices
combined matrix is similar

$$K = \frac{8EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \text{ etc}$$

$$\Gamma = \frac{L}{8}E_f \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 2/3 \end{bmatrix} \text{ etc.}$$

$$P = \begin{bmatrix} P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

solving,

$$\boxed{u = \frac{P}{EL}\begin{bmatrix} 131.17 \\ 119.67 \\ 110.04 \\ 102.13 \\ 95.82 \\ 91.02 \\ 87.64 \\ 85.63 \\ 84.96 \end{bmatrix} \text{ in}}$$

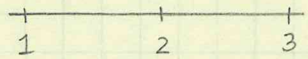
→ 0.10% error (131.17)

eight linear elements

→ 0.15% error (84.96)

ASSIGNMENT #1

3. (cont'd) Quadratic Approximation



$$N = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \delta u = \begin{bmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \end{bmatrix}$$

using the same governing equation,

$$K = \begin{bmatrix} EA \end{bmatrix} \int_0^L \begin{bmatrix} \frac{dN_1}{dx} \\ \frac{dN_2}{dx} \end{bmatrix} \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} dx$$

$$K = \frac{EA}{L} \begin{bmatrix} 7/3 & -8/3 & 1/3 \\ -8/3 & 16/3 & -8/3 \\ 1/3 & -8/3 & 7/3 \end{bmatrix}$$

$$\Gamma = E_f \int_0^L N^T N \, dx$$

$$\Gamma = LE_f \begin{bmatrix} 2/15 & 1/15 & -1/30 \\ 1/15 & 8/15 & 1/15 \\ -1/30 & 1/15 & 2/15 \end{bmatrix}$$

$$P = \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix}$$

$$(K+\Gamma)U = P$$

$$(K+\Gamma)^{-1} P = U$$

$$\boxed{u = \frac{P}{LE} \begin{bmatrix} 131.27 \\ 95.9 \\ 85.12 \end{bmatrix} \text{ in}}$$

$\rightarrow$ 0.02% error

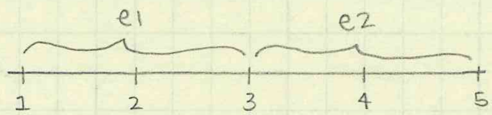
singe element quadratic approximation

$\rightarrow$ 0.04% error

✓

ASSIGNMENT #1

3. (cont'd) Two quadratic elements



$$K_{22} = \frac{EA}{L/2}\begin{bmatrix} 7/3 & -8/3 & 1/3 & 0 & 0 \\ -8/3 & 16/3 & -8/3 & 0 & 0 \\ -1/3 & -8/3 & \frac{7}{3}\cdot 2 & -8/3 & 1/3 \\ 0 & 0 & -8/3 & 16/3 & -8/3 \\ 0 & 0 & 1/3 & -8/3 & 7/3 \end{bmatrix}$$

$$\Gamma_{22} = \frac{L}{2}Ef\begin{bmatrix} 2/15 & 1/15 & -1/30 & 0 & 0 \\ 1/15 & 8/15 & 1/15 & 0 & 0 \\ -1/30 & 1/15 & 2/15\cdot 2 & 1/15 & -1/30 \\ 0 & 0 & 1/15 & 8/15 & 1/15 \\ 0 & 0 & -1/30 & 1/15 & 2/15 \end{bmatrix}, \quad P = \begin{bmatrix} P \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

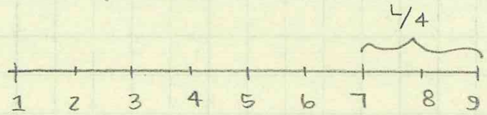
$$\boxed{u = \frac{P}{EL}\begin{bmatrix} 131.30 \\ 110.16 \\ 95.95 \\ 87.76 \\ 85.09 \end{bmatrix} \text{in}}$$

two quadratic element approx.

error ~0%

ASSIGNMENT #1

3. (cont'd) Four quadratic elements



$$K = \frac{EA}{L/4}\begin{bmatrix} 7/3 & -8/3 & 1/3 & & \\ -8/3 & 16/3 & -8/3 & & \\ 1/3 & -8/3 & \frac{7}{3}\cdot 2 & -8/3 & \\ & & -8/3 & 16/3 & \\ & & & & \text{etc.} \end{bmatrix}$$

$$\Gamma = \frac{L}{4} E_f \begin{bmatrix} 2/15 & 1/15 & -1/30 & & \\ 1/15 & 8/15 & 1/15 & & \\ -1/30 & 1/15 & 4/15 & 1/15 & -1/30 \\ & & 1/15 & 8/15 & \text{etc.} \end{bmatrix}$$

$$\boxed{u = \frac{P}{EL}\begin{bmatrix} 131.30 \\ 119.80 \\ 110.17 \\ 102.26 \\ 95.95 \\ 91.15 \\ 87.76 \\ 85.76 \\ 85.09 \end{bmatrix} \text{in}}$$

four quadratic elements

error is very close to 0%

✓

ASSIGNMENT #1

Summary of Part 3

	# of elem	u_o value, in	u_o Δval	u_L value, in	u_L Δval
LINEAR	1	123.076923	8.226605	76.92308	8.168736
	2	129.199372	2.104156	83.045526	2.046287
	4	130.774172	0.529356	84.579534	0.512279
	8	131.170977	0.132552	84.963692	0.128121

* u values multiplied by $\frac{P}{EL}$

Exact solution: $u_o = 131.3035285 \frac{P}{EL}$

$u_L = 86.09181282 \frac{P}{EL}$

	# of elem	u_o value, in	u_o Δval	u_L value, in	u_L Δval
QUADRATIC		131.273644	0.029884	85.119798	0.027985
		131.301153	0.002376	85.093123	0.00131
		131.303372	0.000157	85.091888	0.000075

4. Order of the error

$$\lim_{h \to 0} \frac{\text{error}(h)}{h^x} = \text{constant}$$

error is of order x

Linear elements, error on
u_o is of order 2.0
u_L is of order 2.0

Quadratic elements, error on
u_o is of order 3.9
u_L is of order 4.2

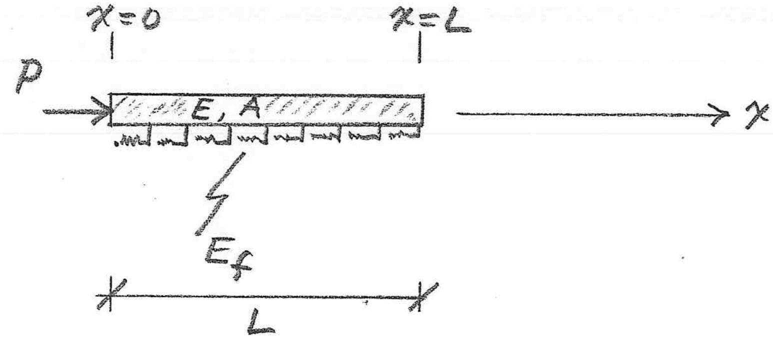
due: Friday 9/22

CE 381R

FALL 2006

Assignment 1

Consider the rod on "elastic foundation" shown below:



The governing differential equation is:

$$\frac{d\sigma}{dx} - \frac{E_f}{A} u = 0$$

At $x=0$ and $x=L$, the boundary conditions must be satisfied:

$$\sigma(0) = -\frac{P}{A}, \quad \sigma(L) = 0$$

1. Verify that
$$u(x) = \frac{P}{\lambda EA} \, \frac{e^{\lambda(x-L)} + e^{-\lambda(x-L)}}{e^{\lambda L} - e^{-\lambda L}} \qquad \left(\lambda = \sqrt{\frac{E_f}{EA}}\right)$$
satisfies the differential equation and boundary conditions.
2. Derive the Principle of Virtual Work.
3. Apply the finite-element method with meshes of 1, 2, 4 and 8 equal-size linear finite elements and 1, 2 and 4 equal-size quadratic finite elements. Record the approximations of $u(0)$ and $u(L)$ in each finite-element solution.
4. Examine the errors in $u(0)$ and $u(L)$ as functions of the element size in the linear case. Estimate the order of the errors as the element size decreases. Carry out the same tasks in the quadratic case.

Use: $E_f = \frac{1}{100} E$, $A = \frac{1}{100} L^2$

CATHERINE HOVELL
29 September 06

1/5

100

CE 381R: Finite Element Analysis

Summary of Results for Assignment 2

Eight linear elements, original stiffness matrix:

NODE	U1
1	131.2
2	119.7
3	110.0
4	102.1
5	95.82
6	91.02
7	87.64
8	85.63
9	84.96

lumped stiffness matrix:

NODE	U1
1	131.1
2	119.6
3	110.0
4	102.1
5	95.83
6	91.04
7	87.67
8	85.66
9	85.00

**

Four quadratic elements, original stiffness matrix:

NODE	U1
1	131.3
2	119.8
3	110.1
4	102.2
5	95.92
6	91.12
7	87.74
8	85.73
9	85.06

lumped stiffness matrix:

NODE	U1
1	131.4
2	119.9
3	110.2
4	102.3
5	96.01
6	91.20
7	87.82
8	85.81
9	85.15

	u_0		u_L	
	regular	lumped	regular	lumped
LINEAR	131.2	131.1	84.96	85.00
QUADRATIC	131.3	131.4	85.06	85.15

$u_{0exact} = 131.3035$

$u_{Lexact} = 86.0918$

✓

C.HOVELL
29 Sept 06

2/5


```
*heading
finite homework 2, 8 linear elements
**
*preprint, echo=yes, model=yes, history=yes
**
*node, nset=rod
1, 0
9, 1.0
*ngen, nset=rod
1,9,1
**
*user element, type=u1, nodes=2, linear
1
**
*matrix, type=stiffness
0.08041667
-0.07979167,0.08041667
**
*element, type=u1, elset=rod
1, 1,2
*elgen, elset=rod
1, 8,1,1
**
*uel property, elset=rod
**
*step, perturbation
*static
*cload
1,1,1
**
*node file, nset=rod
U
*node print, nset=rod
U
**
*end step
```

C.HOVELL
29 Sept 06

3/5

HW2linear2.txt

```
*heading
finite homework 2, 8 linear elements, lumped matrix
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**
*node, nset=rod
1, 0
9, 1.0
*ngen, nset=rod
1,9,1
**
*user element, type=u1, nodes=2, linear
1
**
*matrix, type=stiffness
0.080625
-0.08,0.080625
**
*element, type=u1, elset=rod
1, 1,2
*elgen, elset=rod
1, 8,1,1
**
*uel property, elset=rod
**
*step, perturbation
*static
*cload
1,1,1
**
*node file, nset=rod
U
*node print, nset=rod
U
**
*end step
```

C.HOVELL
29Sept06

4/5


```
*heading
finite homework 2, 4 quadratic elements
**
*preprint, echo=yes, model=yes, history=yes
**
*node, nset=rod
1, 0
9, 1.0
*ngen, nset=rod
1,9,1
**
*user element, type=u1, nodes=3, linear
1
**
*matrix, type=stiffness
0.093667
-0.1065,0.2146667
0.01325,-0.1065,0.093667
**
*element, type=u1, elset=rod
1, 1,2,3
*elgen, elset=rod
1, 4,2,1
**
*uel property, elset=rod
**
*step, perturbation
*static
*cload
1,1,1
**
*node file, nset=rod
U
*node print, nset=rod
U
**
*end step
```

C.HOVELL
29Sept 06

5/5


```
*heading
finite homework 2, 4 quadratic elements, lumped matrix
**
*preprint, echo=yes, model=yes, history=yes
**
*node, nset=rod
1, 0
9, 1.0
*ngen, nset=rod
1,9,1
**
*user element, type=u1, nodes=3, linear
1
**
*matrix, type=stiffness
0.09375
-0.106667,0.215
0.013333,-0.1066667,0.09375
**
*element, type=u1, elset=rod
1, 1,2,3
*elgen, elset=rod
1, 4,2,1
**
*uel property, elset=rod
**
*step, perturbation
*static
*cload
1,1,1
**
*node file, nset=rod
U
*node print, nset=rod
U
**
*end step
```

CE 381R

FALL 2006

Assignment 2

Reconsider the rod on "elastic foundation" examined in Assignment 1. Specifically,

1. Use ABAQUS with input files similar to the ones discussed in class to analyze the rod with 8 equal-size linear finite elements and 4 equal-size quadratic finite elements. Specify the stiffness matrix of the elements using the *MATRIX keyword.

2. Repeat the tasks of Part 1 after replacing the matrix contributed by the elastic foundation with its "lumped" version, i.e., the one resulting from adding all off-diagonal entries in each row to the diagonal entry and then replacing the off-diagonal entries with 0's.

3. Submit copies of your input files along with a summary of results extracted from the ABAQUS .dat files for the displacements at $x=0$ and $x=L$.

ASSIGNMENT #3

CATHERINE HOVELL
09 October 06

1/2

100

CE 381R: Finite Element Analysis

Summary of Results for Assignment 3

1 Element:

Node Label	U.U1 @Loc 1
1	80.
2	120.

2 Element:

Node Label	U.U1 @Loc 1
1	83.6601
3	128.105

4 Element:

Node Label	U.U1 @Loc 1
1	84.7221
5	130.476

8 Element:

Node Label	U.U1 @Loc 1
1	84.9986
9	131.095

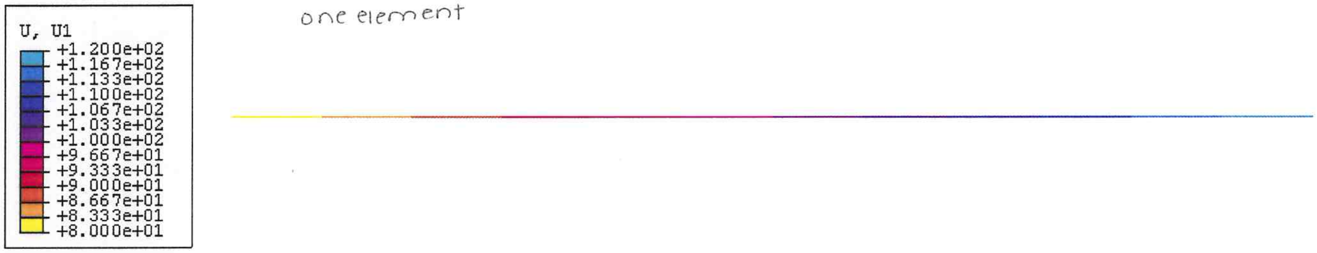
All values multiplied by P / EL

CE381R: FINITE

ASSIGNMENT #3

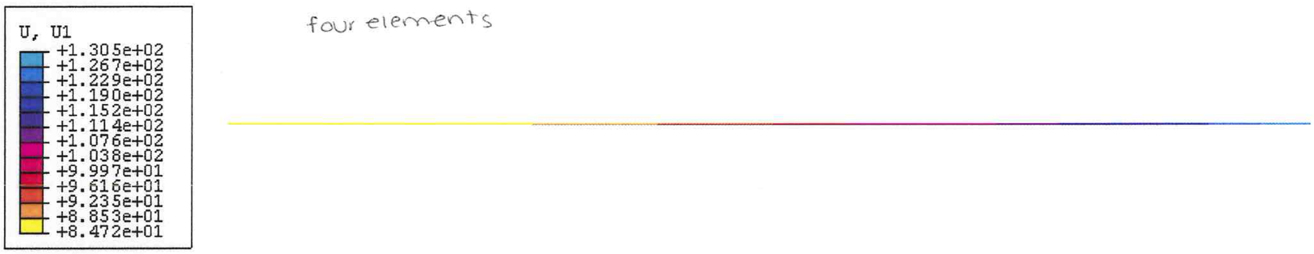
C. HOVELL
09 October 06

one element



two elements

four elements



eight elements

due 10/9

CE 381R

FALL 2006

Assignment 3

Apply ABAQUS CAE to the calculation of the displacements at $x=0$ and $x=L$ of the rod on elastic foundation examined in Assignments 1 and 2. Specifically,

* use the "lumped" version of the matrix contributed by the elastic foundation
* select the "T2D2" element for "meshing"
* using the "Springs/Dashpots" item (accessible through "Engineering Features" of the Part Module), specify springs at the points where the nodes are located with stiffnesses equal to the diagonal entries of the assembled elastic foundation stiffness matrix
* analyze the rod with meshes of 1, 2, 4 and 8 elements (and corresponding numbers of springs)
* use the "Visualization" module to extract the results for the displacements at $x=0$ and $x=L$
* submit a summary of the results

CATHERINE HOVELL
18 October 06

1/1

100

$u \times \frac{qR}{E}$

$\sigma \times q$

$q = $ load

CE381R: Finite Element Analysis

Assignment #4

Linear Analysis, seeding at 1/4R

u(3R,0): -1.92991
v(0,5R): 6.1098
Sig_y(R,0): 2.9223

**

Linear Analysis, seeding at 1/8R

u(3R,0): -1.98341
v(0,5R): 6.15504
Sig_y(R,0): 3.43727

**

Linear Analysis, seeding at 1/16R

u(3R,0): -1.9985
v(0,5R): 6.1675
Sig_y(R,0): 3.33328

**
**

Quadratic Analysis, seeding at 1/4R

u(3R,0): -1.94316
v(0,5R): 6.12568
Sig_y(R,0): 3.51794

**

Quadratic Analysis, seeding at 1/8R

u(3R,0): -1.99206
v(0,5R): 6.16266
Sig_y(R,0): 3.57173

**

Quadratic Analysis, seeding at 1/16R

u(3R,0): -2.00153
v(0,5R): 6.16944
Sig_y(R,0): 3.56669

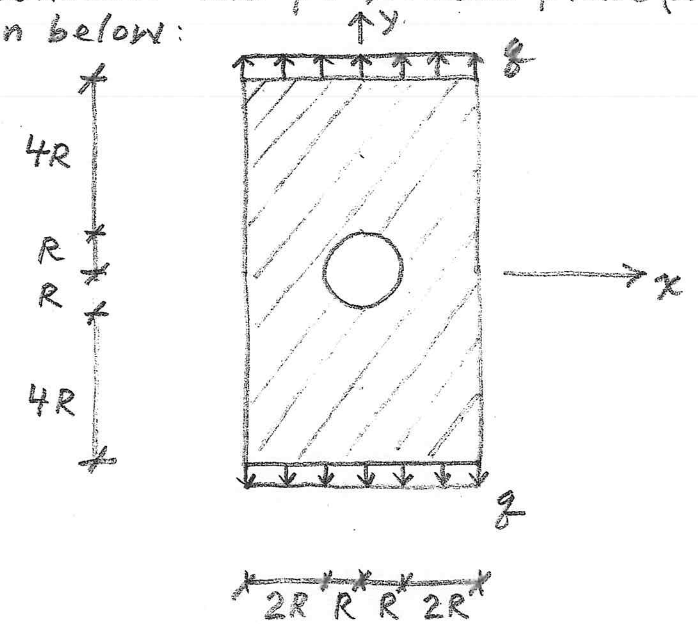
due 10/18

CE 381R

FALL 2006

Assignment 4

Consider the perforated plate (in plane stress) shown below:



E

$\nu = 0.3$

Using ABAQUS CAE and taking advantage of symmetry conditions, analyze the plate with meshes of triangles of size $\frac{1}{4}R$, $\frac{1}{8}R$ and $\frac{1}{16}R$ (specified through the "Seed Part" selection in the "Seed" Menu). Repeat the analysis with meshes of quadrilaterals of the same size. Extract and summarize the results for

$$u(3R, 0)$$

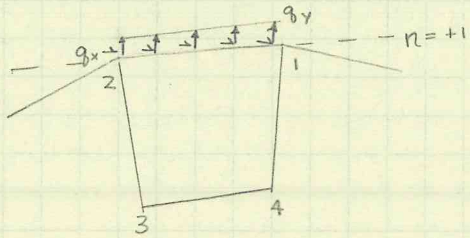
$$v(0, 5R)$$

$$\sigma_y(R, 0)$$

ASSIGNMENT #5

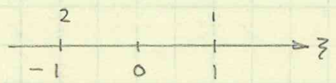
98

1.



$$T = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$$ – constant values when boundary is straight

$$P = d\int_B N^T T \, dB \quad \text{or} \quad d\int_0^L N^T T \, dL$$



$$N_1 = \frac{1}{2}(1+\xi)$$
$$N_2 = \frac{1}{2}(1-\xi)$$

$$L = \left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right]^{1/2}$$

$$\frac{dL}{d\xi} = \left[\frac{dx}{d\xi}^2 + \frac{dy}{d\xi}^2\right]^{1/2}$$

$$x = \sum_{i=1}^{n} (Nx)_i$$

$$x = \frac{1}{2}\left[(1+\xi)x_1 + (1-\xi)x_2\right]$$

$$\frac{dx}{d\xi} = \frac{1}{2}x_1 - \frac{1}{2}x_2$$

Similarly,

$$y = \sum (Ny)_i$$

$$y = \frac{1}{2}y_1 + \frac{\xi}{2}y_1 + \frac{1}{2}y_2 - \frac{\xi}{2}y_2$$

$$\frac{dy}{d\xi} = \frac{y_1}{2} - \frac{y_2}{2}$$

$$\frac{dL}{d\xi} = \left[\frac{1}{4}(x_1 - x_2)^2 + \frac{1}{4}(y_1 - y_2)^2\right]^{1/2} = \frac{1}{2}L$$

$dL = \frac{dL}{d\xi} \cdot d\xi$, so original equation is now:

$$P = d\int_{-1}^{1} N^T T \cdot \frac{L}{2} \, d\xi$$

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix}$$

ASSIGNMENT #5

1. (cont'd)

$$P = \frac{dL}{2} \int_{-1}^{1} N^T T \, d\xi$$

$$P = \frac{L}{2} \cdot d \int_{-1}^{1} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} q_x \\ q_y \end{bmatrix} d\xi = \frac{L}{2} d \int_{-1}^{1} \begin{bmatrix} N_1 q_x \\ N_1 q_y \\ N_2 q_x \\ N_2 q_y \end{bmatrix} d\xi$$

$$N_1 q_x = \left(\frac{1}{2} + \frac{\xi}{2}\right) q_x$$

$$\int N_1 q_x \, d\xi = \frac{1}{2} q_x \left(\xi + \xi^2/2\right)$$

$$N_2 q_y = \left(\frac{1}{2} - \frac{\xi}{2}\right) q_y$$

$$\int N_2 q_y \, d\xi = \frac{1}{2} q_y \left(\xi - \xi^2/2\right)$$

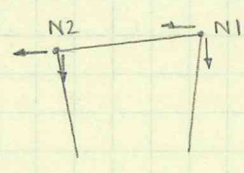
$$P = \frac{L}{2} \cdot d \begin{bmatrix} (-1 + 1/2 - 1 - 1/2) \frac{q_x}{2} \\ -q_y \\ \frac{q_y}{2}(-1 - 1/2 - 1 + 1/2) \\ -q_y \end{bmatrix}$$

← evaluation of integral from $\xi = -1$ to $\xi = +1$

Nodal Forces:

$$P = \frac{\overset{+}{\cancel{-}}\, d \cdot L}{2} \begin{bmatrix} q_x \\ q_y \\ q_x \\ q_y \end{bmatrix} \begin{matrix} \}\, N_1 \\ \\ \}\, N_2 \end{matrix}$$

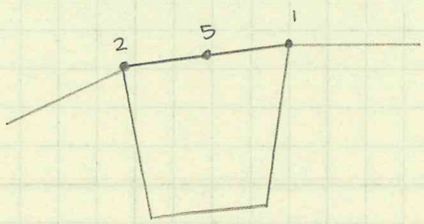
✓



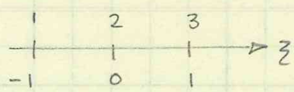
−2

ASSIGNMENT #5

1. Part 2 – three nodes



$$T = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$$



$$N_1 = \frac{1}{2}\xi(\xi-1) \qquad \int N_1\,d\xi = \frac{\xi^3}{6} - \frac{\xi^2}{4}$$

$$N_2 = 1-\xi^2 \qquad \int N_2\,d\xi = \xi - \xi^3/3$$

$$N_3 = \frac{1}{2}\xi(\xi+1) \qquad \int N_3\,d\xi = \frac{\xi^3}{6} + \frac{\xi^2}{4}$$

$$\frac{dL}{d\xi} = \left[\frac{dx}{d\xi}^2 + \frac{dy}{d\xi}^2\right]^{1/2}, \quad x = \sum Nx = \xi/2(\xi-1)x_1 + (1-\xi^2)x_2 + \frac{\xi}{2}(\xi-1)x_3$$

$$= \frac{\xi^2}{2}(x_1+x_3) - \frac{\xi}{2}(x_1+x_3) + x_2 - \xi^2 x_2$$

considering $x_2 = \dfrac{x_3-x_1}{2}$

$$x = x_2 - \xi x_2$$

similarly, $y = y_2 - \xi y_2$

$$\frac{dx}{d\xi} = -x_2, \quad \frac{dy}{d\xi} = -y_2$$

$$\frac{dL}{d\xi} = \sqrt{\frac{1}{4}(x_3-x_1)^2 + \frac{1}{4}(y_3-y_1)^2}$$

$$L = \sqrt{(x_3-x_1)^2 + (y_3-y_1)^2}$$

so, $\dfrac{dL}{d\xi} = \dfrac{L}{2}$

$$N^T T = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} q_x \\ q_y \end{bmatrix}$$

$$\int_{-1}^{1} N_1 q_x = {}^{-1}/_3\, q_x$$

$$\int_{-1}^{1} N_2 q_x = {}^{-4}/_3\, q_x$$

$$\int_{-1}^{1} N_3 q_x = {}^{-1}/_3\, q_x$$

same for q_y

Summation of forces $= -2$ in each direction, same as in 2-node example.

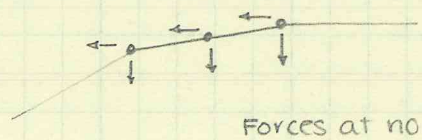
ASSIGNMENT #5

1. Part 2 (cont'd)

Putting it all together,

$$P = d \int N^T T \, dL$$

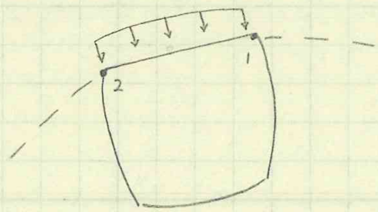
$$\boxed{P = \overset{+}{-} d \, \frac{L}{6} \begin{bmatrix} q_x \\ q_y \\ 4q_x \\ 4q_y \\ q_x \\ q_y \end{bmatrix}}$$



Forces at node 2 are 4x those at nodes 1,3

ASSIGNMENT #5

2.



$$T = \begin{bmatrix} -pn_x \\ -pn_y \end{bmatrix}$$

$\vec{n}$ is normal to surface between points

$$v_{12} = \left(\frac{dx}{dL} \quad \frac{dy}{dL} \quad 0 \right)$$

$$S_{\substack{out \\ of\ pg}} = (0 \quad 0 \quad 1)$$

$$n = S \times v_{12}$$

$$n = \det \begin{bmatrix} i & j & k \\ 0 & 0 & 1 \\ \frac{dx}{dL} & \frac{dy}{dL} & 0 \end{bmatrix}$$

$$n = -\frac{dy}{dL} i + \frac{dx}{dL} j + 0k$$

$$\frac{dy}{dL} = \frac{y_1 - y_2}{L}, \quad \frac{dx}{dL} = \frac{x_1 - x_2}{L}$$

$$n = \begin{bmatrix} \frac{y_2 - y_1}{L} \\ \frac{x_1 - x_2}{L} \end{bmatrix}, \quad T = \frac{-p}{L} \begin{bmatrix} y_2 - y_1 \\ x_1 - x_2 \end{bmatrix}$$

Now, consider Gauss approximations

order = 1, so $2n - 1 > 1$

$n = 1$ — one gauss integration pt.

$$\int_{-1}^{1} F(\xi)\, d\xi = 2 \cdot F(0)$$

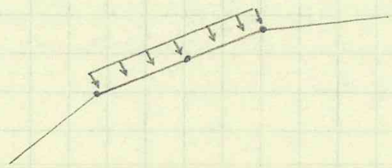
$$P = \frac{-dp}{4} \int_{-1}^{1} \begin{bmatrix} (1+\xi)(y_2 - y_1) \\ (1+\xi)(x_1 - x_2) \\ (1-\xi)(y_2 - y_1) \\ (1+\xi)(x_1 - x_2) \end{bmatrix} d\xi$$

→ general equation modified from problem #1

$$P = \frac{-dp}{2} \begin{bmatrix} y_2 - y_1 \\ x_1 - x_2 \\ y_2 - y_1 \\ x_1 - x_2 \end{bmatrix} \quad \text{or} \quad \boxed{P = \frac{dp}{2} \begin{bmatrix} y_1 - y_2 \\ x_2 - x_1 \\ y_1 - y_2 \\ x_2 - x_1 \end{bmatrix}}$$

ASSIGNMENT #5

2. 3 nodes along straight edge



$$T, \text{ from 2-node} = \frac{P}{L}\begin{bmatrix} y_1 - y_3 \\ x_3 - x_1 \end{bmatrix}$$

$N_1 = \frac{1}{2}\xi(\xi - 1)$

$N_2 = 1 - \xi^2$

$N_3 = \frac{1}{2}\xi(\xi + 1)$

since ξ has order 2, must increase number of gauss integration pts.

$$2n - 1 > 2, \quad n = 2$$

$$\int_{-1}^{1} F(\xi)\,d\xi = F\left(\frac{-1}{\sqrt{3}}\right) + F\left(\frac{1}{\sqrt{3}}\right)$$

$$P = \frac{dp}{2}\begin{bmatrix} \frac{1}{2}\xi(\xi-1)(y_1 - y_3) \\ (x_3 - x_1) \\ (1-\xi^2)(y_1 - y_3) \\ (x_3 - x_1) \\ \frac{1}{2}\xi(\xi+1)(y_1 - y_3) \\ (x_3 - x_1) \end{bmatrix}$$

evaluated at $-1/\sqrt{3}$, $1/\sqrt{3}$

$$\frac{1}{2}\xi(\xi - 1) = \frac{1}{2}\left[\left(-1/\sqrt{3}\right)^2 + 1/\sqrt{3} + \left(1/\sqrt{3}\right)^2 - 1/\sqrt{3}\right]$$

$$= \frac{1}{2}\left(2/3\right) = 1/3$$

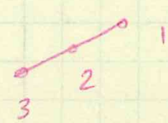
$$(1 - \xi^2) = 1 - \left(-1/\sqrt{3}\right)^2 + 1 - \left(1/\sqrt{3}\right)^2$$

$$= 2 - 2/3 = 4/3$$

$$\frac{1}{2}\xi(\xi + 1) = \frac{1}{2}\left(2/3\right) = 1/3$$

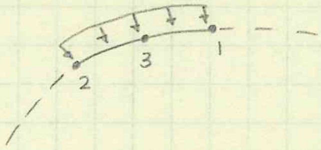
thus,

$$\boxed{P = \frac{dp}{6}\begin{bmatrix} y_1 - y_3 \\ x_3 - x_1 \\ 2(y_1 - y_3) \\ 2(x_3 - x_1) \\ y_1 - y_3 \\ x_3 - x_1 \end{bmatrix}}$$



ASSIGNMENT #5

2. curved surface



Previously, found $\frac{dL}{d\xi}$, said $x_2 = \frac{x_3 - x_1}{2}$, etc.

Not true anymore.

So, $\frac{dL}{d\xi} = \frac{L}{2}$ no longer applicable

$$\frac{dL}{d\xi} = \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}$$

$$P = d \int N^T T \, dL$$

$N_1 = \frac{1}{2}\xi(\xi+1)$

$N_2 = 1 - \xi^2$

$N_3 = \frac{1}{2}\xi(\xi-1)$

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

$$T = p \begin{bmatrix} \frac{dy}{dL} \\ \frac{-dx}{dL} \end{bmatrix}$$

$$P = d \int_{-1}^{1} \begin{bmatrix} \frac{1}{2}\xi(\xi+1) p \, dy/dL \\ \frac{1}{2}\xi(\xi+1) p \, -dx/dL \\ (1-\xi^2) p \, dy/dL \\ (1-\xi^2) p \, -dx/dL \\ \frac{1}{2}\xi(\xi-1) p \, dy/dL \\ \frac{1}{2}\xi(\xi-1) p \, -dx/dL \end{bmatrix} dL$$

Looking at each term individually,

$$\frac{1}{2}\xi(\xi+1) \frac{-dy}{dL} \cdot p \cdot dL$$

can be rewritten as $\frac{dy}{d\xi} \cdot d\xi$

include $dy/d\xi$ in integral, integrate in $d\xi$

$$x = \frac{\xi}{2}(\xi+1)x_1 + (1-\xi^2)x_2 + \frac{\xi}{2}(\xi-1)x_3$$

$$\frac{dx}{d\xi} = \xi x_1 + \frac{x_1}{2} - 2\xi x_2 + \xi x_3 - \frac{x_3}{2}$$

$$\frac{dy}{d\xi} = \xi y_1 + \frac{y_1}{2} - 2\xi y_2 + \xi y_3 - \frac{y_3}{2}$$

ASSIGNMENT #5

2. Part 3 (cont'd)

$$-\tfrac{1}{2}\xi(\xi+1)\cdot p\cdot(\xi y_1 + y_1/2 - 2\xi y_2 + \xi y_3 - y_3/2)\,d\xi$$

Equation is 3rd order, use 2 gauss integration points

$$\int_{-1}^{1} F(\xi)\,d\xi = F(1/\sqrt{3}) + F(-1/\sqrt{3})$$

To make equations neater,
- remove constant $\tfrac{1}{2}p$
- use alternate constants:

$$a = y_1 - 2y_2 + y_3$$
$$b = \tfrac{1}{2}(y_1 - y_3)$$

$$(\xi + \xi^2)(\xi a + b) = a\xi^2 + b\xi^2 + b\xi + a\xi^3$$

Evaluate at $1/\sqrt{3}, -1/\sqrt{3}$

$$a(1/3) + b(1/3) + b(-1/\sqrt{3}) + a(-1/\sqrt{3}) +$$
$$a(1/3) + b(1/3) + b\cancel{(1/\sqrt{3})} + a\cancel{(1/\sqrt{3})}$$

$$= 2/3a + 2/3b \quad \text{all} \times -1/2p$$

First term in matrix: $1/3p\left[(y_1 - 2y_2 + y_3) + \tfrac{1}{2}y_1 - \tfrac{1}{2}y_3\right]$

or $1/3p\left[3y_1/2 - 2y_2 + y_3/2\right] = \underline{p/6\,(3y_1 - 4y_2 + y_3)}$

similarly, second term $= \underline{-p/6\,(3x_1 - 4x_2 + x_3)}$

Third term: $p(1-\xi^2)(\xi a + b) = a\xi - a\xi^3 - b\xi^2 + b$

$$a\cancel{(-1/\sqrt{3})} - a\cancel{(-1/\sqrt{3})} - b(1/3) + b + a\cancel{(1/\sqrt{3})} - a\cancel{(1/\sqrt{3})} - b/3 + b$$

$$= p\cdot(2b - 2/3b)$$
$$= p\cdot\tfrac{4}{3}\cdot\tfrac{1}{2}(y_1 - y_3) = \underline{\tfrac{2}{3}p(y_1 - y_3)}$$

and the 4th term $= \underline{-\tfrac{2}{3}p(x_1 - x_3)}$

ASSIGNMENT #5

2. Part 3 (cont'd)

Fifth term: $\frac{1}{2}p \cdot (\xi^2 - \xi)(\xi a + b)$

$$a\xi^3 - a\xi^2 + b\xi^2 - b\xi, \quad \text{from } -1/\sqrt{3} \text{ to } 1/\sqrt{3}:$$

$$\left[-a(1/3) + b(1/3)\right](2) = 2/3\,b - 2/3\,a$$

$$2/3 \cdot \frac{1}{2}(y_1 - y_3) - 2/3\,(y_1 - 2y_2 + y_3)$$

$$= \frac{2}{3}\left(3/2\,y_1 - 2y_2 + 1/2\,y_3\right)$$

$$= \frac{1}{3}(3y_1 - 4y_2 + y_3) \quad \text{all} \times 1/2\,p$$

in summation,

$$\boxed{P = \frac{p \cdot d}{6}\begin{bmatrix} 3y_1 - 4y_2 + y_3 \\ -(3x_1 - 4x_2 + x_3) \\ 4(y_1 - y_3) \\ -4(x_1 - x_3) \\ 3y_1 - 4y_2 + y_3 \\ -(3x_1 - 4x_2 + x_3) \end{bmatrix}}$$

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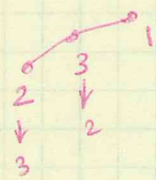

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STAEDTLER® No. 937 811E Engineer's Computation Pad

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CE 381R

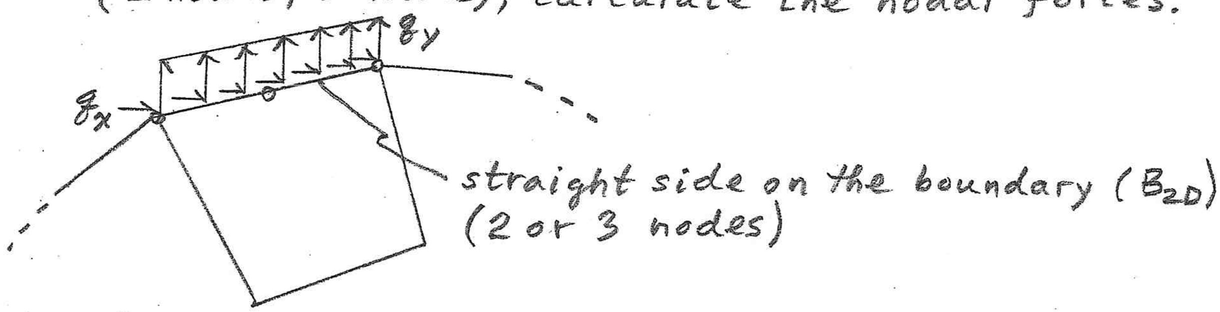
FALL 2006

Assignment 5

1. Consider constant traction (uniformly distributed load):

$$T_x = q_x \ , \ T_y = q_y$$

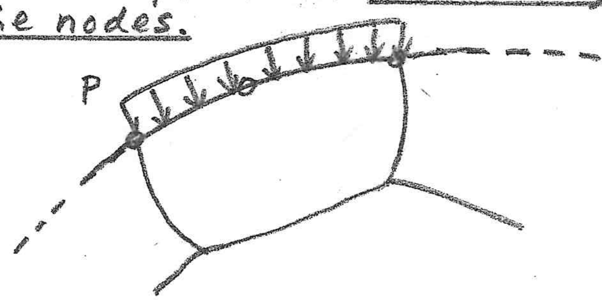
on a straight side of a finite element. Let there be 2 nodes (one at each end) or 3 nodes (one at each end and one at the middle) on the side and let L be the length of the side. In each case (2 nodes, 3 nodes), calculate the nodal forces.



2. Consider constant pressure (uniformly distributed load perpendicular to the boundary):

$$T_x = -p \cdot n_x \ , \ T_y = -p \cdot n_y$$

where n_x, n_y are the components of the outward unit vector, normal to the boundary (p is the level of pressure, positive in the inward direction). The pressure is acting on a side of a finite element. Examine the case of 2 nodes on the side (straight side) and the case of 3 nodes (straight or curved side). Obtain expressions for the nodal forces. Using the minimum number of Gauss-Legendre integration points, evaluate these expressions in terms of the coordinates of the nodes.



CE 381R

FALL 2006

Assignment 5: Solution

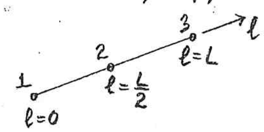
1. The nodal forces due to traction are given by

$$d \cdot \int_{B_{2D}} \underset{\sim}{N}^T \underset{\sim}{T} \, dB_{2D}$$

Along a side of an element, the integral can be evaluated as

$$d \cdot \int_0^L \underset{\sim}{N}^T \underset{\sim}{T} \, dl$$

where l denotes distance along the side measured from one end, say, node 1:



For the purpose of carrying out the integration, it is convenient to map the $[0, L]$ interval onto the "standard" interval $[-1, 1]$:

$$d \cdot \int_0^L \underset{\sim}{N}^T \underset{\sim}{T} \, dl = d \cdot \int_{-1}^{1} \underset{\sim}{N}^T \underset{\sim}{T} \frac{dl}{d\xi} d\xi$$

In the case of a straight side (with node 2 at the middle of the segment 13), the scale factor is constant:

$$\frac{dl}{d\xi} = \frac{L}{2}$$

The matrix of interpolation functions is:

2 nodes

$$\underset{\sim}{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix}$$

$$N_1 = \frac{1}{2}(1-\xi)$$

$$N_2 = \frac{1}{2}(1+\xi)$$

3 nodes

$$\underset{\sim}{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

$$N_1 = \frac{1}{2}\xi \cdot (\xi - 1)$$

$$N_2 = 1 - \xi^2$$

$$N_3 = \frac{1}{2}\xi \cdot (\xi + 1)$$

Considering constant traction:

$$\underset{\sim}{T} = \begin{bmatrix} g_x \\ g_y \end{bmatrix}$$

the nodal forces are found to be:

2 nodes

$$\begin{bmatrix} d \cdot \frac{L}{2} \int_{-1}^{1} N_1 g_x \, d\xi \\ d \cdot \frac{L}{2} \int_{-1}^{1} N_1 g_y \, d\xi \\ d \cdot \frac{L}{2} \int_{-1}^{1} N_2 g_x \, d\xi \\ d \cdot \frac{L}{2} \int_{-1}^{1} N_2 g_y \, d\xi \end{bmatrix} = \begin{bmatrix} g_x \cdot d \cdot \frac{L}{2} \\ g_y \cdot d \cdot \frac{L}{2} \\ g_x \cdot d \cdot \frac{L}{2} \\ g_y \cdot d \cdot \frac{L}{2} \end{bmatrix}$$

$$\left(\text{since } \int_{-1}^{1} N_1 \, d\xi = 1, \ \int_{-1}^{1} N_2 \, d\xi = 1 \right)$$

3 nodes

$$\begin{bmatrix} d \cdot \frac{L}{2} \int_{-1}^{1} N_1 g_x \, d\xi \\ d \cdot \frac{L}{2} \int_{-1}^{1} N_1 g_y \, d\xi \\ d \cdot \frac{L}{2} \int_{-1}^{1} N_2 g_x \, d\xi \\ d \cdot \frac{L}{2} \int_{-1}^{1} N_2 g_y \, d\xi \\ d \cdot \frac{L}{2} \int_{-1}^{1} N_3 g_x \, d\xi \\ d \cdot \frac{L}{2} \int_{-1}^{1} N_3 g_y \, d\xi \end{bmatrix} = \begin{bmatrix} \frac{1}{6} g_x \cdot d \cdot L \\ \frac{1}{6} g_y \cdot d \cdot L \\ \frac{2}{3} g_x \cdot d \cdot L \\ \frac{2}{3} g_y \cdot d \cdot L \\ \frac{1}{6} g_x \cdot d \cdot L \\ \frac{1}{6} g_y \cdot d \cdot L \end{bmatrix}$$

$$\left(\text{since } \int_{-1}^{1} N_1 \, d\xi = \frac{1}{3}, \ \int_{-1}^{1} N_2 \, d\xi = \frac{4}{3}, \ \int_{-1}^{1} N_3 \, d\xi = \frac{1}{3} \right)$$

2. The calculation of nodal forces due to pressure can be carried out in the same manner:

$$d\cdot\int_0^L \underset{\sim}{N}^T \underset{\sim}{T}\, dl = d\cdot\int_{-1}^{1} \underset{\sim}{N}^T \underset{\sim}{T}\,\frac{dl}{d\xi}\, d\xi$$

In the case of a <u>straight side</u>, the traction is constant (since the outward unit vector, normal to the boundary, is constant) and the calculation proceeds as in Part 1 with

$$g_x = -p\cdot n_x$$

$$g_y = -p\cdot n_y$$

To determine n_x and n_y, we take the cross product* of $\begin{bmatrix}0\\0\\1\end{bmatrix}$ and a vector <u>tangent to the side</u>:

<u>2 nodes</u>

$$\begin{bmatrix}0\\0\\1\end{bmatrix}\times\begin{bmatrix}x_2-x_1\\y_2-y_1\\0\end{bmatrix}=\begin{bmatrix}-(y_2-y_1)\\(x_2-x_1)\\0\end{bmatrix}$$

<u>3 nodes</u>

$$\begin{bmatrix}0\\0\\1\end{bmatrix}\times\begin{bmatrix}x_3-x_1\\y_3-y_1\\0\end{bmatrix}=\begin{bmatrix}-(y_3-y_1)\\(x_3-x_1)\\0\end{bmatrix}$$

Dividing by the length of this vector (since $\underset{\sim}{n}$ is a unit vector), the components, n_x, n_y are found to be:

<u>2 nodes</u>

$$n_x = -\frac{y_2-y_1}{L},\quad n_y = \frac{x_2-x_1}{L}\quad\left(L=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\right)$$

<u>3 nodes</u>

$$n_x = -\frac{y_3-y_1}{L},\quad n_y = \frac{x_3-x_1}{L}\quad\left(L=\sqrt{(x_3-x_1)^2+(y_3-y_1)^2}\right)$$

*We are assuming that the exterior of the domain is to the left as we traverse the side from the first node to the last.

The nodal forces are then given by

<u>2 nodes</u>

$$\begin{bmatrix} d\cdot\frac{L}{2}\int_{-1}^{1} N_1\cdot(-pn_x)\,d\xi \\ d\cdot\frac{L}{2}\int_{-1}^{1} N_1\cdot(-pn_y)\,d\xi \\ d\cdot\frac{L}{2}\int_{-1}^{1} N_2\cdot(-p\cdot n_x)\,d\xi \\ d\cdot\frac{L}{2}\int_{-1}^{1} N_2\cdot(-pn_y)\,d\xi \end{bmatrix} = \begin{bmatrix} \frac{1}{2}p\cdot d\cdot(y_2-y_1) \\ -\frac{1}{2}p\cdot d\cdot(x_2-x_1) \\ \frac{1}{2}p\cdot d\cdot(y_2-y_1) \\ -\frac{1}{2}p\cdot d\cdot(x_2-x_1) \end{bmatrix}$$

The polynomials to be integrated in this case, N_1 and N_2, are of degree 1. Therefore, a single Gauss-Legendre point ($\xi_1=0$, $w_1=2$) will suffice for exact integration:

$$\left(\int_{-1}^{1} N_1\,d\xi = N_1(0)\cdot 2 = \frac{1}{2}\cdot 2 = 1\right.$$

$$\left.\int_{-1}^{1} N_2\,d\xi = N_2(0)\cdot 2 = \frac{1}{2}\cdot 2 = 1\right)$$

<u>3 nodes</u>

$$\begin{bmatrix} d\cdot\frac{L}{2}\int_{-1}^{1} N_1\cdot(-pn_x)\,d\xi \\ d\cdot\frac{L}{2}\int_{-1}^{1} N_1\cdot(-pn_y)\,d\xi \\ d\cdot\frac{L}{2}\int_{-1}^{1} N_2\cdot(-pn_x)\,d\xi \\ d\cdot\frac{L}{2}\int_{-1}^{1} N_2\cdot(-pn_y)\,d\xi \\ d\cdot\frac{L}{2}\int_{-1}^{1} N_3\cdot(-pn_x)\,d\xi \\ d\cdot\frac{L}{2}\int_{-1}^{1} N_3\cdot(-pn_y)\,d\xi \end{bmatrix} = \begin{bmatrix} \frac{1}{6}p\cdot d\cdot(y_3-y_1) \\ -\frac{1}{6}p\cdot d\cdot(x_3-x_1) \\ \frac{2}{3}p\cdot d\cdot(y_3-y_1) \\ -\frac{2}{3}p\cdot d\cdot(x_3-x_1) \\ \frac{1}{6}p\cdot d\cdot(y_3-y_1) \\ -\frac{1}{6}p\cdot d\cdot(x_3-x_1) \end{bmatrix}$$

The polynomials to be integrated in this case, N_1, N_2 and N_3, are of degree 2. Therefore, two Gauss-Legendre points ($\xi_1=-1/\sqrt{3}$, $\xi_2=1/\sqrt{3}$, $w_1=1$, $w_2=1$) will be required for exact integration (recall that nip Gauss-Legendre points will integrate exactly polynomials of degree up to $2\cdot nip-1$):

$$\int_{-1}^{1} N_1\,d\xi = N_1\left(-\tfrac{1}{\sqrt3}\right)\cdot 1 + N_1\left(\tfrac{1}{\sqrt3}\right)\cdot 1$$
$$= \tfrac{1}{2}\cdot\left(-\tfrac{1}{\sqrt3}\right)\left(-\tfrac{1}{\sqrt3}-1\right)\cdot 1 + \tfrac{1}{2}\cdot\left(\tfrac{1}{\sqrt3}\right)\cdot\left(\tfrac{1}{\sqrt3}-1\right)\cdot 1$$
$$= \tfrac{1}{2}\cdot\tfrac{1}{3} + \tfrac{1}{2}\cdot\tfrac{1}{\sqrt3} + \tfrac{1}{2}\cdot\tfrac{1}{3} - \tfrac{1}{2}\cdot\tfrac{1}{\sqrt3} = \tfrac{1}{3}$$

$$\int_{-1}^{1} N_2\,d\xi = N_2\left(-\tfrac{1}{\sqrt3}\right)\cdot 1 + N_2\left(\tfrac{1}{\sqrt3}\right)\cdot 1$$
$$= \left[1-\left(-\tfrac{1}{\sqrt3}\right)^2\right]\cdot 1 + \left[1-\left(\tfrac{1}{\sqrt3}\right)^2\right]\cdot 1$$
$$= \left(1-\tfrac{1}{3}\right) + \left(1-\tfrac{1}{3}\right) = \tfrac{4}{3}$$

$$\int_{-1}^{1} N_3\,d\xi = N_3\left(-\tfrac{1}{\sqrt3}\right)\cdot 1 + N_3\left(\tfrac{1}{\sqrt3}\right)\cdot 1$$
$$= \tfrac{1}{2}\cdot\left(-\tfrac{1}{\sqrt3}\right)\left(-\tfrac{1}{\sqrt3}+1\right)\cdot 1 + \tfrac{1}{2}\cdot\left(\tfrac{1}{\sqrt3}\right)\cdot\left(\tfrac{1}{\sqrt3}+1\right)\cdot 1$$
$$= \tfrac{1}{2}\cdot\tfrac{1}{3} - \tfrac{1}{2}\cdot\tfrac{1}{\sqrt3} + \tfrac{1}{2}\cdot\tfrac{1}{3} + \tfrac{1}{2}\cdot\tfrac{1}{\sqrt3} = \tfrac{1}{3}$$

In the more general case of a <u>curved side</u> (and 3 nodes), the unit-vector components, n_x, n_y, and the scale factor $dl/d\xi$ are <u>not</u> constant! Consider two points infinitesimally close:

$$\xi \text{ and } \xi + d\xi$$

The images of these points are:

$$\underset{\sim}{x} \quad\text{and}\quad \underset{\sim}{x} + \frac{d\underset{\sim}{x}}{d\xi}\,d\xi \qquad \left(\underset{\sim}{x} = \begin{bmatrix} x \\ y \end{bmatrix}\right)$$

Thus,

$$\frac{d\underset{\sim}{x}}{d\xi}$$

is a vector tangent to the side at $\underset{\sim}{x}$. We take the cross product

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} \frac{dx}{d\xi} \\ \frac{dy}{d\xi} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{dy}{d\xi} \\ \frac{dx}{d\xi} \\ 0 \end{bmatrix}$$

and divide by the length of this vector to determine n_x and n_y:

$$n_x = -\frac{\frac{dy}{d\xi}}{\sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}}$$

$$n_y = \frac{\frac{dx}{d\xi}}{\sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}}$$

where:

$$\frac{dx}{d\xi} = \frac{dN_1}{d\xi}x_1 + \frac{dN_2}{d\xi}x_2 + \frac{dN_3}{d\xi}x_3$$
$$= \left(\xi - \tfrac{1}{2}\right)x_1 - 2\xi x_2 + \left(\xi + \tfrac{1}{2}\right)x_3$$
$$= \xi\cdot(x_1 - 2x_2 + x_3) + \tfrac{1}{2}(x_3 - x_1)$$

$$\frac{dy}{d\xi} = \frac{dN_1}{d\xi}y_1 + \frac{dN_2}{d\xi}y_2 + \frac{dN_3}{d\xi}y_3$$
$$= \left(\xi - \tfrac{1}{2}\right)y_1 - 2\xi y_2 + \left(\xi + \tfrac{1}{2}\right)y_3$$
$$= \xi\cdot(y_1 - 2y_2 + y_3) + \tfrac{1}{2}(y_3 - y_1)$$

The length of the (infinitesimal) vector from $\underset{\sim}{x}$ to $\underset{\sim}{x} + \frac{d\underset{\sim}{x}}{d\xi}d\xi$ is:

$$dl = \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}\,d\xi$$

and, therefore, the scale factor is given by

$$\frac{dl}{d\xi} = \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}$$

Substituting the above expressions for n_x, n_y and $dl/d\xi$ in the integrals for the nodal forces, we find:

$$
\begin{bmatrix}
-p\cdot d\cdot\int_{-1}^{1} N_1\cdot\left(-\frac{dy}{d\xi}\right)d\xi \\
-p\cdot d\cdot\int_{-1}^{1} N_1\cdot\left(\frac{dx}{d\xi}\right)d\xi \\
-p\cdot d\int_{-1}^{1} N_2\cdot\left(-\frac{dy}{d\xi}\right)d\xi \\
-p\cdot d\int_{-1}^{1} N_2\cdot\left(\frac{dx}{d\xi}\right)d\xi \\
-p\cdot d\int_{-1}^{1} N_3\cdot\left(-\frac{dy}{d\xi}\right)d\xi \\
-p\cdot d\int_{-1}^{1} N_3\cdot\left(\frac{dx}{d\xi}\right)d\xi
\end{bmatrix}
=
\begin{bmatrix}
p\cdot d\cdot\left[-\frac{1}{3}(y_1-2y_2+y_3)+\frac{1}{6}(y_3-y_1)\right] \\
p\cdot d\cdot\left[\frac{1}{3}(x_1-2x_2+x_3)-\frac{1}{6}(x_3-x_1)\right] \\
p\cdot d\left[\frac{2}{3}(y_3-y_1)\right] \\
p\cdot d\left[-\frac{2}{3}(x_3-x_1)\right] \\
p\cdot d\left[\frac{1}{3}(y_1-2y_2+y_3)+\frac{1}{6}(y_3-y_1)\right] \\
p\cdot d\left[-\frac{1}{3}(x_1-2x_2+x_3)-\frac{1}{6}(x_3-x_1)\right]
\end{bmatrix}
$$

The polynomials to be integrated in this case (products of interpolation functions and their derivatives) are at most of degree 3. Therefore, two Gauss-Legendre points ($\xi_1=-1/\sqrt{3}$, $\xi_2=1/\sqrt{3}$, $W_1=1$, $W_2=1$) will be required for exact integration:

$$
\begin{aligned}
\int_{-1}^{1} N_1\frac{dy}{d\xi}\,d\xi &= \frac{1}{2}\cdot\left(-\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}-1\right)\cdot\left[\left(-\frac{1}{\sqrt{3}}\right)(y_1-2y_2+y_3)+\frac{1}{2}(y_3-y_1)\right]\cdot 1 \\
&\quad+\frac{1}{2}\cdot\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}-1\right)\cdot\left[\left(\frac{1}{\sqrt{3}}\right)(y_1-2y_2+y_3)+\frac{1}{2}(y_3-y_1)\right]\cdot 1 \\
&= (y_1-2y_2+y_3)\left(-\frac{1+\sqrt{3}}{6\sqrt{3}}+\frac{1-\sqrt{3}}{6\sqrt{3}}\right) \\
&\quad+(y_3-y_1)\cdot\left(\frac{1+\sqrt{3}}{12}+\frac{1-\sqrt{3}}{12}\right) \\
&= -\frac{1}{3}(y_1-2y_2+y_3)+\frac{1}{6}(y_3-y_1)
\end{aligned}
$$

The rest of the integrals are evaluated in the same manner.

CATHERINE HOVELL
17 November 06

100

CE381R: Finite Element Analysis

Assignment #6

Reduced Integration: CAX8R

Seed: 0.25R
u(3R,0) = -1.0426 $\times \frac{qR}{E}$
w(0,5R) = 5.19254

sig_z(R,0) = 2.12479 $\times q$
sig_th(R,0) = 0.11067

***

Seed: 0.1R
u(3R,0) = -1.04263
w(0,5R) = 5.19259

sig_z(R,0) = 2.12707
sig_th(R,0) = 0.112998

***

Seed: 0.05R
u(3R,0) = -1.04263
w(0,5R) = 5.19259

sig_z(R,0) = 2.13309
sig_th(R,0) = 0.114495

++++++

Full Integration: CAX8

Seed: 0.25R
u(3R,0) = -1.04253
w(0,5R) = 5.1925

sig_z(R,0) = 2.15166
sig_th(R,0) = 0.13833

***

Seed: 0.1R
u(3R,0) = -1.04263
w(0,5R) = 5.19259

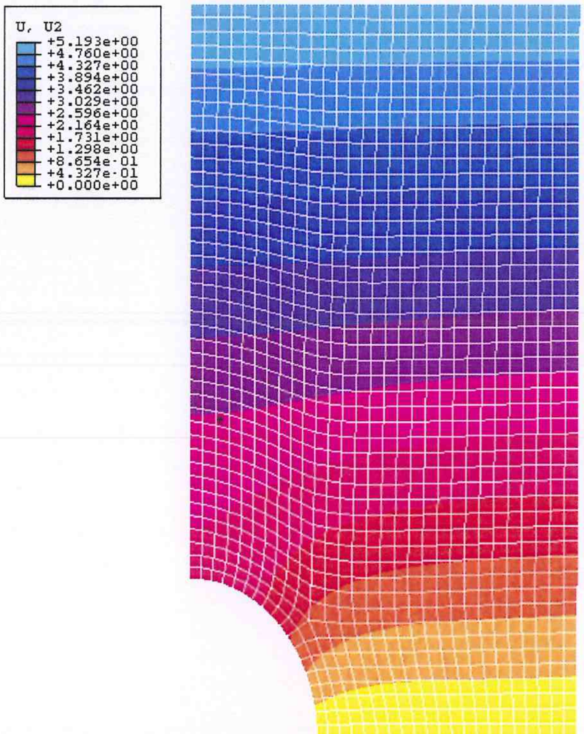
sig_z(R,0) = 2.14471
sig_th(R,0) = 0.124315

***

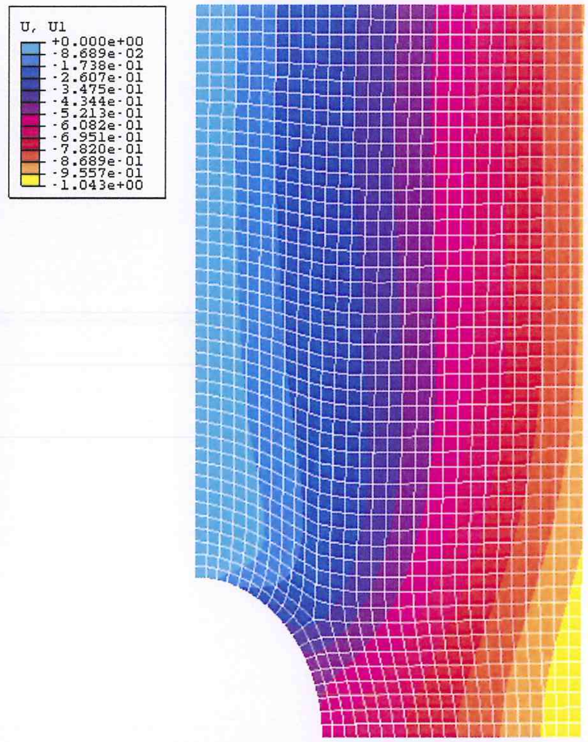
Seed: 0.05R
u(3R,0) = -1.0429
w(0,5R) = 5.19259

sig_z(R,0) = 2.1389
sig_th(R,0) = 0.118344

ASSIGNMENT #6



u_2
w



u_1
u

Example contour plots
from ABAQUS/CAE

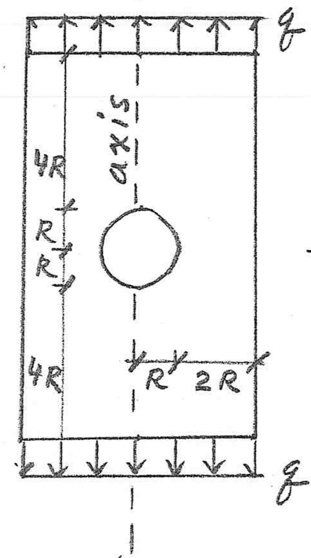
due: 11/17

CE 381R

FALL 2006

Assignment 6

Consider the cylinder with a spherical hole shown below:



E
$\nu = 0.3$

(The section on the plane $\theta = 0$ is identical to the perforated plate examined in Assignment 4.) Using ABAQUS CAE and taking advantage of axial symmetry as well as symmetry about the plane $z=0$, analyze the cylinder and report results for

$$u(3R,0),\ w(0,5R),\ \sigma_z(R,0),\ \sigma_\theta(R,0)$$

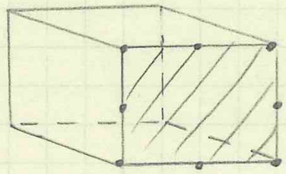
Use meshes of 8-node, axisymmetric finite elements and evaluate the relative performance of full integration vs. reduced integration.

Try more than one mesh density
reduced & full integrations

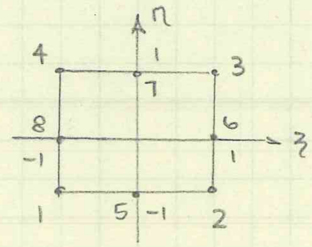
ASSIGNMENT #7

100

1.



8 nodes, mapped into $\xi \cdot \eta$



$$\det J = \frac{A\text{ (real space)}}{A(\xi \cdot \eta \text{ space})} = \frac{W \cdot L}{2 \cdot 2} = \frac{WL}{4}$$

$T_x = T_x$ at each of the points

using:

$$F = \int_\Omega N^T T_x \det J \, d\Omega,$$

calculate forces (in mathcad)

check:

sum of nodal forces = total load,

$T_x WL$

Interpolation functions:

$$N_1 := \frac{-1}{4}(1-\xi)\cdot(1-\eta)\cdot(\xi+\eta+1)$$

$$N_2 := \frac{-1}{4}(\xi+1)(1-\eta)(1-\xi+\eta)$$

$$N_3 := \frac{-1}{4}(1+\xi)(1+\eta)(1-\xi-\eta)$$

$$N_4 := \frac{-1}{4}(1-\xi)(1+\eta)(1+\xi-\eta)$$

$$N_5 := \frac{1}{2}\cdot(1-\xi)\cdot(1+\xi)(1-\eta)$$

$$N_6 := \frac{1}{2}(1+\xi)(1-\eta)(1+\eta)$$

$$N_7 := \frac{1}{2}\cdot(1-\xi)(1+\xi)(1+\eta)$$

$$N_8 := \frac{1}{2}(1-\xi)(1+\eta)(1-\eta)$$

ASSIGNMENT #7

$$F\_explain1 := \int_{-1}^{1}\int_{-1}^{1} N_i \cdot T_x \cdot det\_J \, d\xi \, d\eta$$

$$F\_explain2 := \frac{T_x \cdot W \cdot L}{4} \cdot \int_{-1}^{1}\int_{-1}^{1} N_i \, d\xi \, d\eta$$

$$F := \frac{T_x \cdot W \cdot L}{4} \begin{bmatrix} \int_{-1}^{1}\int_{-1}^{1} \frac{-1}{4}(1-\xi)\cdot(1-\eta)\cdot(\xi+\eta+1) \, d\xi \, d\eta \\ \int_{-1}^{1}\int_{-1}^{1} \frac{-1}{4}\cdot(\xi+1)\cdot(1-\eta)\cdot(1-\xi+\eta) \, d\xi \, d\eta \\ \int_{-1}^{1}\int_{-1}^{1} \frac{-1}{4}(1+\xi)(1+\eta)(1-\xi-\eta) \, d\xi \, d\eta \\ \int_{-1}^{1}\int_{-1}^{1} \frac{-1}{4}(1-\xi)(1+\eta)(1+\xi-\eta) \, d\xi \, d\eta \\ \int_{-1}^{1}\int_{-1}^{1} \frac{1}{2}\cdot(1-\xi)\cdot(1+\xi)(1-\eta) \, d\xi \, d\eta \\ \int_{-1}^{1}\int_{-1}^{1} \frac{1}{2}(1+\xi)(1-\eta)(1+\eta) \, d\xi \, d\eta \\ \int_{-1}^{1}\int_{-1}^{1} \frac{1}{2}\cdot(1-\xi)(1+\xi)(1+\eta) \, d\xi \, d\eta \\ \int_{-1}^{1}\int_{-1}^{1} \frac{1}{2}(1-\xi)(1+\eta)(1-\eta) \, d\xi \, d\eta \end{bmatrix}$$

NODAL FORCES

$$\boxed{F \rightarrow \begin{pmatrix} \frac{-1}{12}\cdot T_x\cdot W\cdot L \\ \frac{-1}{12}\cdot T_x\cdot W\cdot L \\ \frac{-1}{12}\cdot T_x\cdot W\cdot L \\ \frac{-1}{12}\cdot T_x\cdot W\cdot L \\ \frac{1}{3}\cdot T_x\cdot W\cdot L \\ \frac{1}{3}\cdot T_x\cdot W\cdot L \\ \frac{1}{3}\cdot T_x\cdot W\cdot L \\ \frac{1}{3}\cdot T_x\cdot W\cdot L \end{pmatrix}}$$

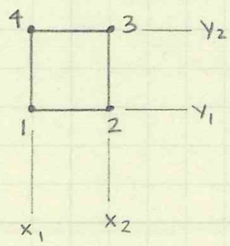
✓

Check: $T_x\cdot W\cdot L\cdot\left[4\cdot\left(\frac{-1}{12}\right)+4\cdot\left(\frac{1}{3}\right)\right] \rightarrow T_x\cdot W\cdot L$ ✓

as expected

ASSIGNMENT #7

2.



$$N_1 = (y - y_2)(x - x_2) \,/\, (y_1 - y_2)(x_1 - x_2)$$

$$N_2 = \frac{(y - y_2)(x - x_1)}{(y_1 - y_2)(x_2 - x_1)}$$

$$N_3 = \frac{(x - x_1)(y - y_1)}{(x_2 - x_1)(y_2 - y_1)}$$

$$N_4 = \frac{(x - x_2)(y - y_1)}{(x_1 - x_2)(y_2 - y_1)}$$

assign

$x_1 = 0 \qquad x_2 = L$

$y_1 = 0 \qquad y_2 = W$

considering partial derivatives:

$$\frac{dN_1}{dx} = \frac{y - W}{WL} \qquad \frac{dN_3}{dx} = \frac{y}{WL}$$

$$\frac{dN_1}{dy} = \frac{x - L}{WL} \qquad \frac{dN_3}{dy} = \frac{x}{WL}$$

$$\frac{dN_2}{dx} = \frac{W - y}{WL} \qquad \frac{dN_4}{dx} = \frac{-y}{WL}$$

$$\frac{dN_2}{dy} = \frac{-x}{WL} \qquad \frac{dN_4}{dy} = \frac{L - x}{WL}$$

$$B_b = \left[\frac{\partial N_1}{dx} \;\; \frac{\partial N_1}{\partial y} \;\; \frac{\partial N_2}{\partial x} \;\; \frac{\partial N_2}{\partial y} \;\; \frac{\partial N_3}{dx} \;\; \frac{\partial N_3}{\partial y} \;\; \frac{\partial N_4}{\partial x} \;\; \frac{\partial N_4}{\partial y} \right]$$

$N_p = [1]$, as there's only one point

$$G = -\int_0^W \int_0^L B_b^T N_p \, dx \, dy$$

solve in Mathcad

CE 381R: FINITE

C. HOVELL
29 NOV 06

4/4

ASSIGNMENT #7

2. calculations:

$$G1 := \int_0^W \int_0^L \frac{y - W}{W \cdot L} \cdot N_p \, dx \, dy \qquad G5 := \int_0^W \int_0^L \frac{y}{W \cdot L} \cdot N_p \, dx \, dy$$

$$G2 := \int_0^W \int_0^L \frac{x - L}{W \cdot L} \cdot N_p \, dx \, dy \qquad G6 := \int_0^W \int_0^L \frac{x}{W \cdot L} \cdot N_p \, dx \, dy$$

$$G3 := \int_0^W \int_0^L \frac{-y + W}{W \cdot L} \cdot N_p \, dx \, dy \qquad G7 := \int_0^W \int_0^L \frac{-y}{W \cdot L} \cdot N_p \, dx \, dy$$

$$G4 := \int_0^W \int_0^L \frac{-x}{W \cdot L} \cdot N_p \, dx \, dy \qquad G8 := \int_0^W \int_0^L \frac{-x + L}{W \cdot L} \cdot N_p \, dx \, dy$$

$$\boxed{G \to \begin{pmatrix} W \\ L \\ -W \\ L \\ -W \\ -L \\ W \\ -L \end{pmatrix} \cdot \frac{1}{2}} \; x d$$

includes optional negative sign

thickness

Given bulk modulus K, $N_p = 1$,

calculate F (= F_m in mathcad)

$$F\_m := + \int_0^W \int_0^L N_p \cdot \frac{1}{K} \cdot N_p \, dx \, dy$$

$$F\_m \to W \cdot \frac{L}{K} \qquad \boxed{F = \frac{+WL}{K} \times d}$$

✓

As defined, augmented matrix would be:

$$\begin{bmatrix} \underset{\sim}{K_s} & \underset{\sim}{G} \\ \underset{\sim}{G}^T & - \underset{\sim}{F} \end{bmatrix}$$

✓

due 11/29

CE 381R

FALL 2006

Assignment 7

1. Assume that one of the faces of a 20-node "brick" (hexahedron) is subjected to uniform traction in the x-direction:

$$T_x = q$$

Also, for the purposes of this exercise, assume that the face is a rectangle of length L and width W. Obtain the nodal forces.

2. The matrix of a 4-node quadrilateral in plane strain, augmented with a pressure node (constant pressure over the element) can be written in the form:

$$\begin{bmatrix} \underset{\sim}{K}_s & \underset{\sim}{G} \\ \underset{\sim}{G}^T & \underset{\sim}{F} \end{bmatrix}$$

where the element degrees of freedom have been arranged as:

$$\begin{bmatrix} \underset{\sim}{U} \\ p \end{bmatrix}$$

Assuming that the element is a rectangle (length: L, width: W) aligned with the coordinate axes, obtain $\underset{\sim}{G}$ and F. Take K as the bulk modulus (constant over the element).

CE 381R

FALL 2006

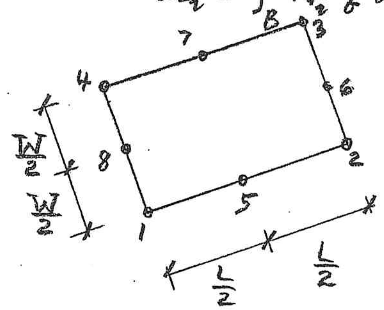
Assignment 7: Solution

1. The nodal forces due to traction are given by

$$\int_B \underset{\sim}{N}^T \underset{\sim}{T}\, dB$$

In the case under consideration, only the forces at nodes on the face in the x-direction will be different from zero. With the node numbering shown below, the forces are:

$$X_i = \int_B N_i\, g\, dB$$



with

$$N_1(\xi,\eta) = -\frac{1}{4}(1-\xi)(1-\eta)(\xi+\eta+1)$$

$$N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1)$$

$$N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1)$$

$$N_4(\xi,\eta) = -\frac{1}{4}(1-\xi)(1+\eta)(\xi-\eta+1)$$

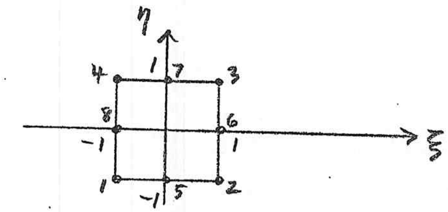
$$N_5(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1-\eta)$$

$$N_6(\xi,\eta) = \frac{1}{2}(1-\eta^2)(1+\xi)$$

$$N_7(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1+\eta)$$

$$N_8(\xi,\eta) = \frac{1}{2}(1-\eta^2)(1-\xi)$$

The face has been mapped onto the "parent" square:



Since the shape in the physical domain is a perfect rectangle, the determinant of the Jacobian matrix is constant:

$$\det(\underset{\sim}{J}) = \frac{WL}{4}$$

Thus, we find:

$$X_i = \int_{-1}^{1}\int_{-1}^{1} N_i\, g\, \frac{WL}{4}\, d\xi\, d\eta$$

Using 2 Gauss-Legendre integration points in the ξ-direction and the same number in the η direction, we obtain:

$$X_1 = \frac{gWL}{4}\Big[N_1\Big(-\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}\Big)\cdot 1 + N_1\Big(\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}\Big)\cdot 1 + N_1\Big(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\Big)\cdot 1 + N_1\Big(-\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\Big)\cdot 1\Big]$$

$$= \frac{gWL}{4}\Big[-\frac{1}{4}\frac{(\sqrt{3}+1)^2}{3}\frac{\sqrt{3}-2}{\sqrt{3}} - \frac{1}{4}\frac{(\sqrt{3}-1)(\sqrt{3}+1)}{3} - \frac{1}{4}\frac{(\sqrt{3}-1)^2}{3}\frac{\sqrt{3}+2}{\sqrt{3}} - \frac{1}{4}\frac{(\sqrt{3}+1)(\sqrt{3}-1)}{3}\Big]$$

$$= -\frac{gWL}{12}$$

$$X_5 = \frac{qWL}{4}\left[N_5\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)\cdot 1 + N_5\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)\cdot 1 \right.$$

$$\left. + N_5\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\cdot 1 + N_5\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\cdot 1\right]$$

$$= \frac{qWL}{4}\left[\frac{\sqrt{3}+1}{3\sqrt{3}} + \frac{\sqrt{3}+1}{3\sqrt{3}} \right.$$

$$\left. + \frac{\sqrt{3}-1}{3\sqrt{3}} + \frac{\sqrt{3}-1}{3\sqrt{3}}\right]$$

$$= \frac{qWL}{3}$$

Similarly, we find:

$$X_2 = X_3 = X_4 = -\frac{qWL}{12}$$

$$X_6 = X_7 = X_8 = \frac{qWL}{3}$$

2. The matrices are given by:

$$\underset{\sim}{G} = -d\int_{\Omega_{2D}} \underset{\sim}{B}_b^T \underset{\sim}{N}_p \, d\Omega_{2D}$$

$$\underset{\sim}{F} = -d\int_{\Omega_{2D}} \underset{\sim}{N}_p^T \frac{1}{K} \underset{\sim}{N}_p \, d\Omega_{2D}$$

(d: element thickness)

In the case under consideration,

$$\underset{\sim}{N}_P = [1]$$

and

$$\underset{\sim}{B}_b = \left[\frac{\partial N_1}{\partial x} \;\; \frac{\partial N_1}{\partial y} \;\; \frac{\partial N_2}{\partial x} \;\; \frac{\partial N_2}{\partial y} \;\; \frac{\partial N_3}{\partial x} \;\; \frac{\partial N_3}{\partial y} \;\; \frac{\partial N_4}{\partial x} \;\; \frac{\partial N_4}{\partial y}\right]$$

where:

$$N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

The element has been mapped onto the (standard) "parent" square. On account of the element (rectangular) alignment with the (physical) x-y coordinate system, we can write:

$$\frac{\partial x}{\partial \xi} = \frac{L}{2}, \quad \frac{\partial y}{\partial \xi} = 0, \quad \frac{\partial x}{\partial \eta} = 0, \quad \frac{\partial y}{\partial \eta} = \frac{W}{2}$$

and, therefore,

$$\frac{\partial \xi}{\partial x} = \frac{2}{L}, \quad \frac{\partial \eta}{\partial x} = 0, \quad \frac{\partial \xi}{\partial y} = 0, \quad \frac{\partial \eta}{\partial y} = \frac{2}{W}$$

Thus, we find:

$$\frac{\partial N_1}{\partial x} = \frac{2}{L}\cdot\left(-\frac{1}{4}\right)\cdot(1-\eta)$$

$$\frac{\partial N_1}{\partial y} = \frac{2}{W}\cdot\left(-\frac{1}{4}\right)(1-\xi)$$

$$\frac{\partial N_2}{\partial x} = \frac{2}{L}\cdot\left(\frac{1}{4}\right)\cdot(1-\eta)$$

$$\frac{\partial N_2}{\partial y} = \frac{2}{W}\cdot\left(-\frac{1}{4}\right)\cdot(1+\xi)$$

$$\frac{\partial N_3}{\partial x} = \frac{2}{L}\cdot\left(\frac{1}{4}\right)\cdot(1+\eta)$$

$$\frac{\partial N_3}{\partial y} = \frac{2}{W}\cdot\left(\frac{1}{4}\right)\cdot(1+\xi)$$

$$\frac{\partial N_4}{\partial x} = \frac{2}{L}\cdot\left(-\frac{1}{4}\right)\cdot(1+\eta)$$

$$\frac{\partial N_4}{\partial y} = \frac{2}{W}\cdot\left(\frac{1}{4}\right)\cdot(1-\xi)$$

Also, the determinant of the Jacobian is constant

$$\det(\underset{\sim}{J}) = \frac{W \cdot L}{4}$$

The integrals in $\underset{\sim}{G}$ and $\underset{\sim}{F}$ can be evaluated exactly using one-point ($\xi_1 = 0$, $\eta_1 = 0$, $w_1 = 4$) Gauss integration. Therefore,

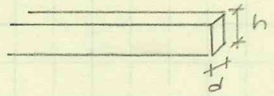
$$\underset{\sim}{G} = \begin{bmatrix} \frac{W}{2} d \\ \frac{L}{2} d \\ \hline -\frac{W}{2} d \\ \frac{L}{2} d \\ \hline -\frac{W}{2} d \\ -\frac{L}{2} d \\ \hline \frac{W}{2} d \\ -\frac{L}{2} d \end{bmatrix}$$

$$\underset{\sim}{F} = -\frac{1}{K} W \cdot L \cdot d$$

ASSIGNMENT #8

100

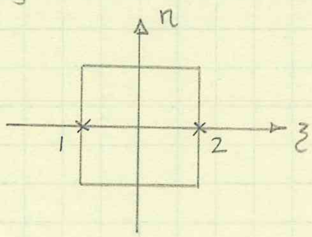
1.



$$K = E \cdot K_\varepsilon + G \cdot K_\gamma$$

L, G, E

Mapping the element to ξ, η



$$N_1 = \tfrac{1}{2}(1-\xi)$$

$$N_2 = \tfrac{1}{2}(1+\xi)$$

K_ε needs the first row of the B matrix (B_ε)
K_γ needs the second (B_γ)

$$B_\varepsilon = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & -\eta \frac{h}{2} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & 0 & -\eta \frac{h}{2} \frac{\partial N_2}{\partial x} \end{bmatrix}$$

$$B_\gamma = \begin{bmatrix} 0 & \frac{\partial N_1}{\partial x} & -\frac{\partial \eta}{\partial y} \frac{h}{2} N_1 & 0 & \frac{\partial N_2}{\partial x} & -\frac{\partial \eta}{\partial y} \frac{h}{2} N_2 \end{bmatrix}$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$\frac{\partial y}{\partial \xi} = 0$, as the height does not change

$\frac{\partial x}{\partial \eta} = 0$

$\frac{\partial x}{\partial \xi} = \frac{L}{2}$, $\frac{\partial y}{\partial \eta} = \frac{h}{2}$

$$J = \begin{bmatrix} L/2 & 0 \\ 0 & h/2 \end{bmatrix}, \quad J^{-1} = \frac{1}{\frac{L}{2} \cdot \frac{h}{2}} \begin{bmatrix} h/2 & 0 \\ 0 & L/2 \end{bmatrix} = \begin{bmatrix} 2/L & 0 \\ 0 & 2/h \end{bmatrix}$$

$\frac{\partial \xi}{\partial x}$

$\frac{\partial \eta}{\partial y}$

ASSIGNMENT #8

1. (cont'd)

$$\frac{\partial \xi}{\partial x} = \frac{2}{L}, \quad \frac{\partial \eta}{\partial y} = \frac{2}{h}$$

$$\frac{\partial N_1}{\partial x} = \frac{\partial N_1}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} = -\frac{1}{2} \cdot \frac{2}{L} = \frac{-1}{L}$$

$$\frac{\partial N_2}{\partial x} = \frac{1}{2} \cdot \frac{2}{L} = \frac{1}{L}$$

$$-\eta \frac{h}{2} \cdot \frac{\partial N_1}{\partial x} = -\eta \cdot \frac{h}{2} \frac{-1}{L} = \frac{h}{2L} \cdot \eta$$

$$= \frac{-h}{2L} \cdot \eta \text{ for } N_2$$

$$-\frac{\partial \eta}{\partial y} \cdot \frac{h}{2} N_1 = \frac{-2}{h} \cdot \frac{h}{2} \cdot \frac{1}{2}(1-\xi) = \frac{-1}{2}(1-\xi)$$

$$= \frac{-1}{2}(1+\xi) \text{ for } N_2$$

$$B_\varepsilon = \begin{bmatrix} \frac{-1}{L} & 0 & \frac{h}{2L} \cdot \eta & \frac{1}{L} & 0 & \frac{-h}{2L} \cdot \eta \end{bmatrix}$$

$$B_\gamma = \begin{bmatrix} 0 & \frac{-1}{L} & \frac{-1}{2}(1-\xi) & 0 & \frac{1}{L} & \frac{-1}{2}(1+\xi) \end{bmatrix}$$

$$\det J = \frac{h \cdot L}{4}$$

$$K_\varepsilon = \iint B_\varepsilon^T B_\varepsilon \det(J)\, d\xi\, d\eta$$

$$K_\varepsilon = \frac{h \cdot L}{4 \cdot L^2} \int_{-1}^{1} \int_{-1}^{1} \begin{bmatrix} 1 & 0 & -\frac{1}{2} h\eta & -1 & 0 & \frac{1}{2} h \cdot \eta \\ & 0 & 0 & 0 & 0 & 0 \\ & & \frac{1}{4} h^2 \eta^2 & \frac{1}{2} h\eta & 0 & -\frac{1}{4} h^2 \eta^2 \\ & & & 1 & 0 & -\frac{1}{2} h\eta \\ & \text{sym.} & & & 0 & 0 \\ & & & & & \frac{1}{4} h^2 \eta^2 \end{bmatrix} d\xi\, d\eta$$

ASSIGNMENT #8

1. (cont'd)

there are only 3 expressions to solve:

$$\int_{-1}^{1}\int_{-1}^{1} 1\, d\xi\, d\eta = \int_{-1}^{1} \xi \Big|_{-1}^{1} d\eta = \int_{-1}^{1} 2\, d\eta = 4$$

$$\int_{-1}^{1}\int_{-1}^{1} \eta\, d\xi\, d\eta = \int_{-1}^{1} \eta\xi \Big|_{-1}^{1} d\eta = \int_{-1}^{1} 2\eta\, d\eta = 0$$

$$\int_{-1}^{1}\int_{-1}^{1} \eta^2\, d\xi\, d\eta = \int_{-1}^{1} \eta^2\xi \Big|_{-1}^{1} d\eta = \int_{-1}^{1} 2\eta^2\, d\eta = \frac{2}{3}\eta^3 \Big|_{-1}^{1} = \frac{4}{3}$$

$$K_\varepsilon = d\frac{h}{L}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & h^2/12 & 0 & 0 & -h^2/12 \\ & & & 1 & 0 & 0 \\ & \text{sym.} & & & 0 & 0 \\ & & & & & h^2/12 \end{bmatrix}$$

$$K_\gamma = \iint B_\gamma^T B_\gamma \det(J)\, d\xi\, d\eta$$

$$K_\gamma = \iint_{\xi,\eta} \frac{h\cdot L}{4}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 1/L^2 & \frac{1}{2L}(1-\xi) & 0 & -1/L^2 & \frac{1}{2L}(1+\xi) \\ & & 1/4(1-\xi)^2 & 0 & \frac{-1}{2L}(1-\xi) & \frac{1}{4}(1-\xi^2) \\ & & & 0 & 0 & 0 \\ & \text{sym.} & & & 1/L^2 & \frac{-1}{2L}(1+\xi) \\ & & & & & \frac{1}{4}(1+\xi)^2 \end{bmatrix}$$

ASSIGNMENT #8

1. (cont'd)

Because $(1-\xi)$ is similar to $(1+\xi)$
and $(1-\xi)^2$ is similar to $(1+\xi)^2$,
there are four integrals to evaluate:

$$\int_{-1}^{1}\int_{-1}^{1} 1\, d\xi\, d\eta = 4, \text{ as before}$$

$$\int_{-1}^{1}\int_{-1}^{1}(1\pm\xi)\, d\xi\, d\eta = \int_{-1}^{1} \xi - \frac{\xi^2}{2}\Big|_{-1}^{1} d\eta = \int_{-1}^{1} 2\, d\eta = 4$$

$$\int_{-1}^{1}\int_{-1}^{1}(1\pm\xi)^2\, d\xi\, d\eta = \int_{-1}^{1}\left[\xi-\xi^2+\tfrac{1}{3}\xi^3\right]\Big|_{-1}^{1} d\eta = \int_{-1}^{1} 8/3\, d\eta = \frac{16}{3}$$

$$\int_{-1}^{1}\int_{-1}^{1}(1-\xi^2)\, d\xi\, d\eta = \int_{-1}^{1} \xi - \frac{1}{3}\xi^3\Big|_{-1}^{1} d\eta = \int_{-1}^{1}\frac{4}{3}\, d\eta = \frac{8}{3}$$

$$K_\gamma = d\,h\cdot L\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 1/L^2 & 1/2L & 0 & -1/L^2 & 1/2L \\ & & 1/3 & 0 & \frac{-1}{2L} & 1/6 \\ & & & 0 & 0 & 0 \\ & \text{sym.} & & & 1/L^2 & -1/2L \\ & & & & & 1/3 \end{bmatrix}$$

1.a) Full integration,

$$K = d\cdot h\cdot L\begin{bmatrix} E/L^2 & 0 & 0 & -E/L^2 & 0 & 0 \\ & G/L^2 & G/2L & 0 & -G/L^2 & G/2L \\ & & \frac{Eh^2}{12L^2}+\frac{G}{3} & 0 & \frac{-G}{2L} & \frac{-Eh^2}{12L^2}+\frac{G}{6} \\ & & & E/L^2 & 0 & 0 \\ & & & & G/L^2 & \frac{-G}{2L} \\ & & & & & \frac{Eh^2}{12L^2}+\frac{G}{3} \end{bmatrix}$$

ASSIGNMENT # 8

1. (cont'd)

For selectively reduced integration, K_ξ is the same

highest order $= \xi^2, \eta$

2×1

reduced, use 1×1

Gauss approximation for 1×1:

$$\int_{-1}^{1} F(x)\,dx = 2 \cdot F(0)$$

Reconsider 4 integrals:

$$\int_{-1}^{1}\int_{-1}^{1} 1 \, d\xi \, d\eta = 4$$

$$\int_{-1}^{1}\int_{-1}^{1} (1-\xi) \, d\xi \, d\eta = 4$$, true for $(1+\xi)$ as well

$$\int_{-1}^{1}\int_{-1}^{1} (1-\xi)^2 \, d\xi \, d\eta = 4$$, true for $(1+\xi)^2$ as well

$$\int_{-1}^{1}\int_{-1}^{1} (1-\xi^2) \, d\xi \, d\eta = 4$$

$$K_\gamma = d \cdot h \cdot L \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 1/L^2 & 1/2L & 0 & -1/L^2 & 1/2L \\ & & 1/4 & 0 & -1/2L & 1/4 \\ & & & 0 & 0 & 0 \\ & & & & 1/L^2 & -1/2L \\ & & & & & 1/4 \end{bmatrix}$$

1. b) selectively reduced integration

$$K = d \cdot h \cdot L \begin{bmatrix} E/L^2 & 0 & 0 & -E/L^2 & 0 & 0 \\ & \frac{G}{L^2} & \frac{G}{2L} & 0 & \frac{-G}{L^2} & \frac{G}{2L} \\ & & \frac{Eh^2}{12L^2} + \frac{G}{4} & 0 & \frac{-G}{2L} & \frac{-Eh^2}{12L^2} + \frac{G}{4} \\ & & & \frac{E}{L^2} & 0 & 0 \\ & \text{sym.} & & & \frac{G}{L^2} & \frac{-G}{2L} \\ & & & & & \frac{Eh^2}{12L^2} + \frac{G}{4} \end{bmatrix}$$

Note: all values are the same or smaller; K is less stiff ✓

ASSIGNMENT #8

2.

cantilever beam, length L

$$\frac{L}{h} = 10,\ \frac{d}{h} = 1,\ \nu = 0.3$$

Beam theory leads to

$$y_{end} = \frac{PL^3}{3EI}, \quad I = \tfrac{1}{12} d h^3 = \tfrac{1}{12} h^4 \qquad \theta_{end} = \frac{-PL^2}{2EI} = \frac{-P(10h)^2}{\tfrac{1}{6} E h^4}$$

$$y_{end} = \frac{P(10h)^3}{\tfrac{3}{12} E h^4} = 4000\ P/Eh = 40000\ P/EL \qquad \theta_{end} = 600\ P/Eh^2 = 60000\ P/EL^2$$

single element approximation:

$$P = K \cdot u, \quad P = \begin{bmatrix} R_{u1} \\ R_{v1} \\ R_{\theta 1} \\ 0 \\ -P \\ 0 \end{bmatrix}, \quad u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ u_2 \\ v_2 \\ \theta_2 \end{bmatrix}$$

by examination, important part of K matrix is the bottom corner
F matrix → bottom 2
u matrix → bottom 2

$$\begin{bmatrix} -P \\ 0 \end{bmatrix} = d \cdot E \begin{bmatrix} \frac{1}{200 \cdot h(1+\nu)} & \frac{-1}{40(1+\nu)} \\ \frac{-1}{40(1+\nu)} & \frac{h}{1200} + \frac{h}{12(1+\nu)} \end{bmatrix} \begin{bmatrix} v_2 \\ \theta_2 \end{bmatrix}$$

using mathcad, (verified with ABAQUS)

one element

$v_2 = 1020\ P/EL$
$\theta_2 = 1520\ P/EL^2$ > Full integration

$v_2 = 30{,}260\ P/EL$
$\theta_2 = 60{,}000\ P/EL^2$ > reduced

As ν increases, the deflection and rotation increase in the full integration case. The θ does not change when selectively reduced.

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ASSIGNMENT #8

Input1F.inp

```
*heading
HW 8, second half
*node
1, 0,0
5, 1,0
*ngen, nset=beam
1,5,1
*user element, type=U1, nodes=2, linear
1,2,3
*matrix, type=stiffness
0.01
0, 0.003846
0, 0.001923, 0.00129
-0.01, 0, 0, 0.01
0, -0.003846, -0.001923, 0,
0.003846
0, 0.001923, 0.0006327, 0,
-0.001923, 0.00129
*element, type=U1, elset=beam
1, 1,5
***elgen, elset=beam
**1, etc
**
*uel property, elset=beam
*boundary
1, 1,3
*step, perturbation
*static
*cloud
5, 2, -1
*node file, nset=beam
U
*node print, nset=beam
U
*end step


**THE FOLLOWING TABLE IS PRINTED FOR NODES BELONGING TO NODE SET BEAM
**
**       NODE FOOT-    U1            U2            U3
**            NOTE
**
**          1         0.000      -1.0000E-36  -1.0005E-36
**          5         0.000       -1021.        -1522.
**
**MAXIMUM             0.000      -1.0000E-36  -1.0005E-36
**AT NODE                    1             1             1
**                               v2            θ2
**MINIMUM             0.000       -1021.        -1522.
**AT NODE                    1    ·P/EL    5   ·P/EL²   5
```

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ASSIGNMENT #8

Input1R.inp

```
*heading
HW 8, second half
*node
1, 0,0
5, 1,0
*ngen, nset=beam
1,5,1
*user element, type=U1, nodes=2, linear
1,2,3
*matrix, type=stiffness
0.01
0, 0.003846
0, 0.001923, 0.0009699
-0.01, 0, 0, 0.01
0, -0.003846, -0.001923, 0,
0.003846
0, 0.001923, 0.0009699, 0,
-0.001923, 0.0009699
*element, type=U1, elset=beam
1, 1,5
***elgen, elset=beam
**1, etc
**
*uel property, elset=beam
*boundary
1, 1,3
*step, perturbation
*static
*cload
5, 2, -1
*node file, nset=beam
U
*node print, nset=beam
U
*end step


**THE FOLLOWING TABLE IS PRINTED FOR NODES BELONGING TO NODE SET BEAM
**
**      NODE FOOT-    U1           U2           U3
**           NOTE
**
**        1         0.000     -1.0000E-36   0.000
**        5         0.000     -3.0022E+04 -5.9524E+04
**
**MAXIMUM           0.000    -1.0000E-36   0.000
**AT NODE                  1            1            1
**                            v2           θ2
**MINIMUM           0.000    -3.0022E+04 -5.9524E+04
**AT NODE                  1            5            5
                           ·P/EL        ·P/EL²
```

ASSIGNMENT #8

2. (cont'd)

Map matrices:

$$K = d \cdot h \cdot L \left[\begin{array}{ccccccccc}
E/L^2 & 0 & 0 & -E/L^2 & 0 & 0 & & & \\
 & G/L^2 & G/2L & 0 & -G/L^2 & G/2L & & & \\
 & & \frac{Eh^2}{12L^2}+\frac{G}{4} & 0 & -\frac{G}{2L} & -\frac{Eh^2}{12L^2}+\frac{G}{4} & & & \\
 & & & 2\cdot\frac{E}{L^2} & 0 & 0 & -\frac{E}{L^2} & 0 & 0 \\
 & & & & 2\frac{G}{L^2} & 0 & 0 & -\frac{G}{L^2} & \frac{G}{2L} \\
 & & & & & \frac{Eh^2}{6L^2}+\frac{G}{2} & 0 & -\frac{G}{2L} & \cdots \\
 & & & & & & \frac{E}{L^2} & 0 & 0 \\
 & & & & & & & \frac{G}{L^2} & -\frac{G}{2L} \\
 & & & & & & & & \cdots
\end{array}\right]$$

Because first three deflections are known to be zero, and forces are unknown, ignore first 3 columns and rows

Reduced:

$v_3 = 37{,}760 \; P/EL$

$\theta_3 = 60{,}000 \; P/EL^2$

Full:

$v_3 = 3792 \; P/EL$

$\theta_3 = 5651 \; P/EL^2$

solved in Excel, confirmed in ABAQUS

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ASSIGNMENT #8

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Input2F.inp

```
*heading
HW 8, second half
*node
1, 0,0
5, 1,0
*ngen, nset=beam
1,5,1
*user element, type=U1, nodes=2, linear
1,2,3
*matrix, type=stiffness
0.02
0, 0.007692
0, 0.001923, 0.0006577
-0.02, 0, 0, 0.02
0, -0.007692, -0.001923, 0,
0.007692
0, 0.001923, 0.0003038, 0,
-0.001923, 0.0006577
*element, type=U1, elset=beam
1, 1,3
2, 3,5
*uel property, elset=beam
*boundary
1, 1,3
*step, perturbation
*static
*cload
5, 2, -1
*node file, nset=beam
U
*node print, nset=beam
U
*end step
```

```
** THE FOLLOWING TABLE IS PRINTED FOR NODES BELONGING TO NODE SET BEAM
**
**      NODE FOOT-    U1          U2           U3
**           NOTE
**
**        1          0.000     -1.0000E-36 -1.0000E-36
**        3          0.000      -1190.       -4238.
**        5          0.000      -3792.       -5651.
**
**MAXIMUM            0.000     -1.0000E-36 -1.0000E-36
**AT NODE                  1            1            1
**                               V3           θ3
**MINIMUM            0.000      -3792.       -5651.
**AT NODE                  1            5            5
                             · P/EL      · P/EL^2
```

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ASSIGNMENT #8

Input2R.inp

```
*heading
HW 8, second half
*node
1, 0,0
5, 1,0
*ngen, nset=beam
1,5,1
*user element, type=U1, nodes=2, linear
1,2,3
*matrix, type=stiffness
0.02
0, 0.007692
0, 0.001923, 0.0004974
-0.02, 0, 0, 0.02
0, -0.007692, -0.001923, 0,
0.007692
0, 0.001923, 0.0004641, 0,
-0.001923, 0.0004974
*element, type=U1, elset=beam
1, 1,3
2, 3,5
*uel property, elset=beam
*boundary
1, 1,3
*step, perturbation
*static
*cload
5, 2, -1
*node file, nset=beam
U
*node print, nset=beam
U
*end step
```

stiffness matrix considering $L_e = 0.5L$

```
** THE FOLLOWING TABLE IS PRINTED FOR NODES BELONGING TO NODE SET BEAM
**
**      NODE FOOT-    U1            U2           U3
**           NOTE
**
**        1         0.000      -1.0000E-36 -1.0000E-36
**        3         0.000      -1.1391E+04 -4.5045E+04
**        5         0.000      -3.7798E+04 -6.0060E+04
**
**MAXIMUM          0.000      -1.0000E-36 -1.0000E-36
**AT NODE                  1            1            1
**                               v3           θ3
**MINIMUM          0.000      -3.7798E+04 -6.0060E+04
**AT NODE                  1    ·P/EL   5   ·P/EL²  5
```

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ASSIGNMENT #8

E=	1	nu=	0.3
h=	0.1	G=	0.384615
d=	0.1		
L=	0.25	p=	1

4 element, full integration

KF(1 elem)=

$$\begin{bmatrix} 0.04 & 0 & 0 & -0.04 & 0 & 0 \\ 0 & 0.015385 & 0.001923 & 0 & -0.01538 & 0.001923 \\ 0 & 0.001923 & 0.000354 & 0 & -0.00192 & 0.000127 \\ -0.04 & 0 & 0 & 0.04 & 0 & 0 \\ 0 & -0.01538 & -0.00192 & 0 & 0.015385 & -0.00192 \\ 0 & 0.001923 & 0.000127 & 0 & -0.00192 & 0.000354 \end{bmatrix}$$

KR(4 elem)=

$$\begin{bmatrix} 0.08 & 0 & 0 & -0.04 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.030769 & 0 & 0 & -0.01538 & 0.001923 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.000708 & 0 & -0.00192 & 0.000127 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.04 & 0 & 0 & 0.08 & 0 & 0 & -0.04 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.01538 & -0.00192 & 0 & 0.030769 & 0 & 0 & -0.01538 & 0.001923 & 0 & 0 & 0 \\ 0 & 0.001923 & 0.000127 & 0 & 0 & 0.000708 & 0 & -0.00192 & 0.000127 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.04 & 0 & 0 & 0.08 & 0 & 0 & -0.04 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.01538 & -0.00192 & 0 & 0.030769 & 0 & 0 & -0.01538 & 0.001923 \\ 0 & 0 & 0 & 0 & 0.001923 & 0.000127 & 0 & 0 & 0.000708 & 0 & -0.00192 & 0.000127 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.04 & 0 & 0 & 0.04 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.01538 & -0.00192 & 0 & 0.015385 & -0.00192 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.001923 & 0.000127 & 0 & -0.00192 & 0.000354 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \qquad U = \begin{bmatrix} 0 \\ -1028.98 \\ -7711.86 \\ 0 \\ -3710.51 \\ -13220.3 \\ 0 \\ -7493.73 \\ -16525.4 \\ 0 \\ -11827.8 \\ -17627.1 \end{bmatrix} \rightarrow$$

$v_5 = 11{,}828\ P/EL$

$\theta_5 = 17{,}627\ P/EL^2$

✓

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ASSIGNMENT #8

E=	1	nu=	0.3
h=	0.1	G=	0.384615
d=	0.1		
L=	0.25	p=	1

4 elements, selectively reduced integration

$$
KR(1\ elem)=\begin{vmatrix}
0.04 & 0 & 0 & -0.04 & 0 & 0 \\
0 & 0.015385 & 0.001923 & 0 & -0.015385 & 0.001923 \\
0 & 0.001923 & 0.000274 & 0 & -0.001923 & 0.000207 \\
-0.04 & 0 & 0 & 0.04 & 0 & 0 \\
0 & -0.015385 & -0.001923 & 0 & 0.015385 & -0.001923 \\
0 & 0.001923 & 0.000207 & 0 & -0.001923 & 0.000274
\end{vmatrix}
$$

$$
KR(4\ elem)=\begin{vmatrix}
0.08 & 0 & 0 & -0.04 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.030769 & 0 & 0 & -0.015385 & 0.001923 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.000547 & 0 & -0.001923 & 0.000207 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.04 & 0 & 0 & 0.08 & 0 & 0 & -0.04 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.015385 & -0.001923 & 0 & 0.030769 & 0 & 0 & -0.015385 & 0.001923 & 0 & 0 & 0 \\
0 & 0.001923 & 0.000207 & 0 & 0 & 0.000547 & 0 & -0.001923 & 0.000207 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.04 & 0 & 0 & 0.08 & 0 & 0 & -0.04 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.015385 & -0.001923 & 0 & 0.030769 & 0 & 0 & -0.015385 & 0.001923 \\
0 & 0 & 0 & 0 & 0.001923 & 0.000207 & 0 & 0 & 0.000547 & 0 & -0.001923 & 0.000207 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.04 & 0 & 0 & 0.04 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.015385 & -0.001923 & 0 & 0.015385 & -0.001923 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.001923 & 0.000207 & 0 & -0.001923 & 0.000274
\end{vmatrix}
$$

$$
P=\begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{vmatrix}
\qquad
U=\begin{vmatrix} 0 \\ -3346.25 \\ -26250 \\ 0 \\ -12317.5 \\ -45000 \\ 0 \\ -25038.75 \\ -56250 \\ 0 \\ -39635 \\ -60000 \end{vmatrix}
$$

$v_5 = 39{,}635\ P/EL$ → very close to beam theory answer

$\theta_5 = 60{,}000\ P/EL^2$

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ASSIGNMENT #8

			Full			Reduced		
			1 elem	2 elem	4 elem	1 elem	2 elem	4 elem
nu=	0.5	V_end	-1173.79	-4317.86	-13070.3	-30300	-37800	-39675
		θ_end	-1747.57	-6428.57	-19459.5	-60000	-60000	-60000
nu=	0.1	V_end	-865.793	-3253.09	-10471.5	-30220	-37720	-39595
		θ_end	-1291.59	-4852.94	-15621.3	-60000	-60000	-60000
nu=	0.7 ?	V_end	-1326.46	-4829.44	-14213.1	-30340	-37840	-39715
		θ_end	-1972.92	-7183.1	-21139.9	-60000	-60000	-60000

Changing ν altered the rotations and deflections at the tip of the beam. The changes were much more significant in the fully integrated case. Additionally, increasing the number of elements increased the influence of ν.

✓

Dec. 8, 2006

CE 381R

FALL 2006

Assignment 8

1. Consider the two-node beam finite element discussed in class. Assume that the cross-section is rectangular with width d and depth h. Also, let L, E and G be the length, Young's modulus and shear modulus of the element. Obtain an expression for the stiffness matrix of the element in the form:

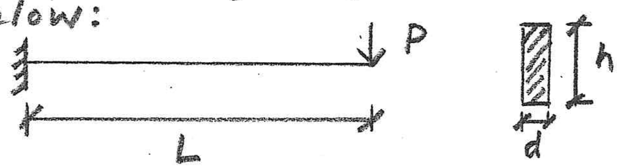
$$E \cdot \underset{\sim}{K}_{\varepsilon} + G \cdot \underset{\sim}{K}_{\gamma}$$

(i.e., obtain expressions for the 6x6 matrices $\underset{\sim}{K}_{\varepsilon}$ and $\underset{\sim}{K}_{\gamma}$) using

a) full integration for both $\underset{\sim}{K}_{\varepsilon}$ and $\underset{\sim}{K}_{\gamma}$

b) full integration for $\underset{\sim}{K}_{\varepsilon}$ and reduced integration for $\underset{\sim}{K}_{\gamma}$

2. Use the two-node beam finite elements derived above in the analysis of the cantilever beam shown below:



$L/h = 10$, $d/h = 1$, $\nu = 0.3$

Report results for meshes of 1, 2 and 4 (equal-size) finite elements (with full and selectively reduced integrations) for the deflection and rotation at the tip (point of load application). Compare with the results from engineering beam theory. Examine the sensitivity of the finite-element results to Poisson's ratio.

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GAUSSIAN QUADRATURE FORMULAS FOR TRIANGLES

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SUMMARY

Several formulas are presented for the numerical integration of a function over a triangular area. The formulas are of the Gaussian type and are fully symmetric with respect to the three vertices of the triangle.

INTRODUCTION

In finite element work, the need sometimes arises to numerically integrate a function over a triangular area. Suitable formulas have been derived by Silvester,[1] Irons[2] and Hammer and co-workers.[3] The formulas of the latter were compiled in a convenient form by Felippa,[4] and the compilation has been reproduced, along with Iron's formulas, in the book by Zienkiewicz.[5] The formulas of Silvester have simple coefficients and are fully symmetric with respect to the three vertices of the triangle, but they are of the Newton–Cotes type and thus are relatively

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inefficient compared with the Gaussian formulas. The formulas of Irons, which make use of successive application of the Gauss and Radau quadrature rules, are highly efficient but have the aesthetically unappealing feature that the sampling points are not arranged symmetrically within the triangle. From the point of view of both numerical efficiency and aesthetics the best formulas are those of Hammer and co-workers which are of the Gaussian type and fully symmetric with respect to the three vertices of the triangle.

Five different formulas were derived by Hammer and co-workers which have truncation errors ranging from second- to sixth-order. Unlike the one-dimensional case, the theory of two- (and three-) dimensional integration is not highly developed. The discovery of additional formulas remains a matter of *ad hoc* investigation in each specific case. The purpose of this note is to present additional formulas of the Hammer and co-workers type which are believed to be new and which were derived in the course of research on triangular finite elements for shells. The formulas will be displayed first, followed by some remarks on their derivation.

FORMULAS

The formulas for numerical integration of a function f over a triangle of area A are all of the form

$$\int\int f \, dA = A \sum_{i=1}^{N} w_i f(\xi_i, \eta_i, \zeta_i) \tag{1}$$

where ξ_i, η_i, ζ_i are the area co-ordinates of the i-th sampling point and w_i is the weight associated with the i-th sampling point. Values of the constants $w_i, \xi_i, \eta_i, \zeta_i$ for the various formulas are listed in Table I. For completeness, three of the formulas due to Hammer and co-workers are also included in the table.* Readers unfamiliar with area co-ordinates may refer to Zienkiewicz's book.[5]

Because of the triangular symmetry of the formulas the sampling points occur in groups of either six, three or one. Thus, if a sampling point has area co-ordinates (α, β, γ), none of which are equal, then there are also five symmetrically disposed points with co-ordinates (α, γ, β), (β, α, γ), (β, γ, α), (γ, α, β) and (γ, β, α), all of which have the same weight. A sampling point which has two equal co-ordinates, for example (α, β, β), occurs as a member of a trio of sampling points with equal weights, the co-ordinates of the other two points of the trio being (β, α, β) and (β, β, α). A single sampling point with co-ordinates $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ may occur at the centroid of the triangle. To save space the weight and co-ordinates of only one point of a group are given in Table I. The entry in the column headed 'Multiplicity' indicates whether the point belongs to a group of one, three or six points. The 'Degree of precision' shown in the table indicates the highest degree polynomial which the formula integrates exactly.

REMARKS ON DERIVATION

The formulas are designed to exactly integrate complete polynomials of a given degree. Since the quadrature-formulas are to be exact for all polynomials of a complete set, one can write (1) for each such polynomial and the relations so obtained are the equations which determine the parameters $w_i, \xi_i, \eta_i, \zeta_i$. The degree of completeness and number of sampling points must of course be selected so that the number of unknown parameters is consistent with the number of available equations.

The equations for w_i, ξ_i, η_i and ζ_i are highly non-linear and their solution is not straightforward, although some steps can be taken to put the equations in a rather manageable form.

* This also affords the opportunity to correct the four-point formula, which is given incorrectly both by Felippa[4] and Zienkiewicz.[5] The 'severe cancellation errors' which are said to occur with this formula may be due simply to use of an incorrect formula.

It was found convenient to work in Cartesian co-ordinates originating at the centroid rather than area co-ordinates, and since the quadrature formulas are independent of the shape of the triangle it was permissible to use the most convenient shape, namely a unit equilateral triangle. The requirement of triangular symmetry means that the sampling points can occur only in symmetric groups of one, three or six points, and imposition of this requirement reduces the number of unknown co-ordinates to a minimum. The number of independent equations was found to be correspondingly reduced. The three polynomials of second degree furnished only one essentially independent equation, as did the polynomials of third, fourth, fifth and seventh degrees. The seven polynomials of the sixth degree gave rise to two independent equations, while the first degree equations were identically satisfied as a result of the imposed symmetry. In some cases it was possible, by judicious manipulation, to solve the equations in closed form. In others, the solution had to be obtained by trial and error, after first eliminating all but one of the unknowns. In all cases the final formulas were checked by using them to integrate a complete set of polynomials.

Although the derivation was carried out for an equilateral triangle, any triangle can be mapped onto an equilateral triangle by a linear transformation. Area co-ordinates are invariant under such mapping, and hence the quadrature formula (1) applies to triangles of any shape.

Table I

w_j	ξ_j	η_j	ζ_j	Multiplicity
	3-point formula	Degree of precision 2		
0·33333 33333 33333	0·66666 66666 66667	0·16666 66666 66667	0·16666 66666 66667	3
	3-point formula	Degree of precision 2		
0·33333 33333 33333	0·50000 00000 00000	0·50000 00000 00000	0·00000 00000 00000	3
	4-point formula	Degree of precision 3		
−0·56250 00000 00000	0·33333 33333 33333	0·33333 33333 33333	0·33333 33333 33333	1
0·52083 33333 33333	0·60000 00000 00000	0·20000 00000 00000	0·20000 00000 00000	3
	6-point formula	Degree of precision 3		
0·16666 66666 66667	0·65902 76223 74092	0·23193 33685 53031	0·10903 90090 72877	6
	6-point formula	Degree of precision 4		
0·10995 17436 55322	0·81684 75729 80459	0·09157 62135 09771	0·09157 62135 09771	3
0·22338 15896 78011	0·10810 30181 68070	0·44594 84909 15965	0·44594 84909 15965	3
	7-point formula	Degree of precision 4		
0·37500 00000 00000	0·33333 33333 33333	0·33333 33333 33333	0·33333 33333 33333	1
0·10416 66666 66667	0·73671 24989 68435	0·23793 23664 72434	0·02535 51345 59132	6
	7-point formula	Degree of precision 5		
0·22500 00000 00000	0·33333 33333 33333	0·33333 33333 33333	0·33333 33333 33333	1
0·12593 91805 44827	0·79742 69853 53087	0·10128 65073 23456	0·10128 65073 23456	3
0·13239 41527 88506	0·47014 20641 05115	0·47014 20641 05115	0·05971 58717 89770	3
	9-point formula	Degree of precision 5		
0·20595 05047 60887	0·12494 95032 33232	0·43752 52483 83384	0·43752 52483 83384	3
0·06369 14142 86223	0·79711 26518 60071	0·16540 99273 89841	0·03747 74207 50088	6
	12-point formula	Degree of precision 6		
0·05084 49063 70207	0·87382 19710 16996	0·06308 90144 91502	0·06308 90144 91502	3
0·11678 62757 26379	0·50142 65096 58179	0·24928 67451 70910	0·24928 67451 70911	3
0·08285 10756 18374	0·63650 24991 21399	0·31035 24510 33785	0·05314 50498 44816	6
	13-point formula	Degree of precision 7		
−0·14957 00444 67670	0·33333 33333 33333	0·33333 33333 33333	0·33333 33333 33333	1
0·17561 52574 33204	0·47930 80678 41923	0·26034 59660 79038	0·26034 59660 79038	3
0·05334 72356 08839	0·86973 97941 95568	0·06513 01029 02216	0·06513 01029 02216	3
0·07711 37608 90257	0·63844 41885 69809	0·31286 54960 04875	0·04869 03154 25316	6

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SHORT COMMUNICATIONS

EFFECTIVE QUADRATURE RULES FOR QUADRATIC SOLID ISOPARAMETRIC FINITE ELEMENTS

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In a recent note,[1] Irons demonstrated several integration formulae for use with solid isoparametric finite elements. The use of different formulae enabled variations in accuracy and running times to be achieved, and clearly one aims for the cheapest rule for a given degree of accuracy.

One of the most successful finite elements is the 20-node isoparametric solid element, and this has been used frequently in the CEGB. This note is mainly concerned with this particular element. The usual integrating rule used is the Gauss $3\times3\times3$ rule, with 27 points per element. In certain circumstances, namely shell-type structures subjected to bending modes, the use of a $2\times2\times2$ rule (8 points per element) has been shown to give good results with rapid convergence[2] in a manner very similar to the results of Zienkiewicz and co-workers[3] using a quadratic thick shell element.[4] However, this reduced rule does not always give satisfactory answers in membrane modes or in shells with solid attachments, and so, as with solid problems, alternative economies in integration techniques are desirable.

The two most economical rules giving the same order of accuracy as the Gauss $3\times3\times3$ rule appear to be the 14 point rule mentioned in Reference 1, and given originally by Hammer and Stroud[5] and the slightly cheaper, slightly less accurate, 13 point rule given originally by Stroud.[6] Details of the 14 point rule may be found in Reference 1 and are not repeated here. The 13 point rule is defined by the following co-ordinates (in the double unit cube) and weighting coefficients:

$$\begin{array}{llll} (0,\ 0,\ 0), & \text{coeff. A} & & \\ \pm(\lambda,\ \xi,\ \xi), & \pm(\xi,\ \lambda,\ \xi), & \pm(\xi,\ \xi,\ \lambda), & \text{coeff. B} \\ \pm(\mu,\ \mu,\ \lambda), & \pm(\mu,\ \gamma,\ \mu), & \pm(\gamma,\ \mu,\ \mu), & \text{coeff. C} \end{array}$$

where

$$\lambda = 0{\cdot}88030430 \qquad \xi = -0{\cdot}49584802$$
$$\mu = 0.79562143, \qquad \gamma = 0{\cdot}025293237$$
$$A = 1{\cdot}68421056, \qquad B = 0{\cdot}54498736, \qquad C = 0{\cdot}507644216$$

One observes that the locations of the integrating points are not completely symmetric in the cube.

In order to demonstrate the relative accuracies of these rules, 3 test problems were run using the $3\times3\times3$ Gauss rule, the 14 point rule and the 13 point rule. The comparisons also include results using the $2\times2\times2$, $4\times4\times4$ and $5\times5\times5$ Gauss rules.

Example 1

A cantilever, encastre at one end, and of dimensions $24\times8\times8$ units, was shear-loaded at the other end. Six 20-node brick elements were used, three along the length and two through the depth. The end deflection is shown in Table I together with the axial stress at the wall for the different integrating rules. The $4\times4\times4$ and $5\times5\times5$ Gauss rules are added to show the correctly integrated values, from which the percentage relative errors of the 13 and 14 point rules are derived. The $3\times3\times3$ rule is shown to integrate exactly, and the 14 point rule is more accurate than the 13 point rule. In this case, the $2\times2\times2$ rule also gives good results since the dominant mode is bending.

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Table I. Comparison of different integrating rules for the cantilever problem

Integrating rule	End deflection	Percentage error	Axial stress at wall	Percentage error
13	0·0933344	0·0220	1830·73	0·2689
14	0·0933570	0·0023	1825·55	0·0148
2×2×2	0·0924773	0·9401	1859·41	1·8397
3×3×3	0·0933549	0·0000	1825·82	0·0000
4×4×4	0·0933549		1825·82	
5×5×5	0·0933549		1825·82	

Example 2

A pressure-loaded cylinder was considered with a radius/thickness ratio of 50 to 1. Here, four 20-node brick elements were used around the half-circumference (a half used with symmetric cuts) and two axially, making eight elements in all. The thinness of the cylinder gave a very distorted set of elements to test the integrating rules, and generally the results were excellent. Table II shows the various results for the hoop stress in the midsurface at the centre of the half-cylinder, the radial deflection at a typical point on the inner surface, and the radial stress at a typical point on the inner surface, these values being sensibly constant over the respective surface. The 4×4×4 and 5×5×5 point rules are again added to show the correctly integrated values, and the percentage relative error of the 13, 14 and 3×3×3 point rules are derived from these (the 5×5×5 point rule specifically, since minor variations between the 4×4×4 and 5×5×5 point rules exist). The results are again very good, although this time the 13 point rule is better than the 14 point rule. The errors with the radial stress are larger than usual because that particular component is of small order compared with other components. The results for the 2×2×2 rule are very inaccurate, particularly for the stresses, since no bending exists.

Table II. Comparison of different integrating rules for moderately thin cylinders

Integrating rule	Hoop stress	Percentage error	Radial deflection	Percentage error	Radial stress	Percentage error
13	$0{\cdot}495400 \times 10^5$	0·0164	2·48089	0·0016	$-0{\cdot}100050 \times 10^4$	1·9435
14	$0{\cdot}495513 \times 10^5$	0·0392	2·48085	0·0032	$-0{\cdot}976273 \times 10^3$	4·3179
2×2×2	$-0{\cdot}344747 \times 10^5$	168·6010	2.78646	12·3151	$-0{\cdot}169223 \times 10^6$	
3×3×3	$0{\cdot}495517 \times 10^5$	0·0400	2.48085	0·0032	$-0{\cdot}974910 \times 10^3$	4·4515
4×4×4	$0{\cdot}495315 \times 10^5$		2·48093		$-0{\cdot}102125 \times 10^4$	
5×5×5	$0{\cdot}495319 \times 10^5$		2·48093		$-0{\cdot}102033 \times 10^4$	

Example 3

The problem of Example 2 was repeated with the radius/thickness ratio increased to 1,000 : 1. This problem was used to test the integrating rules in extremely distorted elements. The same reference values as in Example 2 were used (Table III). This time the relative errors are expressed in terms of the simple results for hoop stress and deflection, with pressure of 1,000 units. The radial stress should equal −1,000 units, and is seen to be poor, particularly in the 3×3×3 case, again because of the relatively small order of this component and roundoff effects. The 13 and 14 point rules give better

Table III. Comparison of different integrating rules for extremely thin cylinders

Integrating rule	Hoop stress	Percentage error	Radial deflection	Percentage error	Radial stress	Percentage error
13	$0{\cdot}100055 \times 10^7$	0·055	$0{\cdot}998858 \times 10^3$	0·1142	$-0{\cdot}669000 \times 10^3$	33.1
14	$0{\cdot}100071 \times 10^7$	0·071	$0{\cdot}998833 \times 10^3$	0·1167	$-0{\cdot}737000 \times 10^3$	26·3
2×2×2	$-0{\cdot}568722 \times 10^9$	—	$0{\cdot}399235 \times 10^5$	—	$-0{\cdot}112392 \times 10^{10}$	—
3×3×3	$0{\cdot}100216 \times 10^7$	0·216	$0{\cdot}998777 \times 10^3$	0·1223	$0{\cdot}116084 \times 10^5$	1,060·84

accuracy in this problem than the $3 \times 3 \times 3$ rule. The results for the $2 \times 2 \times 2$ rule are extremely inaccurate.

The beam problem of Example 1 has been repeated with two 15-node quadratic triangle prism elements in place of each 20-node brick element, and again comparing the results from the $3 \times 3 \times 3$ rule with the 13 and 14 point rules. The accuracy of the different rules were of the same orders as in Example 1. These elements are used frequently with the 20-node elements to accommodate mesh size changes without incurring unduly distorted elements.

A comparison of the 13 point rule with the $3 \times 3 \times 3$ rule has been conducted on a very large three-dimensional structure of complex shape, containing 159 20-node and 15-node elements and 3,912 degrees of freedom. Inspection of the displacements and stresses of largest magnitude showed that the relative errors between the two rules varied by up to 4 per cent. The forward solution execution time was reduced by 75 per cent.

It is concluded from these results that the more economical 13 and 14 point rules are of similar accuracy to the standard $3 \times 3 \times 3$ rule for both very distorted and regular shaped elements with 20 and 15 nodes, and that the quality of accuracy is maintained in very large problems as well as in small test problems. The $2 \times 2 \times 2$ rule is very accurate for certain types of structure (generally shells) subjected to bending modes, but its success in more general applications and loading systems is not guaranteed, and so the rule must be used with care.

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QUADRATURE RULES FOR BRICK BASED FINITE ELEMENTS

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Already isoparametric hexahedral (brick) finite elements with 20 or 32 nodes[1] are highly competitive in practice,[2] despite the observation that 50 per cent of the total computation is often absorbed in numerically integrating the coefficients of the equations.[2] This cost is approximately halved[3] by a method based, essentially, on using a 9×9 [**D**] matrix which operates on $\partial u/\partial x, \partial u/\partial y, \partial u/\partial z, \partial v/\partial x, \ldots \partial w/\partial z$—a technique which, moreover, is more general than the classical $\sum \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B} \times$ constant algorithm.[4]

The purpose of this note is to demonstrate how one may further halve the cost by using simpler integration formulae having the same order of truncation error. We compare certain Gaussian-type rules, some of them new and all of them designed to integrate complete polynomials, with the corresponding product-Gauss rules which are normally used.[5] The former integrate correctly $\sum \mathbf{C}_{ijk} x^i y^j z^k$, $i+j+k \leqslant n$, while the latter integrate correctly a much larger number of terms, those with $i, j, k \leqslant n$. All these rules have been checked by computer. They are now presented in the form:

$$\begin{aligned}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = {} & \mathbf{A}_1 f(0,0,0) \text{ (1 term)} \\ & +\mathbf{B}_6\{f(-b,0,0)+f(b,0,0)+f(0,-b,0)+\ldots \text{6 terms}\} \\ & +\mathbf{C}_8\{f(-c,-c,-c)+f(c,-c,-c)+\ldots \text{8 terms}\} \\ & +\mathbf{D}_{12}\{f(-d,-d,0)+\ldots+f(d,0,-d)+\ldots \text{12 terms}\}\end{aligned}$$

The rules are listed below, labelled according to the number of points they use.

Rule 6 (i.e. a 6-point rule). $\mathbf{B}_6 = 8/6$, $b = 1$. That is, we take the mean of the six mid-face values. This rule is accurate to the complete cubic in x, y, z, i.e. 20 terms. A multiplying constant and x, y, z are freely chosen at 6 points, i.e. 24 constants are chosen. The efficiency is defined as 20/24 so that the rule is nearly Gaussian.

This is an excellent rule. Since the mid-face values are so representative, we should evidently calculate stresses at mid-face in brick elements, rather than at corners—which are the worst possible positions!

Rule 8G (i.e. the $2\times2\times2$ product—Gauss rule). Included for comparison with Rule 6.

Rule 14 with $\mathbf{B}_6 = 0{\cdot}886426593$, $b = 0{\cdot}795822426$, $\mathbf{C}_8 = 0{\cdot}335180055$ and $c = 0{\cdot}758786911$. Accurate to the complete quintic, like Rule 27G below. Another excellent rule, with precisely Gaussian efficiency (56/56), small multiplying constants, and moderately small sextic errors. (The good efficiency is surprising when the term $\mathbf{A}_1$ is absent.)

Rule 15a with $\mathbf{A}_1 = 1{\cdot}564444444$, $\mathbf{B}_6 = 0{\cdot}355555556$, $b = 1$,
$\mathbf{C}_8 = 0{\cdot}537777778$ and $c = 0{\cdot}674199862$.

A slightly less effective rule, whose surplus constant is chosen to make it modular with Rules 1 and 6 above, allowing a flexible strategy.

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Rule 15b with $\mathbf{A}_1 = 0{\cdot}712137436$, $\mathbf{B}_6 = 0{\cdot}686227234$, $b = 0{\cdot}848418011$, $\mathbf{C}_8 = 0{\cdot}396312395$ and $c = 0{\cdot}727662441$. Another rule with surplus constant, now modular with Rule 27a below.

Rule 19 with $\mathbf{A}_1 = 2{\cdot}074074074$, $\mathbf{B}_6 = -0{\cdot}24691358$, $\mathbf{D}_{12} = 0{\cdot}617283951$

and

$$b = d = 0{\cdot}774596669.$$

A rule previously recommended[6] but apparently much less efficient (56/76).

Rule 27a with $\mathbf{A}_1 = 0{\cdot}788073483$, $\mathbf{B}_6 = 0{\cdot}499369002$, $b = 0{\cdot}848418011$, $\mathbf{C}_8 = 0{\cdot}478508449$, $c = 0{\cdot}652816472$, $\mathbf{D}_{12} = 0{\cdot}032303742$ and $d = 1{\cdot}106412899$. A super-efficient rule (120/108) correct to complete heptic like Rule 64G below. (This is unusual among Gaussian rules in that it has sampling points outside the domain.)

Rule 27G (i.e. the $3 \times 3 \times 3$ product—Gauss rule). Included for comparison with Rule 14 etc.

Rule 64G (i.e. the $4 \times 4 \times 4$ product—Gauss rule). Included for comparison with Rule 27a.

Table I. Errors of rules

Rule No.	Quartic terms		Sixth degree terms			Eighth degree terms			
	x^4	$x^2 y^2$	x^6	$x^4 y^2$	$x^2 y^2 z^2$	x^8	$x^6 y^2$	$x^4 y^4$	$x^4 y^2 z^2$
6	1·1	−0·89							
8G	−0·71	0	−0·85	−0·24	0				
14	0	0	−0·18	−0·02	0·22				
15a	0	0	−0·03	−0·13	0·11				
15b	0	0	−0·16	−0·06	0·17				
19	0	0	−0·18	0	−0·30	−0·31	−0·06	0	−0·18
27a	0	0	0	0	0	0·09	0·04	0·10	−0·05
27G	0	0	−0·18	0	0	−0·31	−0·06	0	0
64G	0	0	0	0	0	−0·05	0	0	0

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3. B. M. Irons, 'Economical computer techniques for numerically integrated finite elements', *Int. J. num. Meth. Engng*, **1**, 201 (1969).
4. O. C. Zienkiewicz, *The Finite Element Method*, 2nd edn., McGraw–Hill, New York, 1970, Chap. 19.
5. A. H. Stroud and Don Secrest, *Gaussian Quadrature Formulas*, Prentice–Hall, Englewood Cliffs, 1966, p. 40.
6. J. C. P. Miller, 'Numerical quadrature in two or more dimensions', *Maths Comput.* **14**, 130–138 (1960).

INTRODUCTION DAY

Program to be used: ABAQUS
know some programming (like C++)
linear algebra background is also good

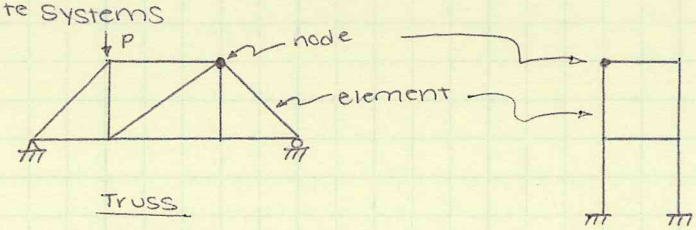
Books (optional)
- Hughes is soft-cover, might be cheaper
- more for reference; class doesn't follow one
- other two are also good, but more expensive

Homeworks - approx. 7
no TA for the class; talk to him directly
MWF 3-4pm, or generally, in the afternoon
around TR, too (not up at lab)

Exams - two 50-minute exams
open book, open notes
plus a final, Dec. 13th, 9-12am

Beginning of the Finite Element Method
- changed systems that appeared continuous into something discrete that can be analyzed more easily

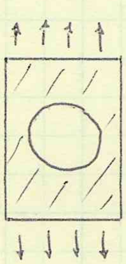
Discrete Systems



Truss
- 9 elements/members
- 6 nodes/joints

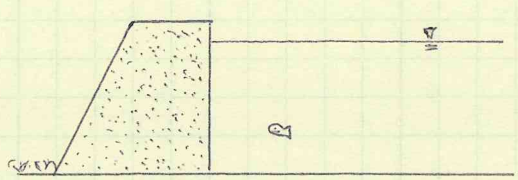
Frame
- 6 elements
- 6 nodes
(connections are different from truss)

Continuous Systems



perforated plate
(hole in the middle)

no obvious beams/columns
or nodes and elements

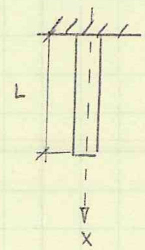


dam/reservoir system
+ rock below
+ sediment in water
+
+ ...

THE FINITE ELEMENT METHOD

one-dimensional example

- demonstrates the concepts underlying the method, but does not reveal the power of the method



A : cross-sectional area, constant

E : modulus of elasticity, constant

γ : unit weight

Loads —

- distributed self-weight load
- applied load Q, vertically down at bottom tip of rod

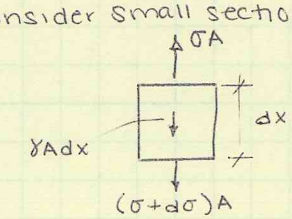
solving for —

- $u(x)$, displacement
- $\varepsilon(x)$, axial strain
- $\sigma(x)$, axial stress

Governing Equations:

(1) Equilibrium equation

consider small section



at a distance x down rod

$\Sigma F = 0: \quad (\sigma + d\sigma)A - \sigma A + \gamma A\, dx = 0$

$$\frac{d\sigma}{dx} + \gamma = 0$$

(2) Strain-displacement equation (NOT compatibility)

deformation in terms of displacement

(+) = stretching, (−) = compressing

$$\varepsilon = \frac{du}{dx}$$

difference in displacements over separation in space (Δx)

(3) constitutive equation (material properties)

$$\sigma = E\varepsilon$$

if material is not elastic, this is the only equation that changes

Boundary conditions

$u(0) = 0$

$\sigma(L)A = Q$, or $\sigma(L) = Q/A$

where you know the displacement, you don't know the stress; if you know the stress, displacement is unknown.

ONE · DIMENSIONAL EXAMPLE

Example, cont'd

Exact solution by the displacement method

$$\frac{d\sigma}{dx} + \gamma = 0 \qquad \text{sub in } \sigma = E\varepsilon$$

$$E\frac{d\varepsilon}{dx} + \gamma = 0 \qquad \text{sub in } \varepsilon = \frac{du}{dx}$$

$$E\frac{d^2u}{dx^2} + \gamma = 0$$

$$\frac{d^2u}{dx^2} + \frac{\gamma}{E} = 0 \quad , \quad \frac{d^2u}{dx^2} = \frac{-\gamma}{E}$$

integrate once: $\frac{du}{dx} = \frac{-\gamma x}{E} + c_1$

integrate again: $\underline{u = \frac{-\gamma}{2E}x^2 + c_1 x + c_2}$

use boundary conditions:

$$u(0) = 0, \quad c_2 = 0$$

$$\sigma(L) = \frac{Q}{A} \longrightarrow E\varepsilon(L) = \frac{Q}{A} \longrightarrow E\left.\frac{du}{dx}\right|_L = \frac{Q}{A}$$

$$\left.\frac{du}{dx}\right|_L = \frac{Q}{EA}$$

$$-\frac{\gamma \overset{L}{x}}{E} + c_1 = \frac{Q}{EA}$$

$$\underline{c_1 = \frac{Q}{EA} + \frac{\gamma L}{E}}$$

Putting it together:

$$\boxed{u(x) = \frac{-\gamma x^2}{2E} + \frac{Qx}{EA} + \frac{\gamma L x}{E}}$$

real FEM does not like to use derivatives

use virtual displacement

$$\frac{d\sigma}{dx} + \gamma = 0 \qquad \text{everywhere in the rod}$$

so does a function that is a multiple

use δu = virtual displacement

$$\delta u\left(\frac{d\sigma}{dx} + \gamma\right) = 0 \quad \text{through rod}$$

$$A\int_0^L \delta u\left(\frac{d\sigma}{dx} + \gamma\right)dx = 0$$

integrate over volume, not length

INTRO TO THE FEM

using virtual displacement

$$A\int_0^L \delta u\left(\frac{d\sigma}{dx}+\gamma\right)dx = 0 \quad \longrightarrow \quad A\int_0^L \delta u\cdot\frac{d\sigma}{dx}\cdot dx + A\int_0^L \delta u\,\gamma\,dx = 0$$

use integration by parts:

$$\frac{d}{dx}(f\cdot g) = \frac{df}{dx}\cdot g + f\frac{dg}{dx}$$

$$\text{or} \quad f\frac{dg}{dx} = \frac{d}{dx}(f\cdot g) - \frac{df}{dx}\cdot g$$

solving through:

$$A\cdot\delta u\cdot\sigma\Big|_0^L - A\int_0^L \frac{d\,\delta u}{dx}\cdot\sigma\,dx + A\int_0^L \delta u\cdot\gamma\cdot dx = 0$$

with $\varepsilon = \frac{du}{dx}$, define $\delta\varepsilon = \frac{d(\delta u)}{dx}$

recall: at $x = L$, force in the x·direction is denoted by P_L

$$P_L = \sigma(L)\cdot A$$

at $x = 0$, force in the x·dir will be denoted by P_0

$$P_0 = -\sigma(0)\cdot A$$

Therefore,

$$A\delta u\cdot\sigma\Big|_0^L = A\delta u(L)\cdot\sigma(L) - A\delta u(0)\cdot\sigma(0)$$

$$= \delta u_0 P_0 + \delta u_L P_L$$

$$\underline{\underline{A\int_0^L \delta\varepsilon\cdot\sigma\,dx = \delta u_0 P_0 + \delta u_L P_L + \cancel{\delta u}\,A\int_0^L \delta u\cdot\gamma\,dx}}$$

Principle of virtual work—
equation holds for arbitrary
δu (and $\delta\varepsilon = d(\delta u)/dx$)

PRINCIPLE OF VIRTUAL WORK

General eq. from last week

$$A\int_0^L \delta\varepsilon\cdot\sigma\cdot dx = \delta u_0 P_0 + \delta u_L P_L + A\int_0^L \delta u\cdot\gamma\cdot dx$$

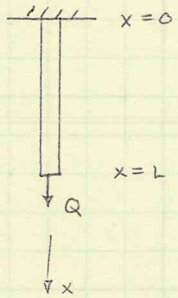
δu: arbitrary virtual disp.

$$\delta\varepsilon = \frac{d(\delta u)}{dx}$$

$\delta u_0 = \delta u(0)$, $\delta u_L = \delta u(L)$

$P_0 = -A\cdot\sigma(0)$

$P_L = A\cdot\sigma(L)$



can you transfer back to the first set of equations?
→ equivalence

σ (+) in tension,
(−) in compression

Returning:

$$\frac{d\sigma}{dx} + \gamma = 0 \qquad (1)$$

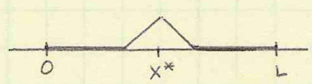
$$A\int_0^L \delta u\left(\frac{d\sigma}{dx} + \gamma\right)dx = 0 \qquad (2)$$

→ Now assume that $\frac{d\sigma}{dx} + \gamma \neq 0$ everywhere along the rod.

Specifically, assume that
$\frac{d\sigma}{dx} + \gamma > 0$ at a point x^*

– Now, ~~assume~~ assuming that $\frac{d\sigma}{dx} + \gamma$ is continuous at x^*, it is possible to identify a small segment $(x^* + \varepsilon, x^* - \varepsilon)$ in which Eq. (1) is positive?

– choose δu such that $\delta u > 0$ in x^* region and $\delta u = 0$ outside the segment



– then,

$$A\int_0^L \delta u\left(\frac{d\sigma}{dx} + \gamma\right)dx > 0$$

contradicting the principle of virtual work

PRINCIPLE OF VIRTUAL WORK

Derivation details

$$A\int_0^L \underbrace{\frac{d}{dx}(\delta u \cdot \sigma)}_{f}\, dx = A\,\delta u \cdot \sigma \Big|_0^L$$

$$\int_a^b \frac{df}{dx}\, dx = f\Big|_a^b = f(b) - f(a)$$

but, $\delta u \cdot \sigma$ must be continuous for this equation to hold

In general, at points of discontinuity of $\delta u \cdot \sigma$, additional terms must be added to the right-hand side of the PVW equation:

$$A\,\delta u(\hat{x}^-)\cdot\sigma(\hat{x}^-) - A\,\delta u(\hat{x}^+)\cdot\sigma(\hat{x}^+)$$

at a point $\hat{x}$ where $\delta u \cdot \sigma$ is discontinuous.

such as: a concentrated load acting at $\hat{x}$

Solution based on the P.V.W.

Galerkin Method:

Assume an approximation for u that satisfies the support condition $u(0)=0$. For example,

$$u(x) \sim ax$$

where a is a constant to be determined from the P.V.W.

$$u(x) = ax$$
$$\varepsilon(x) = du/dx = a$$
$$\sigma(x) = E\varepsilon = Ea$$

$$A\int_0^L \delta\varepsilon \cdot E a\, dx = \delta u_0 \cdot P_0 + \delta u_L P_L + A\int_0^L \delta u \cdot \gamma \cdot dx$$

satisfy this equation for δu of the same shape as u:

$$\delta u = \delta a \cdot x$$

where δa is an arbitrary coefficient

Considering

$$\delta\varepsilon = \frac{d(\delta u)}{dx} = \delta a,$$

$$A\int_0^L \delta a \cdot E \cdot a\, dx = 0 \cdot P_0 + \delta a L \cdot P_L + A\int_0^L \delta a \cdot x \cdot \gamma\, dx$$

METHODS OF SOLVING PVW

Review:

$$A\int_0^L \delta\varepsilon\cdot\sigma\cdot dx = \delta u_0 P_0 + \delta u_L\cdot P_L + A\int_0^L \delta u\cdot\gamma\cdot dx$$

Galerkin method

$u = ax$, a = constant to be determined

$\varepsilon = \frac{du}{dx} = a$

$\sigma = E\cdot\varepsilon = E\cdot a$

$\delta u = \delta a\cdot x$; δu, u are of the same shape

satisfies homogeneous version of support conditions

$\delta\varepsilon = \frac{d(\delta u)}{dx} = \delta a$

Boundaries:

$\delta u_0 = 0$

$\delta u_L = \delta a\cdot L$

Now substitute expressions into P.V.W equation to obtain:

$$A\int_0^L \delta a\cdot E\cdot a\cdot dx = 0\cdot P_0 + \delta a L\cdot P_L + A\int_0^L \delta a\cdot x\cdot\gamma\cdot dx$$

(drops out of equation)

so, equation does not contain (unknown) reaction forces

- all other terms include the constant δa

$$\delta a\cdot E\cdot A\cdot a\cdot L = \delta a\cdot L\cdot P_L + \gamma A\frac{L^2}{2}\cdot\delta a$$

force acting at location L, was previously defined as Q

$$\delta a\left[EALa - LQ - \tfrac{1}{2}\gamma AL^2\right] = 0$$

Since this must be true for an arbitrary δa, it follows that:

$$EALa - LQ - \tfrac{1}{2}\gamma AL^2 = 0$$

So, a is:

$$a = \frac{Q}{EA} + \frac{\gamma L}{2E}$$

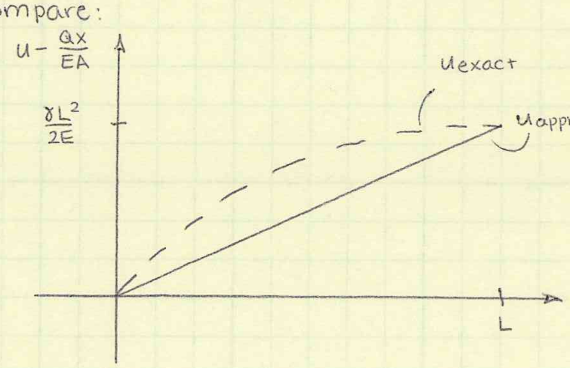
$$u = ax$$

Therefore, the linear approximation is $u = \frac{Qx}{EA} + \frac{\gamma Lx}{2E}$

Compare to exact solution,

$$u = \frac{Qx}{EA} + \frac{\gamma Lx}{E} - \frac{\gamma x^2}{2E}$$

compare:



simple approximation matches at nodes, isn't bad elsewhere

APPROXIMATIONS OF SOL'NS

Quality of approximations

If exact solution is not known, how can accuracy be determined?

How good is good enough?

To check the solution,

- consider the case $Q=0$

$$u_{approx} = \frac{\gamma L x}{2E}$$

check support conditions – $u_0 = 0$, $\sigma(L) = 0$

$$\sigma = E a = \frac{Q}{A} + \frac{\gamma L}{2}, \quad Q = 0$$

$$\sigma = \frac{\gamma L}{2} \longleftarrow \text{error.}$$

makes solution not acceptable

- consider the case $\gamma = 0$

$$u_{approx} = \frac{Qx}{EA}, \quad u_0 = 0$$

$$\sigma_{approx} = \frac{Q}{A}, \quad \sigma_L = Q/A$$

in this case, approximate solution is very accurate (locally perfect)

- in between $(Q, \gamma \neq 0)$ ... not as easy

Toward an improved Galerkin approximation, use

$$u = ax + bx^2$$

at least as good as other approximation because it contains original one – if $b=0$, $u_2 = u_1$ equations

$$\varepsilon = \frac{du}{dx} = a + 2b \cdot x$$

$$\sigma = E \cdot \varepsilon = [E][1 \quad 2x]\begin{bmatrix} a \\ b \end{bmatrix}$$

matrices are fun!

$$\delta u = \delta a \cdot x + \delta b \cdot x^2 = [x \quad x^2][\delta a \quad \delta b]^T$$

$$\delta \varepsilon = [1 \quad 2x][\delta a \quad \delta b]^T$$

$$\delta u_0 = 0, \quad \delta u_L = [L \quad L^2][\delta a \quad \delta b]^T$$

Now,

$$A\int_0^L \delta\varepsilon \cdot \sigma \, dx = A\int [\delta a \quad \delta b]\begin{bmatrix} 1 \\ 2a \end{bmatrix}[E][1 \quad 2x][a \quad b]^T dx$$

$$= A \, [\delta a \quad \delta b] \underbrace{\int [1 \quad 2a]^T [E][1 \quad 2x] \, dx}_{\text{non-constant}} [a \quad b]^T$$

$$\int_0^L \begin{bmatrix} E & 2Ex \\ 2Ex & 4Ex^2 \end{bmatrix} dx$$

ONE. DIM. PROBLEM

$$A\int_0^L \delta\varepsilon\,\sigma\,dx = \delta u_0 P_0 + \delta u_L P_L + A\int_0^L \delta u\cdot\gamma\,dx$$

Galerkin method

with 2nd degree polynomial approximation

$u = ax + bx^2$ — satisfies support condition $u_0 = 0$

— use matrix form for ease later

$$u = \begin{bmatrix} x & x^2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$\varepsilon = \frac{du}{dx} = a + 2bx = \begin{bmatrix} 1 & 2x \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$\sigma = E\cdot\varepsilon = \begin{bmatrix} E \end{bmatrix}\begin{bmatrix} 1 & 2x \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$\delta u = \delta a\cdot x + \delta b\cdot x^2 = \begin{bmatrix} x & x^2 \end{bmatrix}\begin{bmatrix} \delta a \\ \delta b \end{bmatrix}$$

shape functions

$$\delta\varepsilon = \frac{d(\delta u)}{dx} = \delta a + 2\delta b\cdot x = \begin{bmatrix} 1 & 2x \end{bmatrix}\begin{bmatrix} \delta a \\ \delta b \end{bmatrix}$$

$$\delta u_0 = 0$$

$$\delta u_L = \begin{bmatrix} L & L^2 \end{bmatrix}\begin{bmatrix} \delta a \\ \delta b \end{bmatrix}$$

Substitute all that into P.V.W.
(written out on last page ⟵)

$$\int_0^L \begin{bmatrix} E & 2Ex \\ 2Ex & 4Ex^2 \end{bmatrix} dx \ldots \text{ plus some stuff}$$

or,

$$A\int_0^L \begin{bmatrix} 1 & 2x \end{bmatrix}\begin{bmatrix} \delta a \\ \delta b \end{bmatrix}\begin{bmatrix} E \end{bmatrix}\begin{bmatrix} 1 & 2x \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} dx =$$

$$0\cdot P_0 + \begin{bmatrix} L & L^2 \end{bmatrix}\begin{bmatrix} \delta a \\ \delta b \end{bmatrix} P_L + A\int_0^L \begin{bmatrix} x & x^2 \end{bmatrix}\begin{bmatrix} \delta a \\ \delta b \end{bmatrix}\gamma\,dx$$

Make it pretty — $\underbrace{\delta a, \delta b}_{\text{constants}}$ on left; a, b on right

$$\begin{bmatrix} \delta a & \delta b \end{bmatrix}\cdot A\int_0^L \begin{bmatrix} 1 \\ 2x \end{bmatrix}\begin{bmatrix} E \end{bmatrix}\begin{bmatrix} 1 & 2x \end{bmatrix} dx\cdot\begin{bmatrix} a \\ b \end{bmatrix} =$$

$$\begin{bmatrix} \delta a & \delta b \end{bmatrix}\begin{bmatrix} L \\ L^2 \end{bmatrix} P_L + \begin{bmatrix} \delta a & \delta b \end{bmatrix} A\int_0^L \begin{bmatrix} x \\ x^2 \end{bmatrix}\gamma\,dx$$

Now,
$\begin{bmatrix} \delta a & \delta b \end{bmatrix}$ appears in all terms in the same way

EQUATION MASSAGING

Simplifying with Galerkin

$$\begin{bmatrix} \delta a & \delta b \end{bmatrix} \left\{ \int_0^L \begin{bmatrix} 1 \\ 2x \end{bmatrix} \begin{bmatrix} EA \end{bmatrix} \begin{bmatrix} 1 & 2x \end{bmatrix} dx \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} L \\ L^2 \end{bmatrix} \begin{bmatrix} P_L \end{bmatrix} - \int_0^L \begin{bmatrix} x \\ x^2 \end{bmatrix} \begin{bmatrix} \gamma A \end{bmatrix} dx \right\} = \begin{bmatrix} 0 \end{bmatrix}$$

This equation must hold for all arbitrary $\begin{bmatrix} \delta a & \delta b \end{bmatrix}$

Therefore,

$$\int_0^L \begin{bmatrix} EA & 2EAx \\ 2EAx & 4EAx^2 \end{bmatrix} dx \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} QL \\ QL^2 \end{bmatrix} + \int_0^L \begin{bmatrix} x \\ x^2 \end{bmatrix} \begin{bmatrix} \gamma A \end{bmatrix} dx$$

considering the integrals:

$$\begin{bmatrix} EAL & EAL^2 \\ EAL^2 & \frac{4}{3}EAL^3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} QL \\ QL^2 \end{bmatrix} + \begin{bmatrix} \gamma A L^2/2 \\ \gamma A L^3/3 \end{bmatrix} \quad \begin{matrix} \text{Eq. 1} \\ \text{Eq. 2} \end{matrix}$$

Solving ~~for~~ for a, b:

- multiply eq. 1 by $-L$
- add to eq. 2

$$b = -\gamma/2E$$

$$a = \frac{Q}{EA} + \frac{\gamma L}{\cancel{2}E}$$

So, $u = ax + bx^2$

$$u = \frac{Qx}{EA} + \frac{\gamma L x}{\cancel{2}E} + \frac{-\gamma}{2E}x^2$$

compare to exact solution:

$$u = \frac{Qx}{EA} + \frac{\gamma L x}{E} - \frac{\gamma x^2}{2E}$$

SAME EQ!

if exact solution is unknown,
error at $x = L$

$$\sigma_{approx} = E\left(\frac{QL}{EA} + \gamma L^2/E - \frac{\gamma L^2}{2E}\right)$$

$$= \frac{QL}{A} + \gamma L^2 - \gamma L^2/2$$

$$= \frac{QL}{A} + \frac{\gamma L^2}{2}$$

$$\sigma_{approx} = E\left(\frac{Q}{EA} + \frac{\gamma L}{E} - \frac{\gamma L}{E}\right)$$

$$= \frac{Q}{A} \qquad (\sigma_{approx} - \sigma_{exact}) = 0$$

locally very accurate

APPROXIMATIONS

New method

Instead of considering a second-degree approximation as a way to improve the first-degree polynomial approximation, we can instead use a piecewise linear polynomial approximation

For example, use two subregions in $[0, L]$

$(0, L_1)$ sub-1

(L_1, L) sub-2

Use a linear polynomial in each subregion, but ensure that the approximation is continuous.

For example, write these approximations in terms of the displacements at $x=0$, $x=L_1$, $x=L$

$(0, L_1)$: $u(x)$ is a linear combination of u_0, u_{L1}

$$u(x) = \_\_ u_0 + \_\_ u_{L1}$$

$$\frac{L_1 - x}{L_1 - 0} \qquad \frac{x-0}{L_1-0}$$

interpolation functions

N_0, N_{L1}

$N_0(0) = 1, \; N_0(L_1) = 0$

$N_{L_1}(L_1) = 0, \; N_{L1}(0) = 0$

LOCAL expressions – only valid in this one subregion

(L_1, L)

$$u(x) = \frac{L-x}{L-L_1} u_{L_1} + \frac{x-L_1}{L-L_1} u_L$$

In each region, proceed as before (calculate $\varepsilon, \sigma, \delta\varepsilon$, etc) and solve for u_0, u_L, u_{L1}

MISSING NOTES IN HERE

Galerkin –

used linear polynomial approximation defined over $(0, L)$

$u(x) = ax$

or, quadratic polynomial

$u(x) = ax + bx^2$

Alternate approach –

(instead of quadratic), apply lower order approximation in a piece-by-piece manner

why?

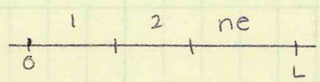
Galerkin approx. must satisfy the support conditions

easy in 1D, not so easy in 2D, 3D

so, partition:

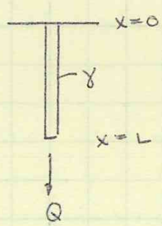
Approximations

use linear polynomial approximations in subregions obtained by partitioning $(0, L)$



each subregion referred to as an element

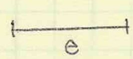
elements 1, 2 ... ne ← number of elements



$$A\int_0^L \delta\varepsilon\,\sigma\,dx = \delta u_0 P_0 + \delta u_L P_L + A\int_0^L \delta u\,\gamma\,dx$$

Apply the procedure used for the entire rod to each of the elements

consider a typical element e



approximation assumed in e: $u(x) = a^e + b^e x \ldots$

a^e, b^e are constants over (and unique to) e

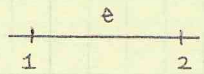
Keep in mind: we are (usually) interested in continuous (and therefore, approximate) solutions for u. This implies that the coefficients $a^e, b^e \ldots$ must be such that continuity is satisfied at common points of elements. | doable, but not convenient

Instead, rewrite the approximation (*) using parameters related to points of the element that are shared with neighboring elements:

$u(x) \approx$ linear polynomial in terms of the values of u at the end points of the element (other pts. may be involved)

ELEMENTAL APPROXIMATIONS

Better approximation



← local names for endpoints of e

x_1, x_2 coordinates of the end points

u_1, u_2 (approximations of) u at endpoints

Now, construct a linear polynomial over (x_1, x_2) such that:

$u(x_1) = u_1$

& $u(x_2) = u_2$

or,

$a^e + b^e x_1 = u_1$ (1) two equations,

$a^e + b^e x_2 = u_2$ (2) two unknowns (a, b)

To solve:

multiply (1) by (-1) and add to (2)

$$-a^e - b^e x_1 + a^e + b^e x_2 = -u_1 + u_2$$

$$b_e = \frac{u_2 - u_1}{x_2 - x_1}$$

$$a_e = u_1 - \frac{u_2 - u_1}{x_2 - x_1} x_1$$

Rewrite the linear approximation

$$u(x) = u_1 - \frac{u_2 - u_1}{x_2 - x_1} x_1 + \frac{u_2 - u_1}{x_2 - x_1} x \quad (\text{from } u(x) = a_e + b_e x)$$

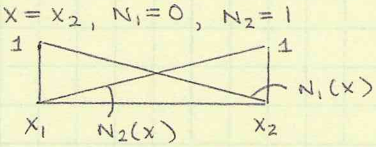
simplified / rearranged:

$$u(x) = \underbrace{\frac{x_2 - x}{x_2 - x_1}}_{N_1(x)} u_1 + \underbrace{\frac{x - x_1}{x_2 - x_1}}_{N_2(x)} u_2$$

interpolation functions

Interpolation functions:

- at $x = x_1$, $N_1 = 1$, $N_2 = 0$
  $x = x_2$, $N_1 = 0$, $N_2 = 1$
-



- $\sum_{i=1}^{2} N_i = 1.0$ everywhere in the element
  consider the constant solution,
  $u_1 = u_2$
  needs to be able to produce constant solutions before trying linear solutions

USING ELEMENTS

Apply P.V.W. to single element e

$$u = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\varepsilon = \frac{du}{dx} = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\sigma = \varepsilon E = [E] \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\frac{dN_1}{dx} = \frac{-1}{x_2 - x_1} \qquad \frac{dN_2}{dx} = \frac{1}{x_2 - x_1}$$

$$\delta u = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta u_2 \end{bmatrix}$$

$$\delta \varepsilon = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta u_2 \end{bmatrix}$$

Plug into equation

$$A \int_0^L \delta\varepsilon \, \sigma \, dx = \delta u_0 P_0 + \delta u_L P_L + A \int_0^L \delta u \, \gamma \, dx$$

$$\downarrow$$

$$A \int_{x_1}^{x_2} \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta u_2 \end{bmatrix} \sigma \, dx = \ldots$$

$$\downarrow$$

$\frac{1}{2} L_e$ (area under N curve)

$$\int_{x_1}^{x_2} \begin{bmatrix} \frac{dN_1}{dx} \\ \frac{dN_2}{dx} \end{bmatrix} [EA] \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} dx \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \int_{x_1}^{x_2} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} [\gamma A] \, dx$$

or,

$$\begin{bmatrix} \frac{EA}{L_e} & \frac{-EA}{L_e} \\ \frac{-EA}{L_e} & \frac{EA}{L_e} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\gamma A L_e \\ \frac{1}{2}\gamma A L_e \end{bmatrix}, \quad L_e = x_2 - x_1$$

shared values between elements

equations to be satisfied within each small element

what happens to forces between neighboring elements?

ERROR CONSIDERATIONS

Error in F.E.A.

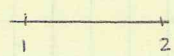
error(h)
└ element size

$$\lim_{h \to 0} \frac{\text{error}(h)}{h^x} = \text{constant not equal to } 0$$

then, the error is of order x.

So, to find the order of the error, calculate the exponent that makes that limit true

Typical element



$$u(x) = N_1(x) u_1 + N_2(x) u_2$$

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}, \quad N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

$$u(x) = \underbrace{\begin{bmatrix} N_1 & N_2 \end{bmatrix}}_{\text{Matrix of Interpolation Functions}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\varepsilon(x) = \underbrace{\begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix}}_{\text{called } \underset{\sim}{B}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

called $\underset{\sim}{B}$ $\longrightarrow \begin{bmatrix} \frac{-1}{x_2 - x_1} & \frac{1}{x_2 - x_1} \end{bmatrix}$

$$\varepsilon(x) = [B][u]$$

$$\sigma = E \cdot \varepsilon$$

$$[\sigma] = [E][\varepsilon]$$

$[E]$: $\underset{\sim}{D}$, the rigidity matrix

$$\underset{\sim}{\delta u} = \begin{bmatrix} \delta u_1 \\ \delta u_2 \end{bmatrix} \qquad \underset{\sim}{\Gamma} = \begin{bmatrix} \frac{1}{2} \gamma L_e A \\ \frac{1}{2} \gamma L_e A \end{bmatrix} = A \int_0^L \hat{N}^T \gamma \, dx$$

Governing equation

$$\underset{\sim}{\delta u}^T \underbrace{\left(A \int_0^L \underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} \, dx \right)}_{\underset{\sim}{K}\text{, the stiffness matrix}} \underset{\sim}{u} = \underset{\sim}{\delta u}^T \underline{P} + \delta u^T \Gamma$$

$$\underset{\sim}{K}\text{, the stiffness matrix} = \begin{bmatrix} \frac{EA}{L_e} & -\frac{EA}{L_e} \\ -\frac{EA}{L_e} & \frac{EA}{L_e} \end{bmatrix} = \frac{EA}{L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

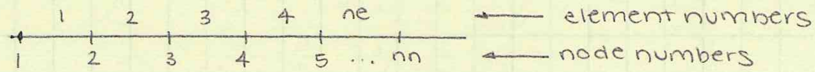
Since δu is arbitrary,

$$\underset{\sim}{K} \underset{\sim}{u} = \underset{\sim}{P} + \underset{\sim}{\Gamma}$$

SIMPLIFYING & GENERALIZING

Matrix form equation

$$\underset{\sim}{K}\,\underset{\sim}{u} = \underset{\sim}{P} + \underset{\sim}{\Gamma}$$



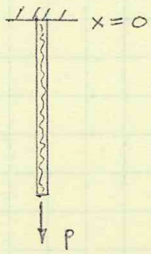
For each element,

$$\frac{EA}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{bmatrix} = \begin{bmatrix} P_1^{(e)} \\ P_2^{(e)} \end{bmatrix}$$

superscripts indicate elem. number

u_1, u_2 for el. 1

u_2, u_3 for el. 2



unknowns u_1 through u_n

(some might be known — boundaries, etc.)

— P_1 through P_{nn}

(again, some known, P_{nn}, eg)

Total number of unknowns:

$nn + 2ne$

Total number of equations:

$2ne + 2$

so, need more equations

Additional equations

- satisfied at nodes shared by the elements

$$P_2^{(1)} = -P_{(2)}^{(2)}$$

- must consider external loads if they exist

$$P_2^{(1)} + P_2^{(2)} = P_{ext}$$

$$A\sigma(x_2) + (-A)\sigma(x_2)$$

can be different; are, when externally applied load exists

- this adds as many equations as internal nodes

$nn - 2$

- new total:

$$2ne + 2 + nn - 2 = 2ne + nn = \#\text{ of unknowns}$$ ✓

Assemble equations!

ASSEMBLY

It's time for some gigantic matrices

$$
\begin{bmatrix}
\frac{EA}{L_1} & -\frac{EA}{L_1} & & & & \\
-\frac{EA}{L_1} & \frac{EA}{L_1}+\frac{EA}{L_2} & -\frac{EA}{L_2} & & 0 & \\
 & -\frac{EA}{L_2} & \frac{EA}{L_2}+\frac{EA}{L_3} & -\frac{EA}{L_3} & & \\
 & & -\frac{EA}{L_3} & \cdots & & \\
 & & & & \cdots & \\
 & 0 & & & & \frac{EA}{L_{nn-1}}+\frac{EA}{L_{nn}}
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{nn} \end{bmatrix}
=
\begin{bmatrix} P_1^{(1)} \\ P_2^{(1)}+P_2^{(2)} \\ P_3^{(2)}+P_3^{(3)} \\ \vdots \\ P_{nn}^{(ne)} \end{bmatrix}
+\frac{1}{2}A\gamma
\begin{bmatrix} L_1 \\ L_1+L_2 \\ L_2+L_3 \\ \vdots \\ L_{ne} \end{bmatrix}
$$

equal to zero, except in locations of applied load

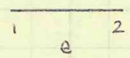
Big system of equations; with support conditions, can be pretty easily solved

$u_1=0$, $P_{nn}=Q$, etc.

tridiagonal system of equations (almost as simple as diagonal)

ONE · DIM. ELEMENT

Brief review



E_e, A_e, L_e, γ_e

$\underset{\sim}{u} = \underset{\sim}{N}\underset{\sim}{U} \longleftarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$\uparrow$

$[N_1 \;\; N_2]$

$\underset{\sim}{K_e} = A_e \int_{x_1}^{x_2} \underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}\, dx \; , \quad \Gamma_e = A \int_{x_1}^{x_2} \underset{\sim}{N}^T [\gamma_e]\, dx$

$\underset{\sim}{B} = \left[\frac{dN_1}{dx} \;\; \frac{dN_2}{dx} \right]$, $\underset{\sim}{D} = [E_e]$

Γ_e: element node vector

K_e: element stiffness matrix

$$\underset{\sim}{K_e}\underset{\sim}{U_e} = \underset{\sim}{P_e} + \underset{\sim}{\Gamma_e} \quad \text{or} \quad \left(\frac{EA}{L}\right)_e \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} P_o \\ P_L \end{bmatrix} + \frac{\gamma A L}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$u_1 = 0, \; P_L = Q$ (in original example)

↳ add $+B$ to 11 val in K_e matrix,

$P_o = -B \frac{EA}{L} u_1$

u_1 solves to a small value

$u_L = \frac{QL}{EA} + \frac{\gamma L^2}{2E}$

(solution from before)

using two elements

$$EA \begin{bmatrix} 1/L_1 & -1/L_1 & \\ -1/L_1 & 1/L_1 + 1/L_2 & -1/L_2 \\ & -1/L_2 & 1/L_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} P_o \\ 0 \\ P_L \end{bmatrix} + \begin{bmatrix} \gamma A(L_1) \\ \gamma A(L_1 + L_2) \\ \gamma A(L_2) \end{bmatrix}$$

EA: assuming $E_1 = E_2$, $A_1 = A_2$

0: would be non-zero if an app. load existed

$$u_2 = \frac{QL}{2EA} + \frac{3}{8}\frac{\gamma L^2}{E}$$

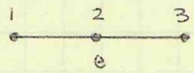
$$u_3 = \frac{QL}{EA} + \frac{\gamma L^2}{2E} \qquad (\text{if } L_1 = L_2)$$

exact values at nodes.

inexact between, because of linear interpolation

QUADRATIC ELEMENTS

In one dimension,



u(x) is a second-degree polynomial
needs three nodes per element
middle node need not be in the middle
higher accuracy occurs at certain locations

Now,

$$N = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}$$

$$N_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$$

$$N_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$$

$$N_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

still satisfy equal to one at node, 0 at other two

$$B = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} & \frac{dN_3}{dx} \end{bmatrix}$$

$$\underset{\sim}{K_e} = A \int_{x_1}^{x_3} \underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} \, dx \quad , \quad \underset{\sim}{\Gamma}_e = A \int_{x_1}^{x_3} N^T [\gamma] \, dx$$

Integrating,

$$K = \frac{EA}{L} \begin{bmatrix} 7/3 & -8/3 & 1/3 \\ -8/3 & 16/3 & -8/3 \\ 1/3 & -8/3 & 7/3 \end{bmatrix} , \quad \Gamma = \begin{bmatrix} 1/6 \, \gamma AL \\ 2/3 \, \gamma AL \\ 1/6 \, \gamma AL \end{bmatrix}$$

$\downarrow$ explained

using Simpson's Rule:

$$\int_a^b f(x)\,dx = (b-a)\left[1/6 f(a) + 2/3 f\left(\frac{a+b}{2}\right) + 1/6 f(b) \right]$$

$$\Gamma = A(x_3-x_1)\left[1/6 N_1(x_1)\gamma + \tfrac{2}{3} \underset{\underset{0}{\uparrow}}{N_1(x_2)}\gamma + 1/6 \underset{\underset{0}{\uparrow}}{N_1(x_3)}\gamma \right]$$

for line 1 of Γ matrix

$= 1/6 \gamma AL$ for N_1

ABAQUS! INTRO

Extra details

*preprint, echo = yes, model = yes, history = yes

*node file , *node print

→ prints into .dat file! try inserting, checking file

interactive, on command line

abaqus, job = ___ , interactive (no commas)

outputs more data right then

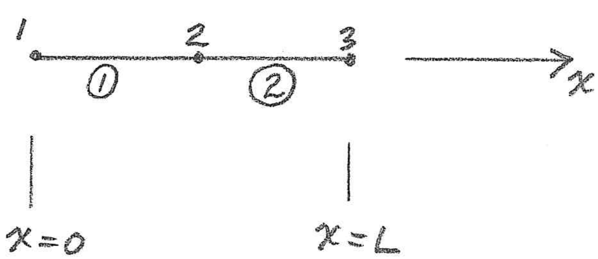
*user element, ul

define your own element

needs *matrix to define stiffness, etc.

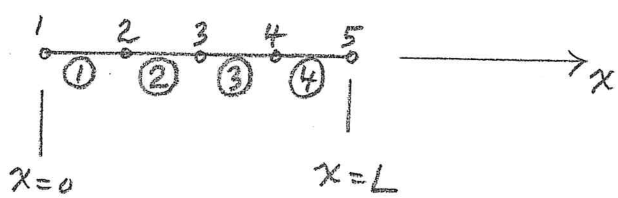
rod2_u1.inp

```
*HEADING
Rod Analysis
*PREPRINT, ECHO=YES, MODEL=YES, HISTORY=YES
*RESTART, WRITE, FREQ=1
*NODE, NSET=ROD
1, 0.
2,  .5
3, 1.
*USER ELEMENT, TYPE=U1, NODES=2, LINEAR
1
*MATRIX, TYPE=STIFFNESS
 2.
-2., 2.
*ELEMENT, TYPE=U1, ELSET=ROD
1, 1, 2
2, 2, 3
*UEL PROPERTY, ELSET=ROD
*BOUNDARY
1, 1, 1
*STEP, PERTURBATION
*STATIC
*CLOAD
1, 1, .25
2, 1, .5
3, 1, .25
*NODE FILE, NSET=ROD
U
RF
CF
*NODE PRINT, NSET=ROD
U
RF
CF
*EL FILE, ELSET=ROD
NFORC
*EL PRINT, ELSET=ROD
NFORC
*END STEP
```



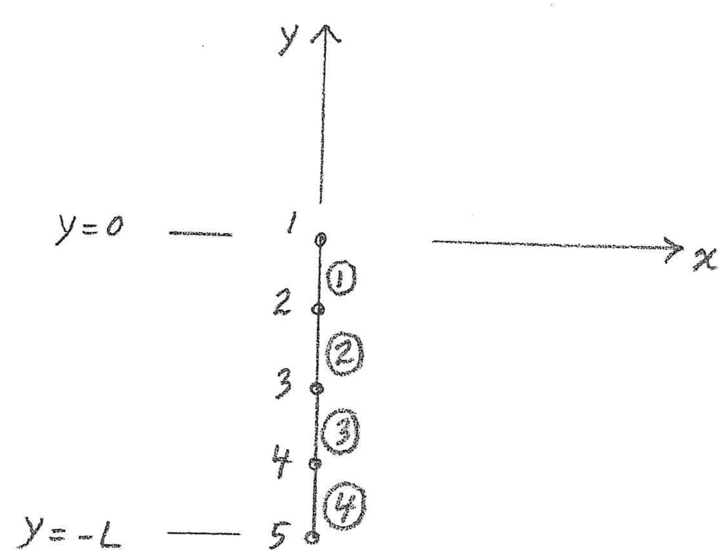
rod4_u1.inp

```
*HEADING
Rod Analysis
*PREPRINT, ECHO=YES, MODEL=YES, HISTORY=YES
*RESTART, WRITE, FREQ=1
*NODE
1, 0.
5, 1.
*NGEN, NSET=ROD
1, 5, 1
*USER ELEMENT, TYPE=U1, NODES=2, LINEAR
1
*MATRIX, TYPE=STIFFNESS
 4.
-4., 4.
*ELEMENT, TYPE=U1
1, 1, 2
*ELGEN, ELSET=ROD
1, 4, 1, 1
*UEL PROPERTY, ELSET=ROD
*BOUNDARY
1, 1, 1
*STEP, PERTURBATION
*STATIC
*CLOAD
1, 1, .125
2, 1, .25
3, 1, .25
4, 1, .25
5, 1, .125
*NODE FILE, NSET=ROD
U
RF
CF
*NODE PRINT, NSET=ROD
U
RF
CF
*EL FILE, ELSET=ROD
NFORC
*EL PRINT, ELSET=ROD
NFORC
*END STEP
```



rod4_t2d2.inp

```
*HEADING
Rod Analysis
*PREPRINT, ECHO=YES, MODEL=YES, HISTORY=YES
*RESTART, WRITE, FREQ=1
*NODE
1, 0.,  0.
5, 0., -1.
*NGEN, NSET=ROD
1, 5, 1
*ELEMENT, TYPE=T2D2
1, 1, 2
*ELGEN, ELSET=ROD
1, 4, 1, 1
*SOLID SECTION, ELSET=ROD, MATERIAL=BEST
1.
*MATERIAL, NAME=BEST
*DENSITY
1.
*ELASTIC, TYPE=ISOTROPIC
1., .2
*BOUNDARY
ROD, 1, 1
1, 2, 2
*STEP, PERTURBATION
*STATIC
*DLOAD
ROD, GRAV, 1., 0., -1.
*NODE FILE, NSET=ROD
U
RF
CF
*NODE PRINT, NSET=ROD
U
RF
CF
*EL FILE, ELSET=ROD
NFORC
*EL PRINT, ELSET=ROD
NFORC
*END STEP
```



A Note on Nodal Interpolation Functions

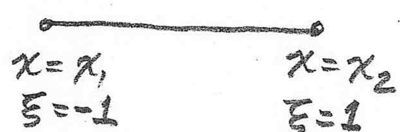
The polynomial interpolation functions we have discussed so far with regard to one dimensional finite elements can be written very concisely by means of a linear mapping of the element domain onto the standard interval $[-1, 1]$. For example, consider the two-node element occupying the interval $[x_1, x_2]$. Denoting the standard coordinate by ξ $(-1 \le \xi \le 1)$, the linear mapping of $[x_1, x_2]$ onto $[-1, 1]$ can be written as

$$x = x_1 + \frac{1+\xi}{2}(x_2 - x_1)$$

and $\qquad (x_2 - x_1 = L_e)$

$$\xi = -1 + \frac{x - x_1}{x_2 - x_1} \cdot 2$$

These relationships ensure that $x = x_1$ and $\xi = -1$ are images of each other. The same holds for $x = x_2$ and $\xi = 1$. Between the two end points, x and ξ vary linearly with each other.



Because of the linearity of the mapping, polymials in x become polynomial in ξ of the same degree. Thus, the nodal interpolation functions are given by

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$

$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

in terms of x and

$$N_1(\xi) = \frac{1-\xi}{2}$$

$$N_2(\xi) = \frac{1+\xi}{2}$$

in terms of ξ. The expressions for N_1 and N_2 in terms of ξ can be obtained in the usual manner by noting that N_1 is equal to 1 for $\xi = -1$ and 0 for $\xi = 1$ while N_2 is equal to 1 for $\xi = 1$ and 0 for $\xi = -1$.

Similarly, for the three-node element, the linear mapping of $[x_1, x_3]$ onto $[-1, 1]$ can be written as

$$x = x_1 + \frac{1+\xi}{2}(x_3 - x_1)$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x_3 - x_1 = L_e)$

$$\xi = -1 + \frac{x - x_1}{x_3 - x_1}\cdot 2$$

These relationships render $x = x_1$ and $\xi = -1$ images of each other and the same holds for $x = x_3$ and $\xi = 1$ as wells as for $x = x_2 = \frac{x_1 + x_3}{2}$ and $\xi = 0$. The nodal interpolation functions are given by

$$N_1(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}$$

$$N_2(x) = \frac{(x - x_3)(x - x_1)}{(x_2 - x_3)(x_2 - x_1)}$$

$$N_3(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

in terms of x and

$$N_1(\xi) = \frac{\xi\cdot(\xi - 1)}{2}$$

$$N_2(\xi) = 1 - \xi^2$$

$$N_3(\xi) = \frac{\xi\cdot(\xi + 1)}{2}$$

in terms of ξ. Again, the expressions in terms of ξ can be obtained in the usual manner by noting that N_1 is equal to 1 for $\xi = -1$ and 0 for $\xi = 0$ and $\xi = 1$ and, similarly, for N_2 and N_3.

The expressions in terms of ξ can be used in computing various integrals that arise with greater simplicity. Both examples below refer to the three-node element.

<u>Example 1</u>

$$\int_{x_1}^{x_3} \frac{dN_1}{dx}\cdot\frac{dN_2}{dx}\,dx = \int_{-1}^{1} \frac{dN_1}{d\xi}\,\frac{d\xi}{dx}\,\frac{dN_2}{d\xi}\,\frac{d\xi}{dx}\,\frac{dx}{d\xi}\,d\xi$$

$$= \int_{-1}^{1} \frac{dN_1}{d\xi}\,\frac{2}{L_e}\,\frac{dN_2}{d\xi}\,\frac{2}{L_e}\,\frac{L_e}{2}\,d\xi = \frac{2}{L_e}\int_{-1}^{1}\frac{dN_1}{d\xi}\,\frac{dN_2}{d\xi}\,d\xi$$

$$= \frac{2}{L_e}\int_{-1}^{1}\left(\xi - \frac{1}{2}\right)\cdot(-2\xi)\,d\xi = \frac{2}{L_e}\int_{-1}^{1}(-2\xi^2 + \xi)\,d\xi$$

$$= \frac{2}{Le}\left[-\frac{2}{3}\xi^3 + \frac{\xi^2}{2}\right]\Bigg|_{-1}^{1} = \frac{2}{Le}\left[-\frac{2}{3} + \cancel{\frac{1}{2}} - \frac{2}{3} - \cancel{\frac{1}{2}}\right] = -\frac{8}{3}\frac{1}{Le}$$

Example 2

$$\int_{x_1}^{x_3} N_1 \cdot N_2 \, dx = \int_{-1}^{1} N_1 \cdot N_2 \, \frac{dx}{d\xi} d\xi = \int_{-1}^{1} N_1 \cdot N_2 \, \frac{Le}{2} \, d\xi$$

$$= \frac{Le}{2} \int_{-1}^{1} \frac{\xi \cdot (\xi - 1)}{2} \cdot (1 - \xi^2) \, d\xi$$

$$= \frac{Le}{2} \int_{-1}^{1} \left(\frac{1}{2}\xi^2 - \frac{1}{2}\xi\right) \cdot (1 - \xi^2) \, d\xi$$

$$= \frac{Le}{2} \int_{-1}^{1} \left(\frac{1}{2}\xi^2 - \frac{1}{2}\xi^4 - \frac{1}{2}\xi + \frac{1}{2}\xi^3\right) d\xi$$

$$= \frac{Le}{2}\left[\frac{1}{6}\xi^3 - \frac{1}{10}\xi^5 - \frac{1}{4}\xi^2 + \frac{1}{8}\xi^4\right]\Bigg|_{-1}^{1}$$

$$= \frac{Le}{2}\left[\frac{1}{6} - \frac{1}{10} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{8}} + \frac{1}{6} - \frac{1}{10} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{8}}\right]$$

$$= \frac{Le}{2}\left[\frac{1}{3} - \frac{1}{5}\right] = \frac{Le}{\cancel{2}} \, \frac{\cancel{2}}{15} = \frac{1}{15} Le$$

ABAQUS AND ETC.

Lumped mass matrix (but not stiffness!)

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \longrightarrow \begin{bmatrix} a+b & 0 \\ 0 & a+b \end{bmatrix}$$

does not lose accuracy
(error is of the same order)
think of lumped mass models
in dynamic analysis

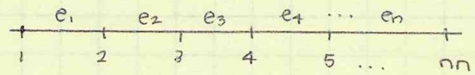
ABAQUS CAE notes and explanation

For simple truss model

- 2D planar, wire config.
- section beam (truss)
- linear perturbation step
- boundary conditions
  - type: displacement
    - restrict y-dir movement
    - (as one-dim analysis not an option)
- loads
  - concentrated load at non-fixed end
  - use $P = 1.0$ for this example
- meshing
  - select element type (linear, quadratic)
  - truss family (T2D2)
- field output
  - stress, strain, displacement

END O' ONE DIMENSION

General knowledge



finite elements – term introduced by Clough and Martin

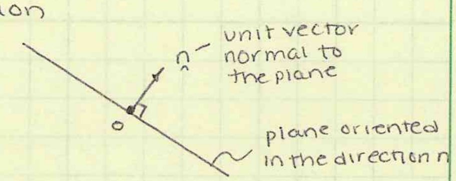
MORE THAN ONE DIMENSION

Stress (tensor)

- describes the load as force per unit area at a point in the continuum
- has six components required (in general)
- can be arranged in a 3×3 matrix

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

σ_{xy}: force in the x direction per unit area on a plane through the material point of interest oriented in the y direction



- one can show that force in the x dir (per unit area) on a plane oriented in the direction n is given by:

$$T_x = \sigma_{xx} n_x + \sigma_{xy} n_y + \sigma_{xz} n_z$$

↳ traction in the x dir

$$T = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ & \sigma_{yy} & \sigma_{yz} \\ & & \sigma_{zz} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}, \quad \underset{\sim}{T} = \underset{\sim}{\sigma}\,\underset{\sim}{n}$$

↑ traction vector

- considering an infinitesimal element in the neighborhood of the material point of interest and applying equilibrium of moments and forces on the element, one finds:

$$\left.\begin{aligned} \sigma_{xy} &= \sigma_{yx} \\ \sigma_{xz} &= \sigma_{zx} \\ \sigma_{yz} &= \sigma_{zy} \end{aligned}\right\}$$ ensure moment equilibrium (in the absense of distributed moments)

thus, symmetry of σ matrix

MULTIPLE DIMENSIONS

Stress calculations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x = \rho \frac{\partial^2 u}{\partial t^2}$$

b_x ← body force density in x·direction

ρ ← mass density

u ← x displacement

Equilibrium of forces in the x·direction (on the element)

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial z} + b_y = \rho \frac{\partial^2 v}{\partial t^2}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + b_z = \rho \frac{\partial^2 w}{\partial t^2}$$

Assuming the direction of the gravitational field coincides with the negative z direction, the body·force density can be written as:

$$\underset{\sim}{b} = \begin{bmatrix} 0 \\ 0 \\ -\gamma \end{bmatrix}, \text{ where } \gamma = \text{unit weight}$$

Applying the principle of virtual work,

- multiply 3 eqs above by $\delta u, \delta v, \delta w$
- integrate across Ω, the domain of the object
- integration by parts

$$\int_\Omega \left[\delta u \cdot \rho \cdot \frac{\partial^2 u}{\partial t^2} + \delta v \cdot \rho \frac{\partial^2 v}{\partial t^2} + \delta w \cdot \rho \cdot \frac{\partial^2 w}{\partial t^2} \right] d\Omega$$

example:

$$\int \delta u \frac{\partial \sigma_x}{dx} d\Omega = \int \left[\frac{d}{dx} (\delta u \cdot \sigma_x) - \frac{d}{dx} \delta u \, \sigma_x \right] d\Omega$$

Green's theorem:

$$\int_\Omega \frac{df}{dx} d\Omega = \int_B f \cdot n_x \, dB$$

, where $\underset{\sim}{n}$ is the outward normal to area Ω, pt B

→ replace derivatives of the stress components with the derivitives of the virtual displacements

STRESS CALCULATIONS

Calculations of vectors, etc

$\sigma_x, \sigma_y, \sigma_z$ – Normal stress components

$\sigma_{xy}, \sigma_{xz}, \sigma_{yz}$ – Shear stress components

in three dimensions,

u, v, w are displacement components

b_x, b_y, b_z are body force components

Principle of virtual work

Multiply the 3 ~~eqns~~ equations of force equilibrium by $\delta u, \delta v, \delta w$ and add. Then integrate over domain Ω

$$\int_\Omega \frac{df}{dx}\, d\Omega = \int_B f \cdot n_x \, dB \qquad (\text{or } y, z)$$

insert lots of integration details

Strain-displacement equations

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \cancel{\frac{\partial y}{\partial x}}\ \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z}$$

extensional strain components

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$ describe distortion

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

distorsional strain components

ε_{xy}: related to the change in angle between two elements

$\gamma_{xy} = 2\times\ \varepsilon$ value

Constitutive equations

– relate stress and strain

– here limited to isotropic, linear, elastic equations

combining equations

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{bmatrix} = \begin{bmatrix} \lambda+2G & \lambda & \lambda & & & \\ \lambda & 2G+\lambda & \lambda & & 0 & \\ \lambda & \lambda & \lambda+2G & & & \\ & & & G & & \\ & 0 & & & G & \\ & & & & & G \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

$$\underset{\sim}{\sigma} \qquad \underset{\sim}{D} \qquad \underset{\sim}{\varepsilon}$$

λ: Lamé modulus

G: shear modulus

E: Young's modulus

ν: Poisson's ratio

conversions:

$$\frac{\lambda+2G}{G} = \frac{2-2\nu}{1-2\nu}, \quad G = \frac{E}{2(1+\nu)} \qquad \underset{\sim}{\sigma} = \underset{\sim}{D}\,\underset{\sim}{\varepsilon}$$

STRESS CALCULATIONS

General equations

$$\sigma = D\,\varepsilon$$

- ε — strain components
- D — material stiffness (or, rigidity) matrix
- σ — stress matrix

simplifying giant integrals (not written)

$$\int_B (\delta u \cdot T_x + \delta v \cdot T_y + \delta \omega \cdot T_z)\,dB - \int_\Omega \begin{pmatrix} \delta\varepsilon_x\sigma_x + \delta\varepsilon_y\sigma_y + \delta\varepsilon_z\sigma_z + \delta\gamma_{xy}\sigma_{xy} + \\ \delta\gamma_{yz}\sigma_{yz} + \delta\gamma_{zx}\sigma_{zx} \end{pmatrix} d\Omega$$

$$+ \int_\Omega (\delta u \cdot b_x + \dots)\,d\Omega = \int_\Omega \left(\delta u \cdot \rho \frac{\partial u^2}{\partial^2 t} + \dots\right)$$

remember:

$$\begin{bmatrix} \sigma_x & \sigma_{xy} & \sigma_{xz} \\ & \sigma_y & \sigma_{yz} \\ & & \sigma_z \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}$$

or n_x, n_y, n_z

$$\int_B \delta u^T T\,dB - \int_\Omega \delta\varepsilon^T \sigma\,d\Omega + \int_\Omega \delta u^T b\,d\Omega = \int_\Omega \delta u^T \rho\,\ddot{u}\,d\Omega$$

$$\ddot{u} = \begin{bmatrix} \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial^2 v}{\partial t^2} \\ \frac{\partial^2 \omega}{\partial t^2} \end{bmatrix}$$

rearranging sides of equation,

$$\underbrace{\int_\Omega \delta\varepsilon^T \sigma\,d\Omega}_{\text{internal virtual work}} = \underbrace{\int_B \delta u^T T\,dB + \int_\Omega \delta u^T b\,d\Omega + \int_\Omega \delta u^T \rho\,\ddot{u}\,d\Omega}_{\text{external virtual work}}$$

dropped in static equations

THREE DIMENSIONS

General equation

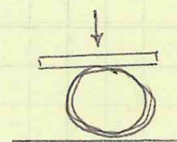
$$\int_{\Omega} \delta \varepsilon^T \sigma \, d\Omega = \int_B \delta u^T T \, dB + \int_{\Omega} \delta u^T (b - \rho \ddot{u}) \, d\Omega$$

In two dimensions, we distinguish three possibilities:

- plane strain
- plane stress
- antiplane shear

Plane strain

body (structure) extends unchanged in one direction
i.e., the z dir ("out of plane" direction)
$\varepsilon_z = 0$



split cylinder test

load is uniform in the out-of-plane direction
$w = 0, \ \partial/\partial z = 0$
(if z is the unchanging dir.)

so, only $\sigma_x, \sigma_y, \sigma_{xy}$ are relevant in the P. of V.W.

$$\delta\varepsilon = \begin{bmatrix} \delta\varepsilon_x \\ \delta\varepsilon_y \\ \delta\varepsilon_z \end{bmatrix}, \quad b = \begin{bmatrix} b_x \\ b_y \end{bmatrix}, \quad \delta u = \begin{bmatrix} \delta u \\ \delta v \end{bmatrix}, \quad \ddot{u} = \begin{bmatrix} \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial^2 v}{\partial t^2} \end{bmatrix}$$

From the equations of isotropic, linear elasticity,

$$\sigma_x = (\lambda + 2G)\varepsilon_x + \lambda\varepsilon_y \ (+\lambda\varepsilon_z)$$
$$\sigma_y = \lambda\varepsilon_x + (\lambda + 2G)\varepsilon_y \ (+\lambda\varepsilon_z)$$
$$\sigma_z = \lambda\varepsilon_x + \lambda\varepsilon_y \ (+ (\lambda + 2G)\varepsilon_z)$$

ε_z terms are zero

σ_z does not equal zero, but is entirely in terms of σ_x, σ_y

$$\tau_{xy} = G\gamma_{xy}, \quad \tau_{yz} = G\cancel{\gamma_{yz}}^{\,0}, \quad \tau_{zx} = G\cancel{\gamma_{zx}}^{\,0}$$

Matrix equations
$\sigma = D\varepsilon$

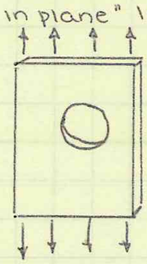
$$D = \begin{bmatrix} \lambda + 2G & \lambda & 0 \\ \lambda & \lambda + 2G & 0 \\ 0 & 0 & G \end{bmatrix}$$

REMEMBER:
σ_z does not necessarily equal 0.

MULTI·DIMENSION

Plane stress

- similar to plane strain
- instead of $\varepsilon_z = 0$, $\sigma_z = 0$
- applies to thin bodies subjected to loads in the plane of the cross·section ("in plane" loads)



perforated plate

Stress/strain equations:

$$\varepsilon_x = \frac{du}{dx}, \quad \varepsilon_y = \frac{dv}{dy}, \quad \varepsilon_z = \frac{\partial w}{\partial z}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0, \quad \gamma_{zx} = 0$$

assume that v is a function of (x, y)
u is a function of (x, y)

Set $\delta w = 0$ ⟶ removes z·equilibrium from P.V.W.

- $\delta\varepsilon_z = 0$
- $\delta\gamma_{yz} = 0$
- $\delta\gamma_{zx} = 0$

] assume that $\delta u, \delta v$ are functions of (x, y)

Equations of elasticity

$$\sigma_x = (\lambda + 2G)\varepsilon_x + \lambda\varepsilon_y + \lambda\varepsilon_z$$

$$\sigma_y = \lambda\varepsilon_x + (\lambda + 2G)\varepsilon_y + \lambda\varepsilon_z$$

$$\sigma_z = 0 = \lambda\varepsilon_x + \lambda\varepsilon_y + (2G + \lambda)\varepsilon_z$$

or, rearranged,

$$\varepsilon_z = \frac{-\lambda}{\lambda + 2G}(\varepsilon_x + \varepsilon_y)$$

$$D = \begin{bmatrix} \lambda + 2G - \frac{\lambda^2}{\lambda + 2G} & \lambda - \frac{\lambda^2}{\lambda + 2G} & 0 \\ \lambda - \frac{\lambda^2}{\lambda + 2G} & \lambda + 2G - \frac{\lambda^2}{\lambda + 2G} & 0 \\ 0 & 0 & G \end{bmatrix}$$

MULTI-DIMENSION

Isotropic linear elasticity

2D problems

- plane strain ⟶ $\frac{\partial}{\partial z}=0$, $\omega=0$, $u(x,y)$, $v(x,y)$
  $b_z=0$, $\varepsilon_z=0$, $\sigma_z\neq 0$
  falls apart near the ends or edges of a specimen (good for thick bodies)
- plane stress ⟶ Assume that
  - $u(x,y)$
  - $v(x,y)$
  - ω

  $f(z)$
  - good for cases of thin bodies
  - only in-plane forces are applied or considered (those that cause "stretching")
  - assumption is more exact as the thickness goes to 0

    $\sigma_z \to 0$ as $d \to 0$
- antiplane shear

More on plane stress

$$d\int_\Omega \delta\varepsilon^T\,\hat{\sigma}\,d\Omega \leftarrow \text{domain}$$

d ↑ thickness; $\hat{\sigma}$ ↑ average stress : $\hat{\sigma}=\frac{1}{d}\int_{-\frac{d}{2}}^{d/2}\sigma\,dz$ for each direction x, y, z

$\tau_{xy} = G\gamma_{xy}$, $\tau_{yz} = G\cancel{\gamma_{yz}}^{0}$, $\tau_{zx} = G\cancel{\gamma_{zx}}^{0}$

$\underset{\sim}{\sigma} = \underset{\sim}{D}\,\underset{\sim}{\varepsilon}$ where

$$\underset{\sim}{D} = \begin{bmatrix} \left[(\lambda+2G) - \frac{\lambda^2}{\lambda+2G}\right] & \lambda - \frac{\lambda^2}{\lambda+2G} & 0 \\ \lambda - \frac{\lambda^2}{\lambda+2G} & \left[(\lambda+2G) - \frac{\lambda^2}{\lambda+2G}\right] & 0 \\ 0 & 0 & G \end{bmatrix}$$

Isotropic Linear Elasticity (2D)

* Plane Strain, Plane Stress, Antiplane Shear

Plane Strain

$\partial/\partial z = 0$, $w = 0$, $u(x,y)$, $v(x,y)$, $b_z = 0$

$\varepsilon_z = \frac{\partial w}{\partial z} = 0$

$\sigma_x = (\lambda+2G)\varepsilon_x + \lambda\varepsilon_y$

$\sigma_y = \lambda\varepsilon_x + (\lambda+2G)\varepsilon_y$

$\sigma_z = \lambda\varepsilon_x + \lambda\varepsilon_y$ (in general, $\sigma_z \neq 0$)

PVW: $\int_\Omega \delta\underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma}\, d\Omega = \int_B \delta\underset{\sim}{u}^T \underset{\sim}{T}\, dB + \int_\Omega \delta\underset{\sim}{u}^T(\underset{\sim}{b} - \rho\ddot{\underset{\sim}{u}})\, d\Omega$

Integrate over a slice of the body (perpendicular to the z-direction), say, of thickness d. Let Ω_{2D} be the cross-section of the body (in the x-y plane) and B_{2D} be the boundary of the cross-section. Then, the PVW becomes:

$$\cancel{d}\int_{\Omega_{2D}} \delta\underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma}\, d\Omega_{2D} = \cancel{d}\int_{B_{2D}} \delta\underset{\sim}{u}^T \underset{\sim}{T}\, dB_{2D} + \cancel{d}\int_{\Omega_{2D}} \delta\underset{\sim}{u}^T(\underset{\sim}{b} - \rho\ddot{\underset{\sim}{u}})\, d\Omega_{2D}$$

(this is possible because of the assumption that $\partial/\partial z = 0$)

$$\underset{\sim}{\sigma} = \begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{bmatrix} \quad \delta\underset{\sim}{\varepsilon} = \begin{bmatrix}\delta\varepsilon_x\\ \delta\varepsilon_y\\ \delta\gamma_{xy}\end{bmatrix} \quad \delta\underset{\sim}{u} = \begin{bmatrix}\delta u\\ \delta v\end{bmatrix} \quad \underset{\sim}{T} = \begin{bmatrix}T_x\\ T_y\end{bmatrix} \quad \underset{\sim}{b} = \begin{bmatrix}b_x\\ b_y\end{bmatrix}$$

$\underset{\sim}{u} = \begin{bmatrix}u\\ v\end{bmatrix}$ $\quad \underset{\sim}{\sigma} = \underset{\sim}{D}\,\underset{\sim}{\varepsilon}$ where $\underset{\sim}{D}$ is as before

z equilibrium:

$$\cancel{\frac{\partial\sigma_{zx}}{\partial x}} + \cancel{\frac{\partial\sigma_{zy}}{\partial y}} + \cancel{\frac{\partial\sigma_z}{\partial z}} + b_z = \cancel{\rho\frac{\partial^2 w}{\partial t^2}} \quad \therefore b_z = 0$$

$$\sigma_{zx} = G\gamma_{zx} = G\left(\cancel{\frac{\partial u}{\partial z}} + \cancel{\frac{\partial w}{\partial x}}\right) = 0$$

$$\sigma_{zy} = G\gamma_{zy} = G\left(\cancel{\frac{\partial v}{\partial z}} + \cancel{\frac{\partial w}{\partial y}}\right) = 0$$

Plane Stress

Assume first that $u(x,y)$, $v(x,y)$ and $w(z)$

$$\sigma_x = (\lambda+2G)\frac{\partial u}{\partial x} + \lambda\frac{\partial v}{\partial y} + \lambda\frac{\partial w}{\partial z}$$

$$\sigma_y = \lambda\frac{\partial u}{\partial x} + (\lambda+2G)\frac{\partial v}{\partial y} + \lambda\frac{\partial w}{\partial z}$$

$$\sigma_z = \lambda\frac{\partial u}{\partial x} + \lambda\frac{\partial v}{\partial y} + (\lambda+2G)\frac{\partial w}{\partial z}$$

assume $\sigma_z = 0$, than $\varepsilon_z = \frac{\partial w}{\partial z} = \frac{-\lambda}{\lambda+2G}\varepsilon_x - \frac{\lambda}{\lambda+2G}\varepsilon_y$

ε_z depends on $(x,y) \Rightarrow w = z f_1(x,y) + f_0(x,y)$

some functions to be determined

Consider the stress components acting on the boundaries parallel to the cross-section:

$$\sigma_z = 0$$

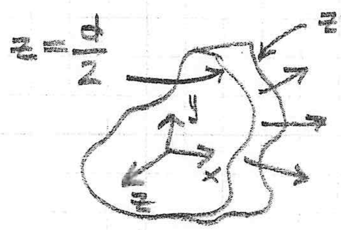
$$\sigma_{zx} = G\left(\cancelto{0}{\partial u/\partial z} + \partial w/\partial x\right) \rightarrow z\frac{\partial f_1}{\partial x} + \frac{\partial f_0}{\partial x}$$

$$\sigma_{zy} = G\left(\cancelto{0}{\partial v/\partial z} + \partial w/\partial y\right) \rightarrow z\frac{\partial f_1}{\partial y} + \frac{\partial f_0}{\partial y}$$

in general $\neq 0$

This would violate boundary conditions on the faces (in the z-direction) of a thin body that is not subjected to any out-of-plane loads.

Instead, apply the condition $\sigma_z = 0$ approximately, ie., assume that the thickness of the body, d, is small and, further, assume that $\sigma_z = 0$, $\sigma_{zx} = 0$, $\sigma_{zy} = 0$ on the faces of the body in the z-direction (these are the planes $z = -d/2$ and $z = d/2$)



only in-plane loads are applied (only loads producing "stretching" of the thin body are to be considered)

z equilibrium: $\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \cancelto{0\ (\text{assumed})}{b_z} = \rho\cancelto{0}{\frac{\partial^2 w}{\partial t^2}}$

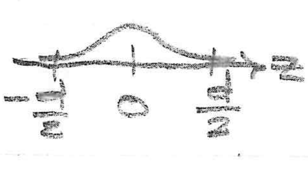
(consider statics problems)

Apply the equil. eq. in the vicinity of each face:

$$\left.\begin{aligned}\frac{\partial \sigma_{zx}}{\partial x} &= 0 \quad \text{at } z = \pm\frac{d}{2}\\ \frac{\partial \sigma_{zy}}{\partial y} &= 0 \quad \text{at } z = \pm\frac{d}{2}\end{aligned}\right\} \Rightarrow \frac{\partial \sigma_z}{\partial z} = 0 \quad \text{at } z = \pm\frac{d}{2}$$

Therefore, at $z = \pm d/2$, $\sigma_z = 0$ and $\partial\sigma_z/\partial z = 0$

On the basis of the "thin-body" assumption, it is reasonable to let $\sigma_z \rightarrow 0$ as $d \rightarrow 0$



Now, consider the PVW and integrate over Ω and B:

$$d \int_{\Omega_{2D}} \delta \underset{\sim}{\varepsilon}^T \hat{\underset{\sim}{\sigma}} \, d\Omega_{2D} \quad \text{replaces} \quad d \int_{\Omega_{2D}} \delta \underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma} \, d\Omega_{2D}$$

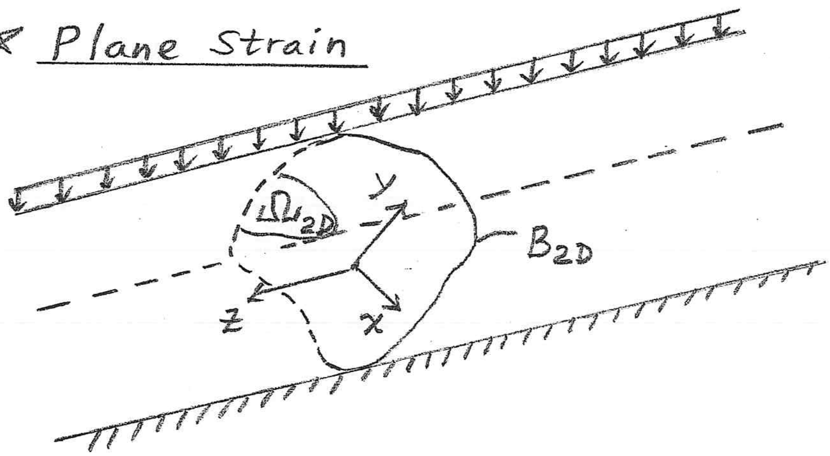
$\hat{\underset{\sim}{\sigma}}$: avg. stress over the thickness

$$\hat{\underset{\sim}{\sigma}} = \frac{1}{d} \int_{-d/2}^{d/2} \underset{\sim}{\sigma} \, dz \quad \text{for each direction } x, y, z$$

Isotropic Linear Elasticity

Two-Dimensional Problems

★ Plane Strain



Assume that

$$u,\ v,\ b_x,\ b_y$$

are functions of the "in-plane" coordinates (x, y) and, possibly, time (t), i.e., they are independent of the "out-of-plane" coordinate (z). Also, assume that

$$w = 0,\quad b_z = 0$$

Finally, the elastic moduli, λ and G, and the mass density, ρ, will be assumed to be, in general, functions of x and y (not z). It then follows that

$$\varepsilon_x,\ \varepsilon_y,\ \gamma_{xy},\ \sigma_x,\ \sigma_y,\ \sigma_z,\ \sigma_{xy}$$

are functions of x, y and t while

$$\varepsilon_z = 0,\quad \gamma_{yz} = 0,\quad \gamma_{zx} = 0$$
$$\sigma_{yz} = 0,\quad \sigma_{zx} = 0$$

Furthermore,

$$\sigma_x = (\lambda + 2G)\,\varepsilon_x + \lambda\,\varepsilon_y$$

$$\sigma_y = \lambda\,\varepsilon_x + (\lambda + 2G)\,\varepsilon_y$$

$$\sigma_z = \lambda\,\varepsilon_x + \lambda\,\varepsilon_y$$

$$\sigma_{xy} = G\,\gamma_{xy}$$

Now, consider a slice of the body parallel to the x-y plane between $z=-\frac{d}{2}$ and $z=\frac{d}{2}$. The thickness (d) of the slice is arbitrary as the body geometry, properties and behavior are independent of z. On the face $z=\frac{d}{2}$, the traction components are:

$$T_x = \sigma_{xz} = 0$$

$$T_y = \sigma_{yz} = 0$$

$$T_z = \sigma_z \text{ (in general, } \neq 0)$$

Similarly, on the face $z=-\frac{d}{2}$, the traction components are:

$$T_x = -\sigma_{xz} = 0$$

$$T_y = -\sigma_{yz} = 0$$

$$T_z = -\sigma_z \text{ (in general, } \neq 0)$$

On the "edge" of the slice (the part of the boundary <u>between</u> the faces $z=-\frac{d}{2}$ and $z=\frac{d}{2}$), the traction components are

$$T_x = \sigma_x n_x + \sigma_{xy} n_y + \cancel{\sigma_{xz}}^{\,0}\, \cancel{n_z}^{\,0}$$

$$T_y = \sigma_{yx} n_x + \sigma_y n_y + \cancel{\sigma_{yz}}^{\,0}\, \cancel{n_z}^{\,0}$$

$$T_z = \cancel{\sigma_{zx}}^{\,0} n_x + \cancel{\sigma_{zy}}^{\,0} n_y + \sigma_z \cancel{n_z}^{\,0} = 0$$

where $\underset{\sim}{n}$ is the outward unit vector normal to the edge. The components n_x, n_y are functions of x and y while $n_z = 0$ everywhere on the edge. Thus, T_x and T_y on the edge are functions of x, y and t while $T_z = 0$ (everywhere on the edge).

For the slice between $z=-\frac{d}{2}$ and $z=\frac{d}{2}$, the Principle of Virtual Work with δu, δv functions of x and y (not z) and $\delta w = 0$ can be rewritten in the equivalent, two-dimensional form:

$$\cancel{d}\cdot\int_{\Omega_{2D}} \delta\underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma}\, d\Omega_{2D} = \cancel{d}\cdot\int_{B_{2D}} \delta\underset{\sim}{u}^T \underset{\sim}{T}\, dB_{2D} + \cancel{d}\cdot\int_{\Omega_{2D}} \delta\underset{\sim}{u}^T(\underset{\sim}{b} - \rho\ddot{\underset{\sim}{u}})\, d\Omega_{2D}$$

where:

$$\delta\boldsymbol{u} = \begin{bmatrix} \delta u \\ \delta v \end{bmatrix}, \quad \delta\boldsymbol{\varepsilon} = \begin{bmatrix} \delta\varepsilon_x \\ \delta\varepsilon_y \\ \delta\gamma_{xy} \end{bmatrix}, \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

$$\boldsymbol{T} = \begin{bmatrix} T_x \\ T_y \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \ddot{\boldsymbol{u}} = \begin{bmatrix} \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial^2 v}{\partial t^2} \end{bmatrix}$$

Note:

$$\boldsymbol{\sigma} = \boldsymbol{D}\,\boldsymbol{\varepsilon}$$

with

$$\boldsymbol{D} = \begin{bmatrix} \lambda+2G & \lambda & 0 \\ \lambda & \lambda+2G & 0 \\ 0 & 0 & G \end{bmatrix}$$

Recall that $\frac{\lambda+2G}{G} = \frac{2-2\nu}{1-2\nu}$ and $G = \frac{E}{2(1+\nu)}$.

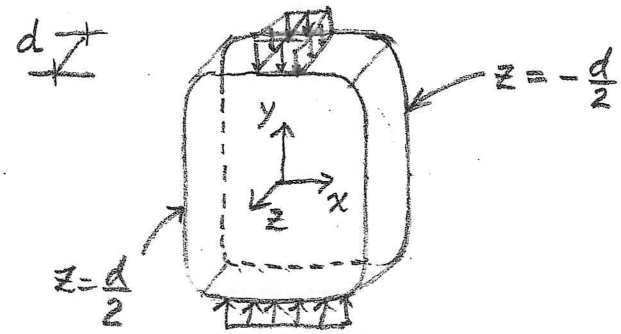
Other representations of $\boldsymbol{D}$:

$$\boldsymbol{D} = \begin{bmatrix} 2G\frac{1-\nu}{1-2\nu} & 2G\frac{\nu}{1-2\nu} & 0 \\ 2G\frac{\nu}{1-2\nu} & 2G\frac{1-\nu}{1-2\nu} & 0 \\ 0 & 0 & G \end{bmatrix}$$

probably the best version to use to get $\boldsymbol{D}$

$$\boldsymbol{D} = \begin{bmatrix} E\frac{1-\nu}{(1+\nu)(1-2\nu)} & E\frac{\nu}{(1+\nu)(1-2\nu)} & 0 \\ E\frac{\nu}{(1+\nu)(1-2\nu)} & E\frac{1-\nu}{(1+\nu)(1-2\nu)} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix}$$

☆ Plane Stress (Approximation)



The out-of-plane dimension (d) is assumed to be much smaller than the in-plane dimensions.
Assume that the faces are free of traction:

$$T_x = T_y = T_z = 0$$

$$\text{on} \quad z = -\frac{d}{2} \text{ and } z = \frac{d}{2}$$

Also, without loss of generality, assume that the traction components on the edge (part of the boundary between $z=-\frac{d}{2}$ and $z=\frac{d}{2}$) are symmetric about the plane $z=0$. Now, define "through-the-thickness" averages of all quantities of interest:

$$\hat{u} = \frac{1}{d}\int_{-\frac{d}{2}}^{\frac{d}{2}} u \, dz \, , \text{ etc.}$$

$$\hat{\varepsilon}_x = \frac{1}{d}\int_{-\frac{d}{2}}^{\frac{d}{2}} \varepsilon_x \, dz, \text{ etc.}$$

$$\hat{\sigma}_x = \frac{1}{d}\int_{-\frac{d}{2}}^{\frac{d}{2}} \sigma_x \, dz, \text{ etc.}$$

etc.

(These averages are functions of only x, y and t.) The elastic moduli, λ and G, and the mass density, ρ, are assumed to be, in general, functions of x and y (not z). Also, the body-force densities

$$b_x \, , \, b_y$$

are functions of x, y and, possibly, t (not z) while

$$b_z = 0$$

Now, introduce the approximation:

$$\hat{\sigma}_z = \frac{1}{d}\int_{-\frac{d}{2}}^{\frac{d}{2}} \sigma_z \, dz = 0$$

$\hat{\sigma}$ is an average σ value

It then follows that

$$\hat{\sigma}_z = \lambda \hat{\varepsilon}_x + \lambda \hat{\varepsilon}_y + (\lambda + 2G)\hat{\varepsilon}_z = 0$$

$$\Rightarrow \quad \hat{\varepsilon}_z = -\lambda \hat{\varepsilon}_x - \lambda \hat{\varepsilon}_y$$

and, therefore,

$$\hat{\sigma}_x = \left(\lambda + 2G - \frac{\lambda^2}{\lambda+2G}\right)\hat{\varepsilon}_x + \left(\lambda - \frac{\lambda^2}{\lambda+2G}\right)\hat{\varepsilon}_y$$

$$\hat{\sigma}_y = \left(\lambda - \frac{\lambda^2}{\lambda+2G}\right)\hat{\varepsilon}_x + \left(\lambda + 2G - \frac{\lambda^2}{\lambda+2G}\right)\hat{\varepsilon}_y$$

The average shearing stress satisfies

$$\hat{\sigma}_{xy} = G\hat{\gamma}_{xy}$$

For the slice between $z = -\frac{d}{2}$ and $z = \frac{d}{2}$, the Principle of Virtual Work with δu, δv functions of x and y (not z) and $\delta w = 0$ can be rewritten in the following form:

$$d \cdot \int_{\Omega_{2D}} \delta\underset{\sim}{\varepsilon}^T \hat{\underset{\sim}{\sigma}} \, d\Omega_{2D} = d \cdot \int_{B_{2D}} \delta\underset{\sim}{u}^T \hat{\underset{\sim}{T}} \, dB_{2D} + d \int_{\Omega_{2D}} \delta\underset{\sim}{u}^T (\hat{\underset{\sim}{b}} - \rho \hat{\ddot{\underset{\sim}{u}}}) \, d\Omega_{2D}$$

where:

$$\delta\underset{\sim}{u} = \begin{bmatrix} \delta u \\ \delta v \end{bmatrix}, \quad \delta\varepsilon = \begin{bmatrix} \delta\varepsilon_x \\ \delta\varepsilon_y \\ \delta\gamma_{xy} \end{bmatrix}, \quad \hat{\underset{\sim}{\sigma}} = \begin{bmatrix} \hat{\sigma}_x \\ \hat{\sigma}_y \\ \hat{\sigma}_{xy} \end{bmatrix}, \quad \hat{\underset{\sim}{\varepsilon}} = \begin{bmatrix} \hat{\varepsilon}_x \\ \hat{\varepsilon}_y \\ \hat{\gamma}_{xy} \end{bmatrix}$$

$$\hat{\underset{\sim}{T}} = \begin{bmatrix} \hat{T}_x \\ \hat{T}_y \end{bmatrix}, \quad \hat{\underset{\sim}{b}} = \begin{bmatrix} b_x \\ b_y \end{bmatrix}, \quad \underset{\sim}{u} = \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix}, \quad \hat{\ddot{\underset{\sim}{u}}} = \begin{bmatrix} \frac{d^2\hat{u}}{dt^2} \\ \frac{d^2\hat{v}}{dt^2} \end{bmatrix}$$

Note:

$$\underset{\sim}{\hat{\sigma}} = \underset{\sim}{D}\,\underset{\sim}{\hat{\varepsilon}}$$

with

$$\underset{\sim}{D} = \begin{bmatrix} \frac{(\lambda+2G)^2-\lambda^2}{\lambda+2G} & \frac{2\lambda G}{\lambda+2G} & 0 \\ \frac{2\lambda G}{\lambda+2G} & \frac{(\lambda+2G)^2-\lambda^2}{\lambda+2G} & 0 \\ 0 & 0 & G \end{bmatrix}$$

Other representations of $\underset{\sim}{D}$:

$$\underset{\sim}{D} = \begin{bmatrix} \frac{2G}{1-\nu} & \frac{2G\nu}{1-\nu} & 0 \\ \frac{2G\nu}{1-\nu} & \frac{2G}{1-\nu} & 0 \\ 0 & 0 & G \end{bmatrix}$$

$$\underset{\sim}{D} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix}$$

Notes:

1. The matrix $\underset{\sim}{D}$ for plane stress can be obtained from the plane-strain version by keeping G the same and replacing ν with

$$\frac{\nu}{1+\nu}$$

2. The matrix $\underset{\sim}{D}$ for plane strain can be obtained from the plane-stress version by keeping G the same and replacing ν with

$$\frac{\nu}{1-\nu}$$

3. The "plane stress approximation" becomes exact in the limit as

$$d \to 0$$

MULTI-DIMENSION

Antiplane shear

- $u=0$, $v=0$
- w is a function of x, y, and t (not z)
- $b_x = b_y = 0$
- b_z is a function of x, y, t

considering these requirements,

$$\varepsilon_x = \frac{du}{dx} = 0 \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = 0 \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \text{ generally} \neq 0 \text{, but } \frac{\partial v}{\partial z} = 0$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0 \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \neq 0 \text{, although } \frac{\partial u}{\partial z} = 0$$

Putting it into matrix form:

$$\varepsilon = \begin{bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}, \quad \sigma = \begin{bmatrix} \sigma_{yz} \\ \sigma_{zx} \end{bmatrix}$$

$$\sigma = D\varepsilon, \qquad D = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}$$

Big equation time:

$$d\int_{\Omega_{2D}} \delta\varepsilon^T \sigma \, d\Omega_{2D} = d\int_{B_{2D}} \delta u^T T \, dB_{2D} + d\int_{\Omega_{2D}} \delta u^T (b - \rho\ddot{u}) \, d\Omega_{2D}$$

$$\delta u = 0, \delta v = 0, \delta z = f(x, y)$$

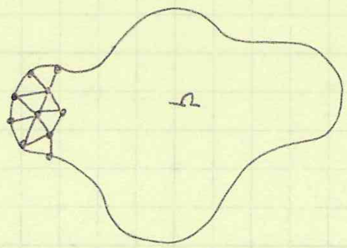
$$\delta\varepsilon = \begin{bmatrix} \delta\gamma_{yz} \\ \delta\gamma_{zx} \end{bmatrix}, \quad \delta u = [\delta w], \quad T = [T_z]$$

$$b = [b_z], \quad \ddot{u} = \left[\frac{\partial^2 w}{\partial t^2}\right]$$

- used in wave propagation in simple media
- wave modeling
- only concerned with shear (specifically, out of plane)

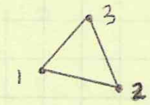
or, anti (as in, antishear)

FEA in TWO DIMENSIONS



simplest element:

constant strain triangle (CST)



- order nodes in the counter-clockwise direction
- in ABAQUS, CPE3, CPS3

C: continuum element; P: plane; E: strain / S: stress (ε = strain)

constant strain triangle

uses polynomial interpolation

$$u = ax + by + cz$$
$$v = a'x + b'y + c'z$$

let u_1, u_2, u_3 be the approximations of u at the three nodes. Then,

~~$ax_1 + bx_2 + cx_3$~~

$$ax_1 + by_1 + c = u_1$$
$$ax_2 + by_2 + c = u_2 \quad \text{etc.}$$

the resulting interpolation is of the form

$$\underline{u = N_1(x,y)u_1 + N_2(x,y)u_2 + N_3(x,y)u_3}$$

where N_1, N_2, N_3 are nodal interpolation functions

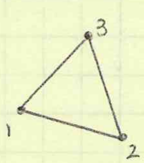
Interpolation functions satisfy:

$$N_1(x_1, y_1) = 1$$
$$N_1(x_2, y_2) = 0$$
$$N_1(x_3, y_3) = 0 \quad \text{etc, for all three functions}$$

solving,
$$N_1 = \frac{(x-x_2)(y_3-y_2) - (y-y_2)(x_3-x_2)}{(x_1-x_2)(y_3-y_2) - (y_1-y_2)(x_3-x_2)}$$

- introduces possibility of dividing by zero
- sign of value is important

what does denominator mean?



$$(x_1-x_2)(y_3-y_2) - (y_1-y_2)(x_3-x_2)$$

$$\vec{21} = \begin{bmatrix} x_1 - x_2 \\ y_1 - y_2 \end{bmatrix}, \quad \vec{23} = \begin{bmatrix} x_3 - x_2 \\ y_3 - y_2 \end{bmatrix}$$

in 3D space, would have 0 line at bottom

$\vec{21} \times \vec{23}$ = denom., in z-direction

so,
$$N_1 = \frac{-1}{2A}\left[x(y_3-y_2) - y(x_3-x_2) - x_2 y_3 + \cancel{x_2 y_2} + y_2 x_3 - \cancel{y_2 x_2}\right]$$

EQUATIONS FOR 2D

Triangle element

Interpolation functions

$$N_1 = \frac{1}{2A}\left[x(y_2-y_3) - y(x_2-x_3) + x_2y_3 - x_3y_2\right]$$

$$N_2 = \frac{1}{2A}\left[x(y_3-y_1) - y(x_3-x_1) + x_3y_1 - x_1y_3\right]$$

$$N_3 = \frac{1}{2A}\left[x(y_1-y_2) - y(x_1-x_2) + x_1y_2 - x_2y_1\right]$$

Encounters problems when:
- A is 0 (points on a line)
- A is negative (nodes ordered incorrectly)

in matrix form:

$$u = \underbrace{\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}}_{\underset{\sim}{N}} \underbrace{\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}}_{\underset{\sim}{u}}$$

$$\varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \partial N_1/\partial x & 0 & \partial N_2/\partial x & 0 & \partial N_3/\partial x & 0 \\ 0 & \partial N_1/\partial y & 0 & \partial N_2/\partial y & 0 & \partial N_3/\partial y \\ \partial N_1/\partial y & \partial N_1/\partial x & \partial N_2/\partial y & \partial N_2/\partial x & \partial N_3/\partial y & \partial N_3/\partial x \end{bmatrix} \underset{\sim}{u}$$

$\underset{\sim}{B}$ matrix

$$\frac{\partial N_1}{\partial x} = \frac{y_2-y_3}{2A}, \quad \frac{\partial N_1}{\partial y} = \frac{x_2-x_3}{2A}, \text{ etc}$$

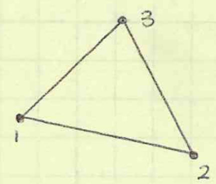
$$\delta\varepsilon = B \cdot \delta u$$

$$K = B^T d D B \cdot A = B^T (dA \cdot D) B$$

thickness

TWO·DIM. FORMULATIONS

Constant Strain Triangle



$+\circlearrowleft$ for numbering

single element contribution to P.V.W.:

$$d\int_{\Omega} \delta\varepsilon^T \sigma \, d\Omega = \delta\bar{u}^T \left[\int_{\Omega} B^T (dD) B \, d\Omega \right] \bar{u}$$

$$= \delta\bar{u}^T \left[B^T (d \cdot A \cdot D) B \right] \bar{u}$$

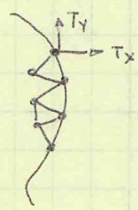
thickness

(assuming that D is constant)

Recall:

- $\delta u = N \delta\bar{u}$
- $u = N\bar{u}$
- $\varepsilon = B\bar{u}$
- $\delta\varepsilon = B\delta\bar{u}$
- $\sigma = D\varepsilon$

Tractions – generally functions of locations on the boundary



$$d\int_B \delta u^T T \, dB = \delta\bar{u}^T \int_B N d \cdot T \, dB$$

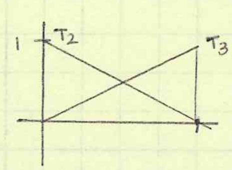
consider the simple case of uniformly·distributed boundary load, i.e., T_x and T_y are constant along B

$$\delta\bar{u}^T \int_B N^T d \underbrace{\begin{bmatrix} T_x \\ T_y \end{bmatrix}}_{\text{constant}} dB$$

$$= \delta\bar{u}^T \begin{bmatrix} 0 \\ 0 \\ \int_{23} N_2 d T_x \, d_{23} \\ \int_{23} N_2 d T_y \, d_{23} \\ \int_{23} N_3 d T_x \, d_{23} \\ \int_{23} N_3 d T_y \, d_{23} \end{bmatrix}$$

(assuming that B is the side 23 of e)

simplifying,



$$\int_{23} N_2 d T_x \, d_{23} = \frac{1}{2} \cdot 1 \cdot L_{23} \cdot d \cdot T_x = \frac{d}{2} L_{23} T_x$$

$$\int_{23} N_3 d T_y \, d_{23} = \frac{1}{2} d \, L_{23} T_y$$

$$= \delta\bar{u}^T \frac{d}{2} L_{23} \begin{bmatrix} 0 \\ 0 \\ T_x \\ T_y \\ T_x \\ T_y \end{bmatrix}$$

lump forces half and half at the nodes involved.

TWO-DIMENSIONAL FORMULATIONS

Constant Strain Triangle

After assembly of element contributions, the equations become:

$$\underbrace{K}_{\text{assembly of } K_e \text{ for all elements}} \mathcal{U} = \underbrace{P}_{\text{assembly of loads due to body force densities and boundary tractions}}$$

Apply support conditions by a method identical to in one dimension

After computing $\mathcal{U}$, process the elements one at a time by:

- extracting the nodal displacements associated with the element, u_e
- calculate the strain components (constant in the CST)

$$\varepsilon_e = B_e u_e$$

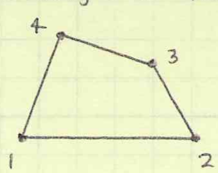
- use strains to calculate stresses

$$\sigma_e = D_e \varepsilon_e$$

More elements

order of interpolation, linear to quad to ...

element shape, triangle to quadrilateral ...

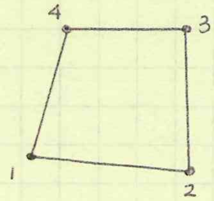


$u = axy + bx + cy + d$ etc

DOES NOT WORK

TWO or THREE DIMENSIONS

Four-sided elements



* see handout for more details

to connect to another element, lines are shared
so, interpolation functions of nodes other than those involved must be zero through entire line

Equations to solve:

$$a_1 + b_1 x_2 + c_1 y_2 + d_1 x_2 y_2 = 1$$
$$a_1 + b_1 x_3 + c_1 y_3 + d_1 x_3 y_3 = 0$$
$$a_1 + b_1 x_4 + c_1 x_4 + d_1 x_4 y_4 = 0$$

} considering all nodes

the only way this works is with rectangle elements aligned with the x, y axis

Further equations

$$\frac{\partial N_1}{\partial x} = \frac{\partial N_1}{\partial \xi}\frac{d\xi}{dx} + \frac{\partial N_1}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{dN_1}{\partial y} = \frac{\partial N_1}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_1}{\partial \eta}\frac{\partial \eta}{\partial y}$$

in matrix form,

$$\begin{bmatrix} \frac{\partial N_1}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial x} \\ \frac{\partial N_1}{\partial y} \end{bmatrix}$$

Jacobian matrix, in FEA calculations

$$x = \Sigma N_i x_i, \quad y = \Sigma N_i y_i$$

$$\frac{\partial x}{\partial \xi} = \Sigma \frac{\partial N_i}{\partial \xi} x_i$$

$$\frac{\partial x}{\partial \eta} = \Sigma \frac{\partial N_i}{\partial \eta} x_i$$

$$\frac{\partial y}{\partial \xi} = \Sigma \frac{\partial N_i}{\partial \xi} y_i, \quad \frac{\partial y}{\partial \eta} = \Sigma \frac{\partial N_i}{\partial \eta} y_i$$

$$\frac{\partial N}{\partial x, y} = J^{-1} \frac{\partial N}{\partial \xi, \eta}$$

as in, inverse of Jacobian matrix

$$\det(J) = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}$$

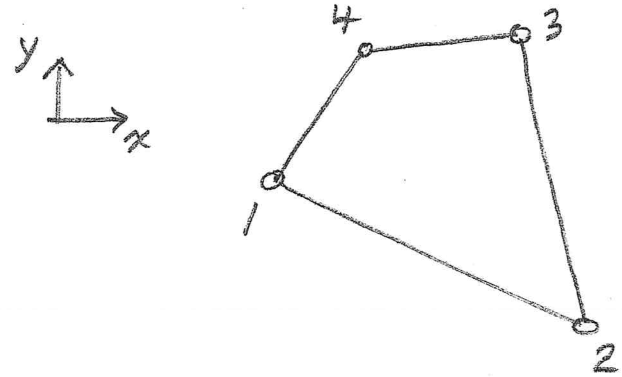
often referred to as the Jacobian, even though it's the determinant

$$\frac{d\Omega}{d\xi d\eta} = |\det(J)|$$

not derivatives, but area, volume, length

11 October 06

Polynomial Interpolation and the Quadrilateral



Assume that no three of the nodes are on the same line. Continuity of displacements across interelement boundaries (element sides) requires that, on any side, the displacements be functions of only the nodes on that side. Let us see whether this requirement can be satisfied with a polynomial interpolation of the form:

$$u = \sum_{i=1}^{4} N_i(x,y)\, u_i \qquad (1)$$

where:

$$N_i(x,y) = a_i + b_i x + c_i y + d_i xy \quad (i=1,\dots,4) \qquad (2)$$

Consider one of these nodal interpolation function, say, and recall that it satisfies:

$$N_1(x_1,y_1) = 1,\ N_1(x_2,y_2) = N_1(x_3,y_3) = N_1(x_4,y_4) = 0 \qquad (3)$$

It can be shown that $d_1 \neq 0$ (by contradiction: $d_1 = 0$ implies that 2, 3 and 4 are on the same line). Now, consider the segment 23 and evaluate N_1 at 2, 3 and the middle of the segment:

$$a_1 + b_1 x_2 + c_1 y_2 + d_1 x_2 y_2 = 0 \qquad (4a)$$

$$a_1 + b_1 x_3 + c_1 y_3 + d_1 x_3 y_3 = 0 \qquad (4b)$$

$$a_1 + b_1 \frac{x_2+x_3}{2} + c_1 \frac{y_2+y_3}{2} + d_1 \frac{x_2+x_3}{2} \cdot \frac{y_2+y_3}{2} \qquad (4c)$$

N_1 at middle of 23

Taking the average of (4a) and (4b), we find:

$$a_1 + b_1 \frac{x_2+x_3}{2} + c_1 \frac{y_2+y_3}{2} + d_1 \frac{x_2 y_2 + x_3 y_3}{2} = 0 \qquad (5)$$

The continuity requirement referred to above implies that N_1 at the middle of 23 be equal to 0:

$$a_1 + b_1 \frac{x_2+x_3}{2} + c_1 \frac{y_2+y_3}{2} + d_1 \frac{x_2+x_3}{2} \cdot \frac{y_2+y_3}{2} = 0 \qquad (6)$$

"Subtracting" (5) from (6), we obtain:

$$d_1 \frac{-x_2y_2 - x_3y_3 + x_2y_3 + x_3y_2}{4} = 0$$

$$\Rightarrow \ (x_3 - x_2)(y_2 - y_3) = 0 \ \ (\text{since } d_1 \neq 0)$$

$$\Rightarrow \ x_2 = x_3 \ \text{ or } \ y_2 = y_3 \qquad (7)$$

Similarly, working with segment 34, we find:

$$x_3 = x_4 \ \text{ or } \ y_3 = y_4 \qquad (8)$$

Also, working with N_3 and segments 41 and 12, we obtain:

$$x_4 = x_1 \ \text{ or } \ y_4 = y_1 \qquad (9)$$

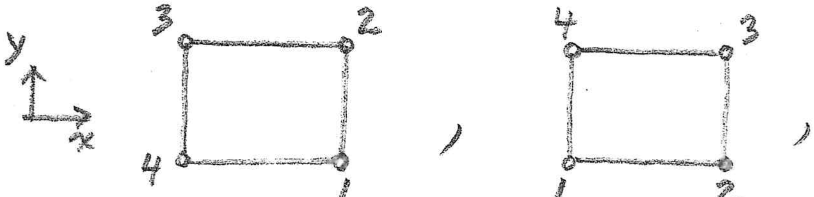
and

$$x_1 = x_2 \ \text{ or } \ y_1 = y_2 \qquad (10)$$

Examining (7), (8), (9) and (10) and keeping in mind the assumption that no three of the nodes are on the same line, we can conclude that

$$\begin{matrix} x_1 = x_2 \\ y_2 = y_3 \\ x_3 = x_4 \\ y_4 = y_1 \end{matrix} \quad \text{or} \quad \begin{matrix} y_1 = y_2 \\ x_2 = x_3 \\ y_3 = y_4 \\ x_4 = x_1 \end{matrix} \qquad (11)$$

Both of the possibilities (11) imply that the quadrilateral is a rectangle aligned with the x and y axes, for example,



, etc. (12)

Indeed, for such rectangles, the continuity requirement is satisfied. It can be shown that the nodal interpolation functions, of the type specified in (1) and (2), for the rectangles in (12), can be written as:

$$N_1(x,y) = \frac{(x-x_3)(y-y_3)}{(x_1-x_3)(y_1-y_3)} \qquad (13)$$

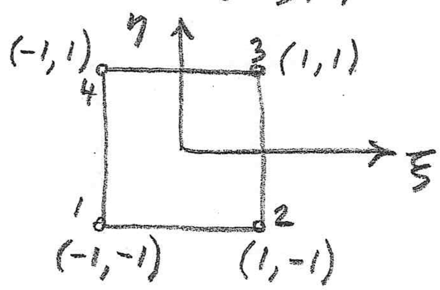
$$N_2(x,y) = \frac{(x-x_4)(y-y_4)}{(x_2-x_4)(y_2-y_4)} \tag{14}$$

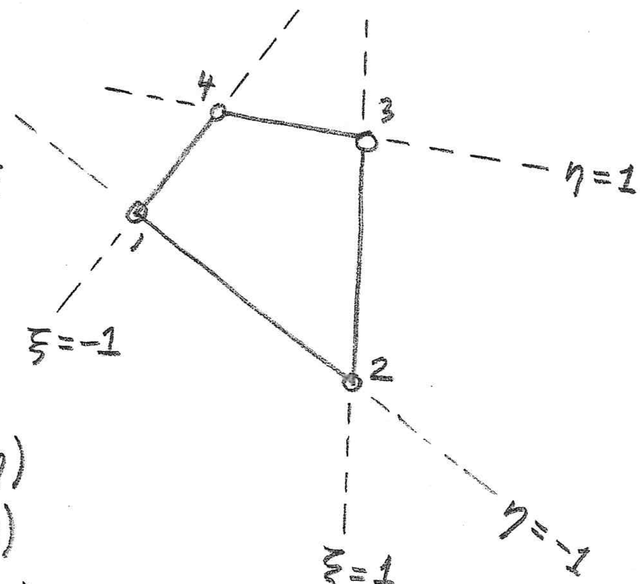
$$N_3(x,y) = \frac{(x-x_1)(y-y_1)}{(x_3-x_1)(y_3-y_1)} \tag{15}$$

$$N_4(x,y) = \frac{(x-x_2)(y-y_2)}{(x_4-x_2)(y_4-y_2)} \tag{16}$$

It is straightforward to check that the continuity requirement is satisfied. For example, N_1 is equal to 0 at all points on segments 23 and 34. Similar observations can be made for N_2, N_3 and N_4.

Arbitrary quadrilaterals can be developed by means of <u>mapping</u> onto a <u>square</u> in another domain. Using a rectangular coordinate system, say, (ξ, η), the interpolation functions can be taken as polynomials in terms of ξ, η:





$$N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta)$$
$$N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta)$$
$$N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

$$x = \sum_{i=1}^{4} N_i(\xi,\eta)\, x_i$$
$$y = \sum_{i=1}^{4} N_i(\xi,\eta)\, y_i$$
$$u = \sum_{i=1}^{4} N_i(\xi,\eta)\, u_i$$
$$v = \sum_{i=1}^{4} N_i(\xi,\eta)\, v_i$$

<u>Isoparametric Quadrilateral</u>

ISOPARAMETRIC ELEMENTS

Overview / catch up

$x = \text{polynomial}(1, \xi, \eta, \xi\eta)$

$y = \text{polynomial}(1, \xi, \eta, \xi\eta)$

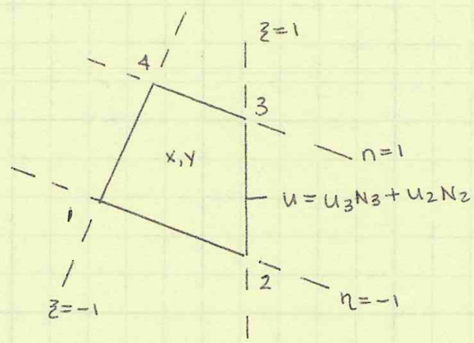
In general, solution for ξ, η in terms of x, y is not possible (explicitly)

$\xi \rightarrow x$ easy

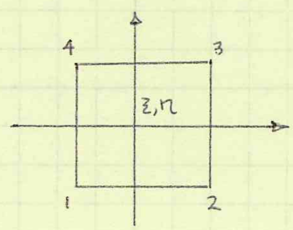
$x \rightarrow \xi$ less easy

$x = \sum N_i(\xi,\eta) x_i$ similarly, $u = \sum N_i(\xi,\eta) u_i$

$y = \sum N_i(\xi,\eta) y_i$ $v = \sum N_i(\xi,\eta) v_i$



considering a quadrilateral



$-1 \le \xi \le 1$

$-1 \le \eta \le 1$

$N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta)$

$N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)$

$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$

$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$

goal: angles between nodes are as large as possible. Small angles = bad
- rectangles, equilateral triangles are the best

Derivatives:

- $\frac{\partial x}{\partial \xi} = \sum \frac{\partial N_i}{\partial \xi} x_i$, $\frac{\partial x}{\partial \eta} = \sum \frac{\partial N_i}{\partial \eta} y_i$
- $\frac{\partial y}{\partial \xi} = \sum \frac{\partial N_i}{\partial \xi} x_i$, $\frac{\partial y}{\partial \eta} = \sum \frac{\partial N_i}{\partial \eta} y_i$
- $\varepsilon_x = \frac{\partial u}{\partial x} = \sum \frac{\partial N_i}{\partial x}(\xi,\eta) u_i$
- $\varepsilon_y = \sum \frac{\partial N_i}{\partial y}(\xi,\eta) v_i$
- $\gamma_{xy} = \sum \frac{\partial N_i}{\partial y}(\xi,\eta) u_i + \sum \frac{\partial N_i}{\partial x}(\xi,\eta) v_i$

ISOPARAMETRIC ELEMENTS

Putting matrices into equations (flip that)

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix}}_{J^{-1}} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix}$$

J^{-1} – the problem is that we don't know those derivatives explicitly

BUT,

$$\begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{\underset{\sim}{J}} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix}$$

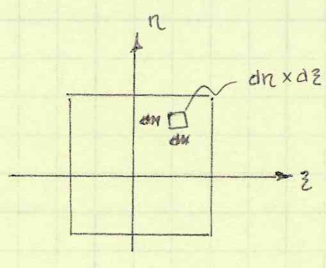
$(\xi,\eta) \rightarrow (x,y)$ explicitly

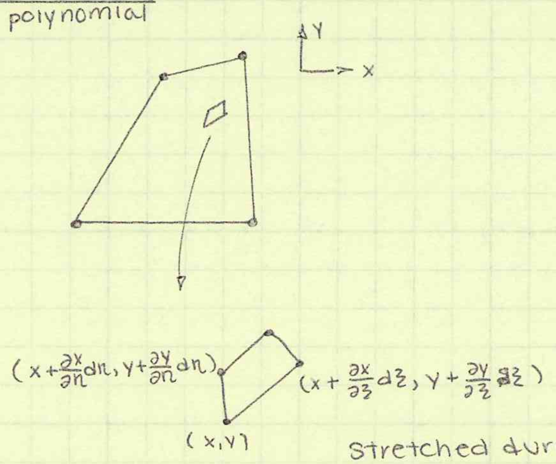
$(x,y) \rightarrow (\xi,\eta)$ not explicit

$$J^{-1} = \frac{1}{\det(J)} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \xi} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \eta} \end{bmatrix}, \quad \det(J) = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}$$

all entries in J^{-1} are, in general, rational functions of ξ, η

$$\frac{\text{polynomial}}{\text{polynomial}}$$

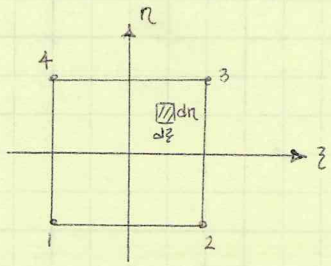




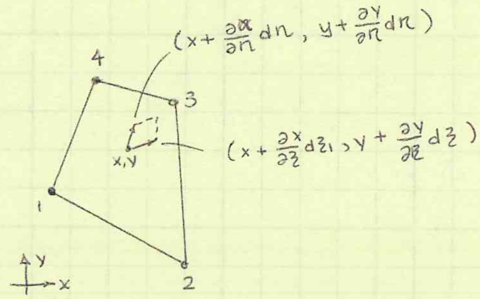
stretched during mapping back to x, y

QUADRILATERAL ELEMS

Mapping between coordinates

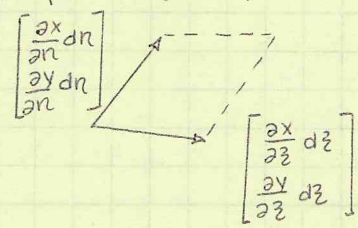


parent domain



physical domain

the mapped parallelogram:



we want the area:

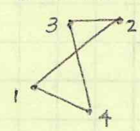
$$A = \begin{bmatrix}\frac{\partial x}{\partial \xi}d\xi \\ \frac{\partial y}{\partial \xi}d\xi \\ 0\end{bmatrix} \times \begin{bmatrix}\frac{\partial x}{\partial \eta}d\eta \\ \frac{\partial y}{\partial \eta}d\eta \\ 0\end{bmatrix} = \frac{\partial x}{\partial \xi}d\xi\frac{\partial y}{\partial \eta}d\eta - \frac{\partial x}{\partial \eta}d\eta\frac{\partial y}{\partial \xi}d\xi$$

add a zero to include third dimension

$$A = \underbrace{\left|\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}\right|}_{\text{equal to } \det(J)} d\xi\, d\eta$$

$$|\det(J)| = \frac{\text{area of}}{\text{area of}}$$

determinant will be positive, unless quadrilateral is poorly formed (defined backwards?)



not so good.

QUADRI. ELEMENTS

Equations and calculations

$$\underset{\sim}{K}^e = \int_{\Omega_e} B^T D B \, d\Omega_e = d \int_{-1}^{1}\int_{-1}^{1} B^T D B \frac{\text{area } \angle}{\text{area } \square} \, d\xi \, d\eta$$

or, det(J)

usually constant

contains derivatives of N_i with respect to x, y (in general, rational functions with det(J) in the denominator)

$$\underset{\sim}{K}^e = \int_{-1}^{1} \left(\sum_{i=1}^{I} \left[B^T D B \det(J) \right]_{\xi=\xi_i} w_i \right) d\eta \cdot d$$

I = number of integration pts. with respect to ξ

continuing,

$$\underset{\sim}{K}^e = d \sum_{j=1}^{J} \sum_{i=1}^{I} \left[B^T D B \det(J) \right]_{\substack{\xi=\xi_i \\ \eta=\eta_j}} w_i w_j$$

weights of integration

Numerical integration

$$\int_{-1}^{1} F(\xi) \, d\xi$$

trapezoidal rule:

$$\int_{-1}^{1} F(\xi) \, d\xi \approx \overset{W_1}{1} \cdot F(-1) + \overset{W_2}{1} \cdot F(1)$$

weight values, must add to total length

"second-order accurate"

Simpson's rule

$$\int_{-1}^{1} F(\xi) \, d\xi \approx \frac{1}{3} F(-1) + \frac{4}{3} F(0) + \frac{1}{3} F(1)$$

three-point formula, fourth-order accurate — up to cubic polynomial functions ~~wt~~ will be estimated accurately

Gauss-Legendre

— one point

$$\int_{-1}^{1} F(\xi) \, d\xi \approx 2 \cdot F(0)$$

2nd-order accuracy

— two point

$$\int_{-1}^{1} F(\xi) \, d\xi \approx 1 \cdot F\left(\frac{-1}{\sqrt{3}}\right) + 1 \cdot F\left(\frac{1}{\sqrt{3}}\right)$$

— three point

$$\int_{-1}^{1} F(\xi) \, d\xi = \frac{5}{9} F\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} F(0) + \frac{5}{9} \left(\sqrt{\frac{3}{5}}\right)$$

sixth-order accuracy!

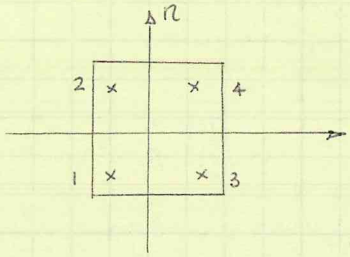
in general, n-point Gauss integration provides 2n order accuracy (accurate for polynomials up to 2n-1)

QUADRATIC ELEMENTS

Calculations and equations

Typically, the locations of the integration points and the corresponding weights are selected from the Gauss · Legendre quadrature family

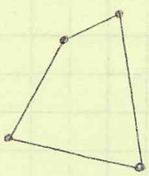
$$K_e = \sum_{ip=1}^{nip} \left[B^T (d \cdot W_{ip} \cdot \det(J) \cdot D) \, B \right]_{\substack{\xi = \xi_{ip} \\ \eta = \eta_{ip}}}$$



$\xi_1 = -1/\sqrt{3}, \ \eta = -1/\sqrt{3}$

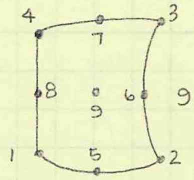
integration point	i	j
1	1	1
2	1	2
3	2	1
4	2	2

Other quadrilaterals



4-node

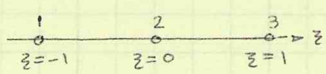
$1, \xi, \eta, \xi\eta$



9-node

$1, \xi, \eta, \xi\eta, \xi^2\eta, \xi\eta^2$
$\eta^2, \xi^2, \xi^2\eta^2$

Lagrangian interpolation functions in each of the ξ, η directions

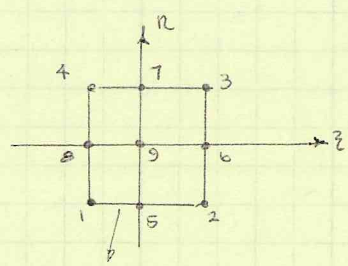


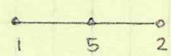
$N_1(\xi) = \frac{1}{2}\xi(\xi - 1)$
$N_2(\xi) = 1 - \xi^2$
$N_3(\xi) = \frac{1}{2}\xi(\xi + 1)$

— one dimensional equations

to get IPs in 2- or 3-D,

node	ξ	η	N_i
1	1	1	$N_1(\xi) N_1(\eta)$
2	3	1	$N_3(\xi) N_1(\eta)$
3	3	3	
4	1	3	
5	2	1	$N_2(\xi) N_1(\eta)$
6	3	2	
7	2	3	
8	1	2	
9	2	2	$N_2(\xi) N_2(\eta)$





node 3 on the line in the ξ direction

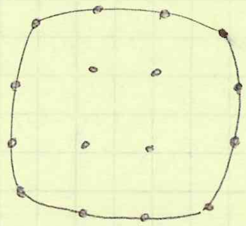
ex. $N_9 = (1 - \xi^2)(1 - \eta^2)$
"bubble" function over the ~~[illegible]~~ square

9 is the only one with $\xi^2\eta^2$ → NO. But if Node 9 were not there, $\xi^2\eta^2$ term doesn't exist for any function

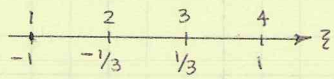
ex. $N_6 = \frac{1}{2}\xi(\xi + 1)(1 - \eta^2)$

MORE and MORE NODES

Higher order Lagrangian elements



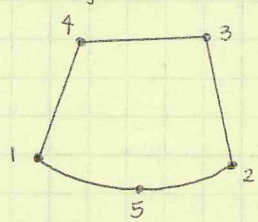
complete third-order interpolation



$N_1(\xi) = \ldots$

$N_2(\xi) = \ldots$

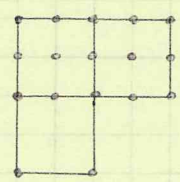
More interesting and useful: serendipity elements



4 standard nodes

1 midside node

good for mesh refinement

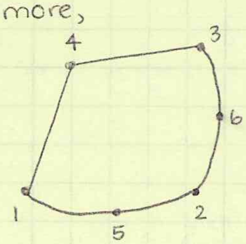


9 node

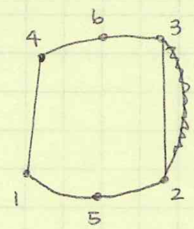
↓

5 node

Even more,



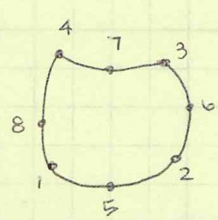
2 consecutive midside nodes



2 non-consecutive midside nodes

variable node elements

Also in this category,

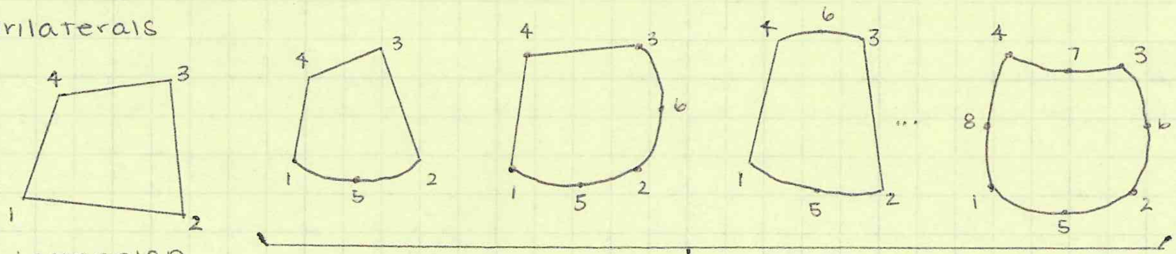


8-node element

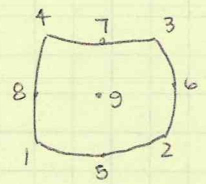
- most useful of the quadrilaterals
- cheap while accurate

FOUR·NODE (+) ELEMENTS

Quadrilaterals

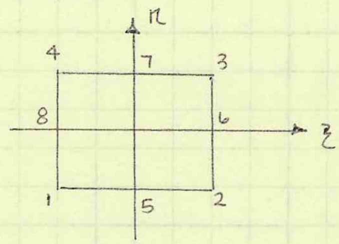


Lagrangian (bilinear) — serendipity elements



(biquadratic)

consider the 8·node element



$N_1(\xi,\eta) = (1-\xi)(1-\eta)(\xi+\eta+1)\,\tfrac{-1}{4}$

- must be zero on all sides that do not include node in question
- also at center nodes on lines point is on (e.g. 5, 8)

$$N_1 = -\tfrac{1}{4}(1-\xi)(1-\eta)(\xi+\eta+1)$$
$$N_2 = -\tfrac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1)$$
...
$$N_5 = \tfrac{1}{2}(1-\xi)(1-\eta)(1+\xi)$$
$$N_6 = \tfrac{1}{2}(1+\xi)(1-\eta)(1+\eta)$$
...

- None of the interpolation functions of the 8·node element contain any $\xi^2\eta^2$ term
- This is the only ~~polynomial in~~ term in the polynomial approximation gained by going to the 9·node element

Numerical integration of K for 8- and 9·node elements

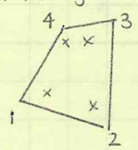
$$K_e = \int_\Omega B^T D B \, d\Omega$$

└ (B) contains polynomial terms of ~~an~~ degree ~~of~~ at ~~least~~ most two in ξ or η

- Therefore, polynomials of at most degree 4 in ξ or η are to be integrated
- 3·point Gauss integration is required (exact up to order 5)

INTEGRATION / APPROXIMATION

Full integration



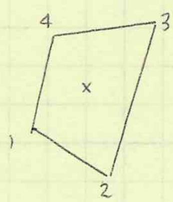
4 integration points,
2×2 gauss

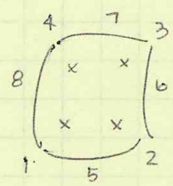
8- or 9-nodes: 9 integration pts, 3×3 gauss ← 9x as expensive to produce the stiffness matrix
$9 B^T D B$ evals vs. $4 B^T D B$ for 4-node, $1 B^T D B$ for triangle

general response:
too stiff, more so than whatever is being modeled

To fix: use reduced integration
2×2 gauss → 1×1
4×4 gauss → 3×3 etc.

Reduced integration





Result: less stiff approximation that balances extra stiffness from discretation

ONE IP
1×1 gauss

Four IP, 2×2 gauss

stiffness matrix is positive definite—
true if and only if, for any vector $x \neq 0$,
$$X^T A X \geq 0$$

HOMEWORK #5 HELP

Homework – nodal force equations

$$d\int_B N^T T \, dB$$

vector due to boundary tractions

traction matrix

matrix o' interpolation functions

$$\begin{bmatrix} -pn_x \\ -pn_y \end{bmatrix}$$

if the side is straight, equations for n_x, n_y are constant
– do cross·products

In non·constant situation (curved), need tangent lines – derivatives
– take normal
– use unit vector

To integrate,
use mapping to a standard interval

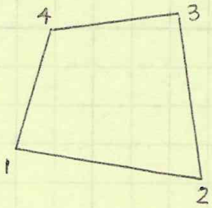
-1 ——— 0 ——— 1 3·nodes

equations in terms of ξ
need determinant of the Jacobian
(one number, scale factor)

use variables for node points, unit thickness d, etc.

QUADRILATERAL ELEMS.

Reduced vs. Full integration



full: 2×2 gauss
reduced: 1×1 gauss

benefits:
- fewer calculations
- less (more accurate) stiffness

$$\underset{\sim}{K} = d \int_\Omega B^T D B \, d\Omega$$

$$= \sum_{ip=1}^{nip} \underline{B^T \left(d \cdot \det(J) \cdot W_{ip} \cdot D \right) B}$$

for the 4-node quad, this is 8×8 symmetric matrix

also an 8×8 symmetric matrix

the columns / rows (eigenvalues and -vectors)

is $\underset{\sim}{K}$ singular?
- determinant = 0
- at least one eigenvalue = 0
- at least one column is a combination of other column(s)
  gauss elimination would fail
- one vector (or more) for $U \neq 0$ results in $KU = 0$

YES, K IS singular, because for U equal to any of the rigid body modes, KU = 0 (no force)

- rigid body modes

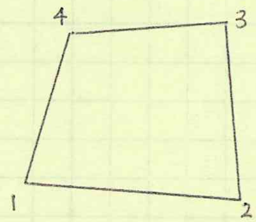
$$x: \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad y: \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad rot: \begin{bmatrix} -y_1 \\ x_1 \\ -y_2 \\ x_2 \\ -y_3 \\ x_3 \\ -y_4 \\ x_4 \end{bmatrix} \quad \times \text{ any number (scaling)}$$

this means 3 of the columns of $\underset{\sim}{K}$ can be written from others

Reduced integration leads to other U vectors which allow for KU = 0

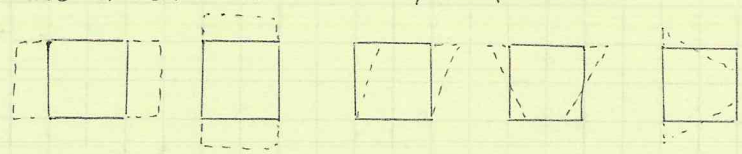
EVALUATION METHODS

Full integration



2×2 gauss points

8×8 K matrix has 3 linearly dependent columns (or rows), corresponding to the 3 rigid·body modes. The other 5 are linearly independent

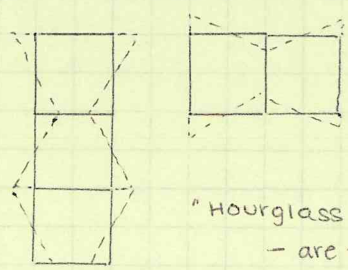


$K = B^T D B \big|_{\xi=0,\ \eta=0}$

$\underset{\sim}{K}$ has 3 zero eigenvalues and 5 positive ones (negative values are imposible ~~d~~-because $\underset{\sim}{D}$ is positive definite)

in the case of reduced integration, $\underset{\sim}{K}$ has (at most) 3 linearly independent equations – thus, only 3 nonzero eigenvalues (and 5 zeros)

– what forms the other two U vectors that result in KU = 0 ?



etc.

"Hourglass modes"

– are two additional modes with KU = 0

– not rigid body modes

"spurious" modes

– can be removed by adding hourglass stiffness to the elem.

Considering the 8·node element – CPS8R or ~~8~~ CPE8R

full integration ⟶ 3×3 gauss, 3 linearly dependent columns

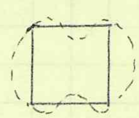
reduced integration ⟶ 2×2 gauss

For each gauss point, one expects 3 additional linearly independent columns

2×2 means 4pts, or 12 lin. ind. columns

has 13 columns + 3 rigid body mode

so, 1 spurious mode



NOT GOOD.

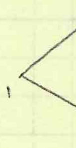
Seems to ruin a model

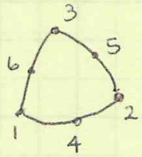
– mode cannot propagate in assemblies of two or more elements

– might happen between elements of significantly different stiffnesses

TRIANGULAR ELEMENTS

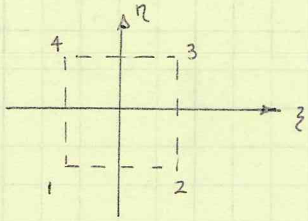
Element details

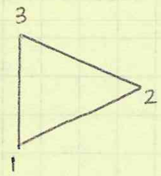




can be derived:
- as degenerate quadrilaterals
- directly by mapping onto a "parent" triangle in ξ-η space

Derive from quadrilateral





Map the entire side 3-4 (in parent domain) onto point 3 in the physical domain

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$x_4 = x_3 ,$$

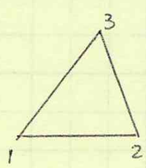
$$x = N_1 x_1 + N_2 x_2 + N_3' x_3 \qquad N_3' = N_3 + N_4$$

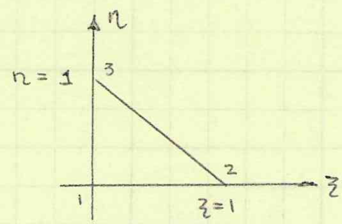
$$y = N_1 y_1 + N_2 y_2 + N_3' y_3$$

$$N_1 = \tfrac{1}{4}(1-\xi)(1-\eta)$$
$$N_2 = \tfrac{1}{4}(1+\xi)(1-\eta)$$
$$\left.\begin{aligned} N_3 &= \tfrac{1}{4}(1+\xi)(1+\eta) \\ N_4 &= \tfrac{1}{4}(1-\xi)(1+\eta) \end{aligned}\right\} \quad N_3' = \tfrac{1}{2}(1+\eta)$$

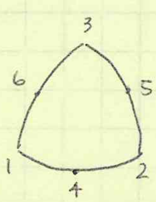
Direct derivation

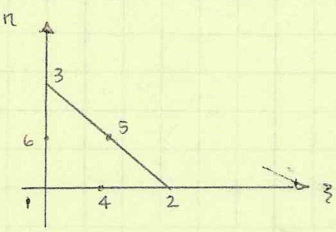
map triangle onto $\xi \cdot \eta$ space





$$N_1 = 1 - \xi - \eta$$
$$N_2 = \xi$$
$$N_3 = \eta$$



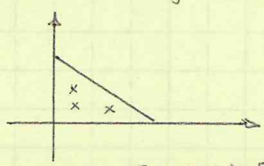


$$N_1 = 2(1-\xi-\eta)(\tfrac{1}{2}-\xi-\eta)$$
$$N_2 = 2\xi(\xi-\tfrac{1}{2})$$
$$N_3 = 2\eta(\eta-\tfrac{1}{2})$$
$$N_4 = 4\xi(1-\xi-\eta)$$
$$N_5 = 4\xi\eta$$
$$N_6 = \eta(1-\xi-\eta)\cdot 4$$

TRIANGLE ELEMENTS

Direct derivation, cont'd

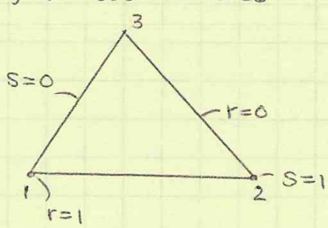
Develop and use special numerical integration schemes for triangle elements



etc.

3-point formula

Triangular coordinates



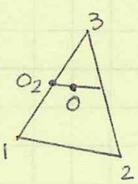
r, s, t satisfying $r+s+t=1$

r: associated with node 1, side 2-3

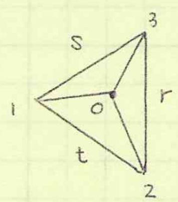
s: node 2, side 3-1

t: node 3, side 1-2

$$\int_\Omega r^\alpha s^\beta t^\gamma \, d\Omega = \frac{\alpha!\,\beta!\,\gamma!}{(\alpha+\beta+\gamma+2)!} \, 2A^e$$



$$1 - \frac{3O_2}{31} = t$$



arbitrary point O

$$\frac{\text{area }(012)}{\text{area }(123)} = t$$

$$s = \frac{A(013)}{A(123)}$$

$$r = \frac{A(023)}{A(123)}$$

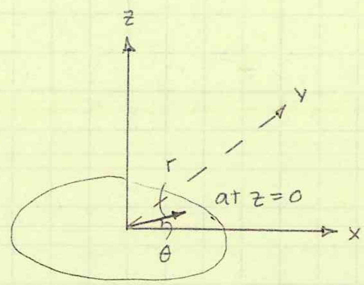
AXISYMMETRIC F.E.A.

Cylindrical coordinates

use instead of cartesian coordinates

r, θ, z . not x, y, z

- z → aligned with the axis of the structure (solid of revolution) axial coordinate
- θ → azimuthal coordinate
- r → radial coordinate

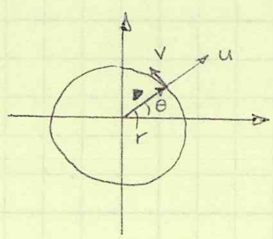


$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

So now, consider a solid of revolution
- occupying the domain Ω
- produced by sweeping over 360°
- cross section in Ω_{2D}
  - independence in geometry from the azimuthal coordinate (θ)
  - independence of loads and resulting response from θ



u: radial displacement — independent of θ (w, too)

v: azimuthal displacement
should = 0

w: axial displacement — out of the page

In cartesian coordinates,

$$u' = u\cos\theta - v\sin\theta$$
$$v' = u\sin\theta + v\cos\theta$$
$$w' = w$$

Strain components

$$\varepsilon_r = \frac{du}{dr} \quad (u = \text{radial disp.})$$
$$\varepsilon_z = \frac{dw}{dz}$$
$$\gamma_{zr} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}$$
$$\varepsilon_\theta = \frac{u}{r}$$

Stresses:

σ_r : radial stress

σ_z = axial stress

τ_{zr} : shear

σ_θ : hoop stress

Stress-strain relationships

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{zr} \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda+2G & \lambda & \lambda & 0 \\ & \lambda+2G & \lambda & 0 \\ & & \lambda+2G & 0 \\ & & & G \end{bmatrix}}_{\underset{\sim}{D}} \begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{zr} \end{bmatrix}$$

$$\int_\Omega \delta\varepsilon^T \sigma \, d\Omega = \int \left[\delta\varepsilon_r \ \delta\varepsilon_\theta \ \delta\varepsilon_z \ \delta\gamma_{zr} \right] \begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{zr} \end{bmatrix} d\Omega$$

$$= \int_{\Omega_{2D}} \int_0^{2\pi} \left[\quad \varepsilon \quad \right] \left[\sigma \right] d\theta \cdot r \cdot d\Omega_{2D}$$

↑ reflects increase in volume as you move away from z-axis

AXISYMMETRIC ANALYSIS

F.E. methods

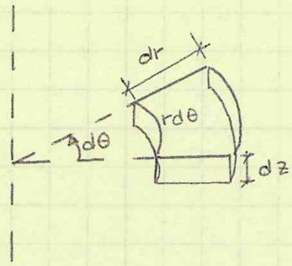
Axially symmetric structure subjected to axially symmetric loads, boundaries, etc.

$$\int_\Omega \delta\varepsilon^T D\, \varepsilon\, d\Omega - \int_\Omega \delta\varepsilon^T D\, \varepsilon_T\, d\Omega$$

gives rise to the thermal load vector

except needs r, 2π, etc:

$$\int_0^{2\pi} \left[\int_\Omega \_\_\_\; r\, d\Omega \right] d\theta = 2\pi \int_\Omega \_\_\; r\, d\Omega$$



volume element:

$$\underbrace{r\,d\theta}_{2\pi \text{ factor}} \cdot \underbrace{dr \cdot dz}_{d\Omega}$$

$$\int_\Omega \delta\varepsilon^T \sigma\, d\Omega = \int \underbrace{\begin{bmatrix} \delta\varepsilon_r \\ \delta\varepsilon_\theta \\ \delta\varepsilon_z \\ \delta\gamma_{xr} \end{bmatrix}^T}_{\delta\varepsilon} \underbrace{\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{xr} \end{bmatrix}}_{\sigma} d\Omega$$

$$\varepsilon_r = \frac{du}{dr}$$

$$\varepsilon_\theta = \frac{u}{r}$$

$$\varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xr} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}$$

$\sigma = D \cdot \varepsilon$

$$D = \begin{bmatrix} \lambda + 2G & \lambda & \lambda & 0 \\ \lambda & \lambda + 2G & \lambda & 0 \\ \lambda & \lambda & \lambda + 2G & 0 \\ 0 & 0 & 0 & G \end{bmatrix}$$

$$u = N \cdot \mathcal{U}$$

$$u = \begin{bmatrix} u \\ w \end{bmatrix}, \quad N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \\ 0 & N_1 & 0 & N_2 & \cdots \end{bmatrix}, \quad \mathcal{U} = \begin{bmatrix} u_1 \\ w_1 \\ u_2 \\ \vdots \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \text{etc.} \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \end{bmatrix}$$

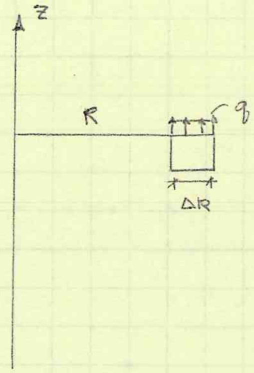
$\varepsilon = B \cdot \mathcal{U}$

AXISYMMETRIC ANALYSIS

General Equations

$$\int_\Omega \delta\varepsilon^T \sigma \, d\Omega = \int_\Omega \delta u^T B^T D \cdot B \cdot u \, d\Omega = \delta u^T \underbrace{\left[\int_\Omega B^T D B \, d\Omega \right]}_{K,\text{ stiffness matrix}} u$$

$\times 2\pi \cdot r$



$T_z = q$

Nodal forces corresponding to q:

- not equally split to two elements

$$2\pi \int N^T T r \, dB$$

$$N_1 = \frac{r - R - \Delta R}{-\Delta R} = \frac{r-R}{\Delta R} + 1$$

$$N_2 = \frac{r-R}{\Delta R}$$

Axial force at

- Node 1:

$$2\pi \int_R^{R+\Delta R} N_1 \cdot q r \, dr$$

q can come out of equation, (integral) is a constant

$$= 2\pi \cdot q \int_R^{R+\Delta R} \left(r - \frac{r(r-R)}{\Delta R} \right) dr$$

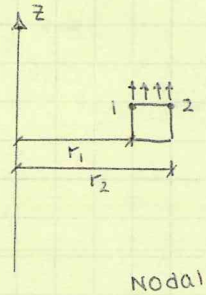
$$= 2\pi \cdot q \left[\frac{r^2}{2} - \frac{r^3}{3\Delta R} + \frac{r^3 R}{2\Delta R} \right] \Bigg|_R^{R+\Delta R}$$

Total load is distributed according to the radial coordinates of nodes 1 + 2

AXISYMMETRIC ANALYSIS

Nodal forces

— due to boundary tractions



$T_z = q$

(boundary segment parallel to x·y plane)

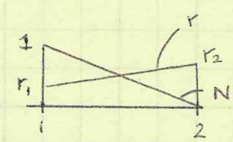
$$N_1 = \frac{r_2 - r}{r_2 - r_1}$$

$$N_2 = \frac{r - r_1}{r_2 - r_1}$$

Nodal force in z·dir at node 1:

$$2\pi \int_{r_1}^{r_2} r N_1 T_z \, dr = 2\pi \int_{r_1}^{r_2} r \cdot \frac{r_2 - r}{r_2 - r_1} q \, dr$$

simpson's rule, gauss w/ two points...



trick for integrating trap w/ triangle

$$(r_2 - r_1)\left[\tfrac{1}{3} r_1 \cdot 1 + \tfrac{1}{6} r_2 \cdot 1\right]$$

$$F_1 = 2\pi (r_2 - r_1)(\tfrac{1}{3} r_1 + \tfrac{1}{6} r_2) q$$

At Node 2, use N_2

$$F_2 = 2\pi (r_2 - r_1)(\tfrac{1}{6} r_1 + \tfrac{1}{3} r_2) \cdot q$$

Total loads

$$z_1 + z_2 = 2\pi (r_2 - r_1)(\tfrac{1}{2} r_1 + \tfrac{1}{2} r_2) \cdot q$$

simplifying...

Total load on the ring

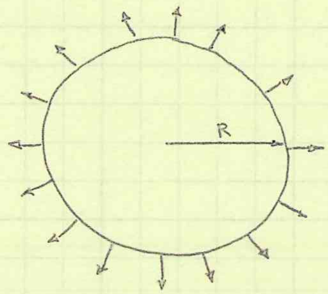
$$\rightarrow \pi \cdot q (r_2 - r_1)(r_2 + r_1)$$

Force matrices show lumping:

$$\begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \text{ "="} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

AXISYMMETRIC ANALYSIS

Nodal Forces



in the r. direction

$$N_1 = \frac{z_2 - z}{z_2 - z_1}$$

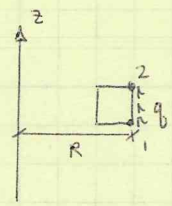
$$N_2 = \frac{z - z_1}{z_2 - z_1}$$

at Node 1

$$2\pi \int_{z_1}^{z_2} R N_1 T_r \, dz$$

$$= 2\pi \cdot R \int_{z_1}^{z_2} N_1 T_r \, dz$$

$$= 2\pi \cdot R \cdot \tfrac{1}{2} q (z_2 - z_1)$$



at Node 2

$$2\pi \cdot R \cdot \tfrac{1}{2} q (z_2 - z_1)$$

Integration over quadrilaterals

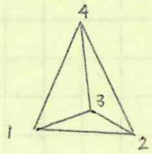
$$2\pi \int_{\Omega} r \cdot B^T D B \, d\Omega$$ – 3rd degree polynomial

2×2 gauss $(2n-1=3)$

No more constant strain triangle (only for 2D)

THREE DIMENSIONS

Basic element: 4-node tetrahedron



$$u, v = a + bx + cy + dz$$

interpolation functions (0 on entire face!)

Node 1 = 1, Face 234 = 0

$$\begin{bmatrix} x_3 - x_2 \\ y_3 - y_2 \\ z_3 - z_1 \end{bmatrix} \times \begin{bmatrix} x_4 - x_2 \\ y_4 - y_2 \\ z_4 - z_2 \end{bmatrix}$$

Result: vector pointing out of the plane 234 (2 × area)

$$\cdot \begin{bmatrix} x - x_2 \\ y - y_2 \\ z - z_2 \end{bmatrix}$$

height

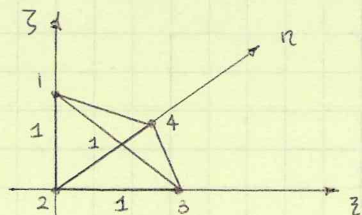
$2A \cdot h = 6V$

$V = \frac{1}{3} A_{234} \cdot h$

$$N_1 = \frac{A \times B \cdot C}{A \times B \cdot \begin{bmatrix} x_4 - x_2 \\ y_4 - y_2 \\ z_4 - z_2 \end{bmatrix}}$$

except 1s, not 4s

Map onto triangle in parent domain



$N_1 = \zeta$

$N_2 = 1 - \zeta - \xi - \eta$

$N_3 = \xi$

$N_4 = \eta$

$$x = \sum_{i=1}^{4} N_i x_i \quad , y, z$$

$$u = \sum N_i u_i \quad , v, w$$

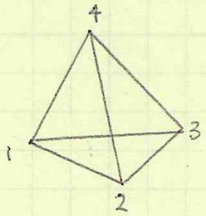
$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix}$$

, det(J) controls relationship between parent and physical domains

$$\int_\Omega B^T D B \, d\Omega = \int_{\Delta \text{ parent}} B^T D B \det(J) \, d\Delta$$

THREE DIMENSIONS

Four-node tetrahedron



Through mapping,

$$N_1 = \zeta$$
$$N_2 = 1 - \zeta - \xi - \eta$$
$$N_3 = \xi$$
$$N_4 = \eta$$

(order changes based on node numbering and configuration)

$$\varepsilon = B \cdot u$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} \quad \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & 0 & \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \\ 0 & 0 & \frac{\partial N_1}{\partial z} & \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & 0 & \cdots \\ 0 & \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial y} & \\ \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_1}{\partial x} & \end{bmatrix} \quad \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ \vdots \\ u_4 \\ v_4 \\ w_4 \end{bmatrix}$$

$$K^e = \int_\Omega B^T D B \, d\Omega = B^T \cdot D \cdot B \cdot V_\Omega$$

volume = ⅓ A(234)·h ; expression in terms of coordinates on last page

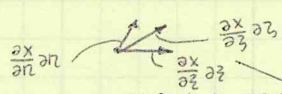
To go from $N(\zeta, \xi, \eta)$ to $N(x, y, z)$, use the Jacobian (and determinant)

$$\begin{bmatrix} \frac{\partial N_1}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial \zeta}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial \xi}{\partial z} \end{bmatrix}}_{J^{-1}} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} \end{bmatrix}$$

Jacobian has terms flipped — $\frac{\partial x}{\partial \xi}$, $\frac{\partial y}{\partial \eta}$ etc.

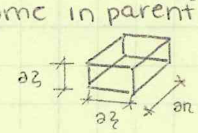
$\partial \xi$ across row, ∂x down col.

|det(J)| = ratio of volume in physical space



where $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

to volume in parent space



$\partial \xi, \partial \eta, \partial \zeta$

Areas:

$$\left(\frac{\partial x}{\partial \xi} \cdot \partial \xi \times \frac{\partial x}{\partial \eta} \partial \eta \right)$$

Volume:

$$A \cdot \frac{\partial x}{\partial \zeta} \partial \zeta$$

$$|\det(J)| = \frac{\left(\frac{\partial x}{\partial \xi} \times \frac{\partial x}{\partial \eta} \right) \cdot \frac{\partial x}{\partial \zeta}}{1}$$

$\partial \xi, \partial \eta, \partial \zeta$ cancel

$$|\det(J)| = \frac{\partial x}{\partial \xi} \times \frac{\partial x}{\partial \eta} \cdot \frac{\partial x}{\partial \zeta}$$

THREE DIMENSIONS

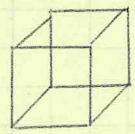
Tetrahedron

Mapping from parent to physical

$\xi \in [-1,1]$

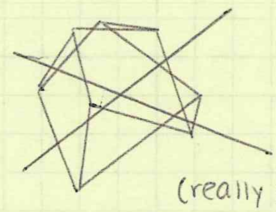
$\eta \in [-1,1]$

$\zeta \in [-1,1]$



$$\int_{\Omega} B^T D B \, d\Omega = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} B^T D B \det(J) \, d\xi \, d\eta \, d\zeta$$

$$= \sum B^T D B \det(J) \Big|_{\substack{w_{ip} \\ \xi = \xi_{ip} \\ \eta = \eta_{ip} \\ \zeta = \zeta_{ip}}}$$



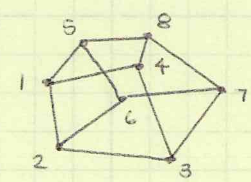
(really sucky 8-node brick elem)

trilinear Lagrange interpolation

$2 \times 2 \times 2$

array for gauss, full integration

Nodes can be added at midpoints of edges, centroids of faces (and in interior of brick) in order to enrich the interpolation.

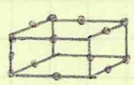


27-node brick is most enriched quadratic brick; needs

$3 \times 3 \times 3$

gauss points for full integration

"Best": 20-node brick



Midpoints on all edges

$8+8+4 = 20$

(c3d20)

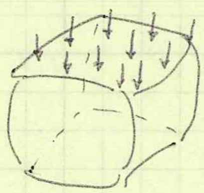
THREE · DIMENSIONAL FEA

Typical elements

- 4·node tetrahedron
- 8·node brick
- 20·node brick (15 pt integration, special scheme)
- Lagrangian & Serendipity families of elements

$$K_e = \int_\Omega B^T D B \, d\Omega = \sum_{ip=1}^{nip} B^T D B \det(J)\big|_{ip} W_{ip}$$

Uniform pressure on a face



nodal forces:

$$\int_B N^T \underset{\uparrow}{T} \, dB$$

$-pn$, n is a unit vector normal to B

to get n:

- find two vectors tangent to B

$$\frac{\partial x}{\partial \xi}, \frac{\partial x}{\partial \eta} \qquad x = (x, y, z), \quad x = \sum N_i x_i \text{ etc.}$$

- calculate n:

$$n = \left(\frac{\partial x}{\partial \xi} \times \frac{\partial x}{\partial \eta} \right)$$

so,

$$\int_B N^T (-pn) \, dB = \int_{-1}^{1} \int_{-1}^{1} N^T (-pn) \left| \frac{\partial x}{\partial \xi} \times \frac{\partial x}{\partial \eta} \right| d\xi \, d\eta$$

ISSUES IN FEA

incompressibility

- incompressible elastic behavior
- metal plasticity

recall stress·strain relationship

$$\sigma = D\varepsilon$$

D: in terms of $\lambda+2G, \lambda, G$

$$\frac{\lambda+2G}{G} = \frac{2-2\nu}{1-2\nu} \longrightarrow \lambda+2G = G\,\frac{2-2\nu}{1-2\nu}$$

$\lambda+2G$ is singular when $\nu=0.5$ (divide by zero)

difficulties as ν nears 0.5

$$\sigma_x = (\lambda+2G)\varepsilon_x + \lambda\varepsilon_y + \lambda\varepsilon_z$$
$$\sigma_y = \lambda\varepsilon_x + (\lambda+2G)\varepsilon_y + \lambda\varepsilon_z$$
$$\sigma_z = \lambda\varepsilon_x + \lambda\varepsilon_y + (\lambda+2G)\varepsilon_z$$

$$\sigma_x+\sigma_y+\sigma_z = (3\lambda+2G)\varepsilon_x + (3\lambda+2G)\varepsilon_y + (3\lambda+2G)\varepsilon_z$$

$$\times\tfrac{1}{3} = \underbrace{(\lambda+{}^{2}/_{3}G)}_{\text{bulk modulus, } K}\underbrace{(\varepsilon_x+\varepsilon_y+\varepsilon_z)}_{\text{volumetric strain}} = \text{mean normal stress, or "hydrostatic" stress}$$

volumetric strain — change in volume per unit volume of a material element in a neighboorhood of the material point

bulk modulus, K

Deviatoric stress and strain

$$\sigma: \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ & & \sigma_z \end{bmatrix} \qquad \varepsilon: \begin{bmatrix} \varepsilon_x & \tfrac{1}{2}\gamma_{xy} & \tfrac{1}{2}\gamma_{xz} \\ & \varepsilon_y & \tfrac{1}{2}\gamma_{yz} \\ & & \varepsilon_z \end{bmatrix}$$

normal σ, ε

deviatoric:

$$S = \sigma - \sigma_m \cdot I$$

σ_m: mean normal stress matrix

volume change is related to the bulk modulus

$$e = \varepsilon - \varepsilon\cdot\tfrac{1}{3}\cdot I \qquad (\text{or, } e = \varepsilon - {}^{1}/_{3}\,\varepsilon\cdot I)$$

ε: $(\varepsilon_x+\varepsilon_y+\varepsilon_z)$

S/E relationship:

$$\sigma_m = K\cdot\varepsilon$$
$$S = 2G\cdot e$$

ISSUES IN FEA

Incompressibility (or near –)

$$\sigma_m = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) \qquad \varepsilon = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

σ_m → mean normal stress

ε → volumetric strain – is zero in non-compressible problem

$$\sigma_m = K\varepsilon \qquad K = \lambda + 2/3\, G$$

K → bulk modulus

Deviatoric stress and strain

s, e

$$s = 2Ge$$

$$\left.\begin{aligned} e &= \varepsilon - 1/3\,\varepsilon I \\ s &= \sigma - \sigma_m I \end{aligned}\right\} \quad \varepsilon = \begin{bmatrix} \varepsilon_x & 1/2\,\gamma_{xy} & 1/2\,\gamma_{xz} \\ & \varepsilon_y & 1/2\,\gamma_{yz} \\ & & \varepsilon_z \end{bmatrix}$$

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ & & \sigma_z \end{bmatrix}$$

I → 3×3 identity matrix

Note: $e_x + e_y + e_z = \varepsilon_x + \varepsilon_y + \varepsilon_z - \varepsilon$

$$\frac{1}{3}(s_x + s_y + s_z) = 0$$

in vector form:

$$e = \begin{bmatrix} e_x \\ e_y \\ e_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}, \quad s = \begin{bmatrix} s_x \\ s_y \\ s_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}, \quad i = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underset{\sim}{s} = \underset{\sim}{\sigma} - \underset{\sim}{i}\cdot\sigma_m$$

$$\underset{\sim}{e} = \underset{\sim}{\varepsilon} - \underset{\sim}{i}\cdot\frac{1}{3}\varepsilon$$

$$\sigma_m = \frac{1}{3}\underset{\sim}{i}^T\underset{\sim}{\sigma}$$

$$\varepsilon = \underset{\sim}{i}^T\underset{\sim}{\varepsilon}$$

Considering virtual work

internal V.W. at a material point

$$\delta\underset{\sim}{\varepsilon}^T\underset{\sim}{\sigma} = \left[\delta\underset{\sim}{e} + \underset{\sim}{i}\cdot(1/3\,\delta\varepsilon)\right]^T\left[\underset{\sim}{s} + \underset{\sim}{i}\sigma_m\right]$$

$$= \delta\underset{\sim}{e}^T\underset{\sim}{s} + \cancelto{0}{\delta\underset{\sim}{e}^T\underset{\sim}{i}}\,\sigma_m + \cancelto{0}{\underset{\sim}{i}^T\underset{\sim}{s}}\,\frac{1}{3}\delta\varepsilon + (\underbrace{\underset{\sim}{i}^T\underset{\sim}{i}}_{3})\frac{1}{3}\delta\varepsilon\,\sigma_m$$

$$= \underbrace{\delta\underset{\sim}{e}^T\underset{\sim}{s}}_{\text{shear internal virtual work}} + \underbrace{\delta\varepsilon\,\sigma_m}_{\text{bulk internal virtual work}}$$

ISSUES IN FEA

Assuming Isotropic Elasticity:

$$\delta \underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma} = \delta\varepsilon \cdot \sigma_m + \delta \underset{\sim}{e}^T \underset{\sim}{S}$$

$$= \delta\varepsilon \cdot K \cdot \varepsilon + \delta \underset{\sim}{e}^T \underset{\sim}{D_s}\, \underset{\sim}{e}$$

shear rigidity matrix

$$D_s = \begin{bmatrix} 2G & 0 & & & & \\ 0 & 2G & & & & \\ & & 2G & & & \\ & & & G & & \\ & & & & G & 0 \\ & & & & 0 & G \end{bmatrix}$$

stiffness matrix

$$\underset{\sim}{K}^e = \int_\Omega B^T D B\, d\Omega = \underbrace{\int_\Omega [K]\, d\Omega}_{K_b} + \underbrace{\int_\Omega \underset{\sim}{D_s}\, d\Omega}_{K_s}$$

$$\int_\Omega \delta \underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma}\, d\Omega = \int_\Omega \delta\varepsilon \cdot \sigma_m\, d\Omega + \int \delta \underset{\sim}{e}^T \underset{\sim}{S}\, d\Omega$$

$$= \int_\Omega \delta U^T B_{b\lambda}^T \overset{\sigma_m}{} d\Omega + \int \delta \underset{\sim}{U}^T \underset{\sim}{B_s}^T S\, d\Omega$$

shear bulk modulus

$$= \int_\Omega \delta \underset{\sim}{U}^T \underset{\sim}{B_b}^T [K] \varepsilon\, d\Omega + \int_\Omega \delta \underset{\sim}{U}^T \underset{\sim}{B_s}^T \underset{\sim}{D_s}\, e\, d\Omega$$

$$= \int_\Omega \delta \underset{\sim}{U}^T \underset{\sim}{B_b}^T [K] \underset{\sim}{B_b} \underset{\sim}{U}\, d\Omega + \int \delta \underset{\sim}{U}^T B_s^T \underset{\sim}{D_s} \underset{\sim}{B_s} \underset{\sim}{U}\, d\Omega$$

$$\varepsilon = \underset{\sim}{B_b} \underset{\sim}{U} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\underset{\sim}{B_b} = \Big[\underbrace{\frac{\partial N_1}{\partial x} \quad \frac{\partial N_2}{\partial y} \quad \frac{\partial N_1}{\partial z}}_{\text{contributions associated with Node 1}} \Big]$$

continues with N_2, N_3

$$e = \underset{\sim}{B_s} \underset{\sim}{U}$$

recall: $e_x = \varepsilon_x - \frac{1}{3}\varepsilon = \frac{2}{3}\varepsilon_x - \frac{1}{3}\varepsilon_y - \frac{1}{3}\varepsilon_z$

$$\underset{\sim}{B_s} = \left[\underbrace{\begin{matrix} \frac{2}{3}\frac{\partial N_1}{\partial x} & -\frac{1}{3}\frac{\partial N_1}{\partial y} & -\frac{1}{3}\frac{\partial N_1}{\partial z} \\ -\frac{1}{3}\frac{\partial N_1}{\partial x} & \frac{2}{3}\frac{\partial N_1}{\partial y} & -\frac{1}{3}\frac{\partial N_1}{\partial z} \\ -\frac{1}{3}\frac{\partial N_1}{\partial x} & -\frac{1}{3}\frac{\partial N_1}{\partial y} & \frac{2}{3}\frac{\partial N_1}{\partial z} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial y} \\ \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_1}{\partial x} \end{matrix}}_{\text{contributions from Node 1}} \quad \cdots \quad N_2, N_3 \right]$$

ISSUES IN FEA

internal work-type calcs

$$\underset{\sim}{K_e} = \underset{\sim}{K_b} + \underset{\sim}{K_s}$$

$\int_\Omega \underset{\sim}{B_s^T}\, \underset{\sim}{D_s}\, \underset{\sim}{B_s}\, d\Omega$

$\int_\Omega \underset{\sim}{B_b^T}\, [K]\, \underset{\sim}{B_b}\, d\Omega$

produces numbers on orders much larger than other term, as $K \to \infty$ whereas G does not change (G controls D_s) as $\nu \to 1/2$, bulk behavior controls — no compressibility

$$K = \lambda + \frac{2}{3}G = \frac{2\nu}{1-2\nu}G + \frac{2}{3}G$$

$$\text{or, } K = \frac{6\nu + 2 - 4\nu}{3(1-2\nu)}G$$

$$K = \frac{2+2\nu}{3(1-2\nu)}G = \frac{E}{3(1-2\nu)}$$

Bulk modulus

as $\nu \Rightarrow 1/2$, $K \to \infty$

In reduced integration, can be selective — only apply to this half, because this is the half that matters / is too stiff

consider simple elements — one integration point. can't be reduced, thus is "doomed" WRT incompressibility

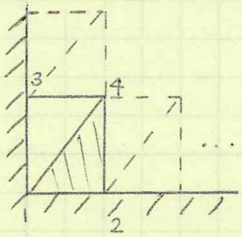
INCOMPRESSIBILITY

Plane stress/strain

$$\underset{\sim}{D}_{\substack{plane\\strain}} = \begin{bmatrix} \lambda+2G & \lambda & 0 \\ \lambda & \lambda+2G & 0 \\ 0 & 0 & G \end{bmatrix}$$

$$D_{\substack{plane\\stress}} = \begin{bmatrix} E/1-\nu^2 & \nu E/1-\nu^2 & 0 \\ \nu E/1-\nu^2 & E/1-\nu^2 & 0 \\ 0 & 0 & E/2(1+\nu) \end{bmatrix}$$

hybrid formulation — allows for perfect incompressibility



$\varepsilon_x+\varepsilon_y+\varepsilon_z \sim 0 \quad (=0, \nu=0.5)$

$\varepsilon_z = 0$ for plane strain

- Area of 124 must stay constant
- 1,2 cannot move because of boundary (3, as well)
- no y displacement for 4 because of incompressibility
- no x displacement for 4
  base of Δ is constant, height must stay, too
- same boundaries march along other elements
  no nodes can move with incompressibility
  only solution is $u=0$

Element is totally useless without hybrid formulation

Hybrid elements/calculations

$$\int_\Omega \delta\underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma}\, d\Omega = \int_\Omega (\delta\varepsilon\cdot\sigma_m + \delta\underset{\sim}{e}^T \underset{\sim}{s})\, d\Omega$$

$$= \int_\Omega (-\delta\varepsilon\cdot \underbrace{p}_{-\sigma_m} + \delta\underset{\sim}{e}^T\underset{\sim}{s})\, d\Omega$$

Functions to approximate u, v, p

Apply volumetric (bulk) behavior via the equation

$$\underbrace{\varepsilon}_{\varepsilon_x+\varepsilon_y\,(+\varepsilon_z)} - \frac{\sigma_m}{K} = 0 \quad \text{in } \Omega$$

$$\text{or } \varepsilon + \frac{p}{K} = 0$$

add to P.V.W. equation

$$+ \int \underbrace{\delta p}_{\text{arbitrary function}} \cdot \left(\varepsilon + \frac{p}{K}\right) d\Omega = 0$$

HYBRID FORMULATION

Augmented Principle of virtual work

↳ by the pressure-volumetric strain relationship

Equilibrium:

$$\int_\Omega \delta\mathbf{\varepsilon}^T \mathbf{\sigma}\, d\Omega = \int_B \delta\mathbf{u}^T \mathbf{T}\, dB + \int_\Omega \delta\mathbf{u}^T \mathbf{b}\, d\Omega$$

volumetric constraint:

$$\cancel{\varepsilon_v}\ \ \varepsilon - \frac{\sigma_m}{K} = 0 \quad , \quad p = -\sigma_m$$

$$\varepsilon + \frac{p}{K} = 0 \longleftarrow$$ must be satisfied everywhere in Ω

$$\int_\Omega \delta p \cdot \left(\varepsilon + \frac{p}{K}\right) d\Omega = 0$$

↑ arbitrary fxn

combining Equations:

$$\int_\Omega \delta\mathbf{\varepsilon}^T \mathbf{\sigma}\, d\Omega + \int_\Omega \delta p\,(\varepsilon + p/K)\, d\Omega = \int_B \delta\mathbf{u}^T \mathbf{T}\, dB + \int_\Omega \delta\mathbf{u}^T \mathbf{b}\, d\Omega$$

equilibrium <u>and</u> volumetric constraint applied at once

Approximated functions

$\mathbf{u}, p$

$$\mathbf{u} = \mathbf{N} \cdot \mathbf{U}$$

$$p = \mathbf{N}_p \mathbf{P}$$

← nodal values of pressure

pressure at any point in element

polynomial interpolation functions (typ. of degree lower than those in $\mathbf{N}$)

LBB condition – precludes arbitrary combinations of $\mathbf{N}$ and $\mathbf{N}_p$ (named for mathematicians)

Simplifying Equation

$$\int_\Omega \delta\mathbf{\varepsilon}^T \mathbf{\sigma}\, d\Omega + \int_\Omega \delta p\,(\varepsilon + p/K)\, d\Omega$$

$$= \int_\Omega \left(\delta\mathbf{e}^T \mathbf{s} + \delta\varepsilon \cdot \overset{-p}{\cancel{\sigma_m}} + \delta p\, \varepsilon + \delta p\, p/K\right) d\Omega$$

$$= \int_\Omega \left(\delta\mathbf{U}^T \mathbf{B}_s^T \mathbf{D}_s \mathbf{B}_s \mathbf{U} - \delta\mathbf{U}^T \mathbf{B}_b^T \mathbf{N}_p \mathbf{P} + \delta\mathbf{P}^T \mathbf{N}_p \mathbf{B}_b \mathbf{U} + \overset{\delta P^T}{\cancel{\delta \mathbf{U}^T}} \mathbf{N}_p^T \frac{1}{K} \mathbf{N}_p \mathbf{P}\right) d\Omega$$

combining terms by $\delta\mathbf{U}$ (arbitrary)

$$\mathbf{K}_s \mathbf{U} - \mathbf{G}\mathbf{P} = \int_B \mathbf{N}^T \mathbf{T}\, dB + \int_\Omega \mathbf{N}^T \mathbf{b}\, d\Omega$$

$$\mathbf{K}_s = \int_\Omega \mathbf{B}_s^T \mathbf{D}_s \mathbf{B}_s\, d\Omega$$

$$\mathbf{G} = \int_\Omega \mathbf{B}_b^T \mathbf{N}_p\, d\Omega$$

HYBRID FORMULATION

Equation massaging, cont'd

combing terms involving $\delta \underset{\sim}{P}$ (also arbitrary):

$$\underset{\sim}{G}^T \underset{\sim}{U} + \underset{\sim}{F}\,\underset{\sim}{P} = 0$$

multiply by -1 to get $-\underset{\sim}{G}^T$

$$G = \int_\Omega \underset{\sim}{B}_b^T \underset{\sim}{N}_P \, d\Omega$$

$$F = \int_\Omega \underset{\sim}{N}_P^T \frac{1}{K} \underset{\sim}{N}_P \, d\Omega$$

$$\begin{bmatrix} \underset{\sim}{K}_S & -\underset{\sim}{G} \\ -\underset{\sim}{G}^T & -F \end{bmatrix} \begin{bmatrix} \underset{\sim}{U} \\ \underset{\sim}{P} \end{bmatrix} = \begin{bmatrix} \int_B \underset{\sim}{N}^T \underset{\sim}{T} \, dB + \int_\Omega \underset{\sim}{N}^T \underset{\sim}{b} \, d\Omega \\ 0 \end{bmatrix}$$

Augmented matrix

$$\begin{bmatrix} \underset{\sim}{K}_S & \underset{\sim}{G} \\ \underset{\sim}{G}^T & \underset{\sim}{F} \end{bmatrix}$$

with $\underset{\sim}{G} = -\int_\Omega \underset{\sim}{B}_b^T \underset{\sim}{N}_P \, d\Omega$

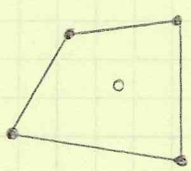
instead of multiplying by -1 (whole equation)

$$B_b = \left[\frac{\partial N_1}{\partial x} \;\; \frac{\partial N_1}{\partial y} \; \cdots \right]$$

$$\underset{\sim}{F} = -\int_\Omega \underset{\sim}{N}_P^T \frac{1}{K} \underset{\sim}{N}_P \, d\Omega$$

$\underset{\sim}{F}$ is symmetric; entire augmented matrix is symmetric, as G and G^T are off diagonals

Hybrid 4-node quadrilateral



• displacement node

○ pressure node

$[N_P] = [1] = \underset{\sim}{N}_P$ = interpolation functions for pressure nodes

$\underset{\sim}{K}_S$: non-negative definite matrix

The ~~aa~~ augmented matrix is <u>not</u> non-negative definite (caused by $\underset{\sim}{F}$)

8-node element

• displacement node

× displacement & pressure node

pressure can be continuous or discontinuous over the mesh

Selectively reduced integration is an option

INCOMPRESSIBILITY

Hybrid Formulation

$\underset{\sim}{u} = \underset{\sim}{N} \cdot \underset{\sim}{U}$ ← nodal values of displacements

$-\sigma_m = p = \underset{\sim}{N}_P \underset{\wedge}{P}$ ← nodal values of pressure

(σ_m: mean normal stress)

Augmented Matrix: $\begin{bmatrix} \underset{\sim}{K}_S & \underset{\wedge}{G} \\ \underset{\sim}{G}^T & \underset{\wedge}{F} \end{bmatrix}$

$$\underset{\sim}{K}_S = \int_\Omega \underset{\wedge}{B}_S^T \underset{\sim}{D}_S \underset{\wedge}{B}_S \, d\Omega$$

$$\underset{\sim}{G} = -\int_\Omega \underset{\sim}{B}_P^T \underset{\sim}{N}_P \, d\Omega$$

$$\underset{\sim}{F} = -\int_\Omega \underset{\sim}{N}_P^T \frac{1}{K} \underset{\sim}{N}_P \, d\Omega$$

K: bulk modulus, $\underset{\sim}{F}$ is usually a small number — zero when incompressible

Selectively Reduced Integration
(alternative to hybrid formulation)

$$\underset{\sim}{K} = \underset{\sim}{K}_S + \underset{\sim}{K}_b = \underbrace{\int_\Omega \underset{\sim}{B}_S^T \underset{\wedge}{D}_S \underset{\wedge}{B}_S \, d\Omega}_{\text{integrate using full integration}} + \underbrace{\int_\Omega \underset{\sim}{B}_b^T \underset{\underset{\sim}{D}_b}{\underbrace{[k]}} \underset{\sim}{B}_b \, d\Omega}_{\text{integrate using reduced integration}}$$

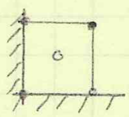
Counting constraints
(Equilibrium and volumetric)

- Heuristic Approach toward evaluating the usefulness of a mesh
- in the limit, the solution should satisfy the governing equations at each material point

  in 2D: 2 equilibrium eq, 1 volumetric constraint > ratio = 2

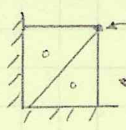
- constraint ratio should be the same in the mesh as it is refined

Ex. 4-node, 1 pressure pt



ratio again = 2 (same as theoretical) (does not satisfy LBB, but can converge)

good mesh, good results



2 equilibrium eq.

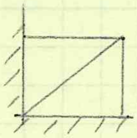
2 constraints (one from each elem)

ratio = 1 (< 2): not good

INCOMPRESSIBILITY

Constraints, etc.

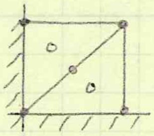
Ex.



$r = 2/2 = 1$

Expect "locking" — the mesh locks into a mode of deformation that cannot represent the sol'n; an overconstrained mesh (from the POV of volumetric behavior)

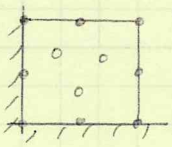
Improvement:



4 equilibrium eqs.
2 volumetric constraints
$r = 2$ ✓

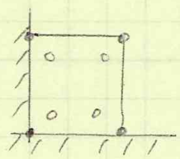
satisfies the LBB condition, convergence ~~is~~ is guaranteed

Good:



$\frac{6}{3} = 2$ ✓

Bad:



$\frac{2}{4} = \frac{1}{2}$ X very bad

Extreme:

$r > 2$ is underconstrained from the POV of volumetric behavior — not applying incompressibility as much as we should (need to up it)

~~[illegible]~~ $r = 2$ is ideal
r close to 2 might work; not definite
Tassoulas says don't bother

CONVERGENCE

Requirements

- smooth approximation in each element
- continuity of (displacement) approximation across elements
- "completeness" of approximation

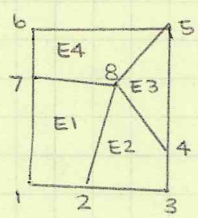
need to be able to find derivatives within each element (just 1st derivative)

→ must be capable of representing the simplest, most basic modes of disp. and deformation {combinations of the polynomials 1, x, y, z}

→ allows for integration by parts "conforming" elements — satisfy continuity across elem. boundaries

→ test for completeness: Patch Test (due to Irons)

- use an arbitrary patch of elements of the type in consideration
- Ex:



- at all nodes on the boundary of the patch, specify u, v, w according to each of the basic monomials

case 1: nodes $i = 1 \dots 7$

$u_i = 1, \; v_i = 0$

solution must provide $u_8 = 1, \; v_8 = 0$

case 2: n $i = 1 \dots 7$

$v_i = 1$, again $v_8 = 1$

$u_i = 0$, $u_8 = 0$

case 3: n1–n7, $u_i = x_i, \; v_i = 0$

$u_8 = x_8, \; v_8 = 0$

Case 4: n1–n7, $u_i = 0, \; v_i = x_i$

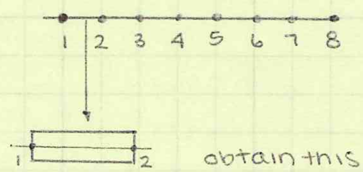
$u_8 = 0, \; v_8 = x_8$

case 5, 6 are the same, but with y_i, y_8

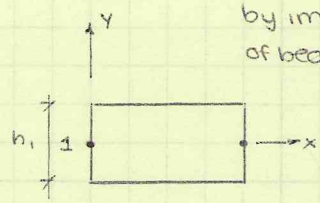
BEAMS, PLATES, SHELLS

Use individual theories
- beam, plate, shell theories exist already
- derive P. of V.W. from those
- introduce approximations of displacements (and rotations)
- apply the Galerkin method, proceed as before

Alternate (better) method
- start with FE formulation based on the continuum theory
- introduce the assumptions and simplifications of beam, plate, and shell theories
  - i.e. plane sections remain plane
  - Example:



obtain this element from a rectangular continuum elem. by imposing the conditions of beam theory
- plane sections remain plane
  - impose at each end
  - use u_1, v_1, θ_1 (infinitesimal rotation of the section at 1 WRT z)



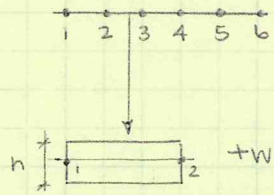
- interpolate u and v at the section (at 1) in terms of u_1, v_1, θ_1

$$u = u_1 - \eta\,\theta_1 \frac{h_1}{2}$$

$$v = v_1$$

BEAMS, PLATES, SHELLS

consider a beam



two node finite element beam

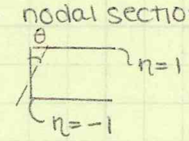
x-axis at center line

variables:

$$u_1, v_1, \theta_1, \quad u_2, v_2, \theta_2$$

1. write expressions for u, v at each nodal section

node 1: $u = u_1 - \eta\theta_1 h/2$
$v = v_1$



variable ranging from −1 to 1

node 2: $u = u_2 - \eta\theta_2 h/2$
$v = v_2$ ← beam moves vertically as a rigid body

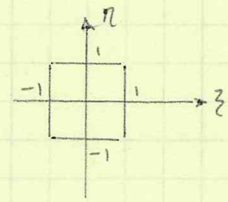
between nodes, interpolate:

$$u = (u_1 - \eta\theta_1 h/2) N_1(\xi) + (u_2 - \eta\theta_2 h/2) N_2(\xi)$$

$$v = v_1 N_1(\xi) + v_2 N_2(\xi)$$

ξ ranges from −1 to 1 (mapped element)

$$\boxed{N_1 = \tfrac{1}{2}(1-\xi), \quad N_2 = \tfrac{1}{2}(1+\xi)}$$



Higher order beam element approximations can be written as:

$$u = \sum_{i=1}^{nne} (u_i - \eta\theta_i h/2) N_i(\xi)$$

$$v = \sum_{i=1}^{nne} v_i N_i(\xi)$$

nne = number of nodes in one element

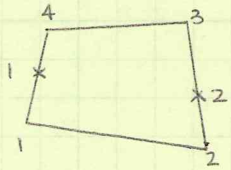
N_i: ith Lagrange polynomial of degree nne−1

more complex: variable h

could be derived from a (4-node) quadrilateral by means of kinematic condensation of DOF at nodal sections

BEAMS, PLATES, and SHELLS

Beam derivation from quadrilateral



x1 is at the middle of 14

x2 is at the middle of 23

variables

$u_{x1}, v_{x1}, \theta_{x1}$

$$u_1 = u_{x1} + (\bar{V}_{x1\,y})\theta_{x1}$$

$$v_1 = v_{x1} - (\bar{V}_{x1\,x})\theta_{x1}$$

u_4, v_4 switch signs in middle

$\bar{V}_{x1}$ x,y are vector components of line 14 from x1 to 4

$$\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & V_{x1y} & 0 & 0 & 0 \\ 0 & 1 & -V_{x1x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & V_{x2y} \\ 0 & 0 & 0 & 0 & 1 & -V_{x2x} \\ 0 & 0 & 0 & 1 & 0 & -V_{x2y} \\ 0 & 0 & 0 & 0 & 1 & V_{x2x} \\ 1 & 0 & -V_{x1y} & 0 & 0 & 0 \\ 0 & 1 & V_{x1x} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1} \\ v_{x1} \\ \theta_{x1} \\ u_{x2} \\ v_{x2} \\ \theta_{x2} \end{bmatrix}$$

$\underset{\sim}{R}$ – transformation matrix

The loads can be similarly transformed

$$\begin{bmatrix} X_{x1} \\ Y_{x1} \\ M_{x1} \\ X_{x2} \\ Y_{x2} \\ M_{x2} \end{bmatrix} = \begin{bmatrix} \\ \\ & R^T & \\ \\ \\ \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ X_3 \\ Y_3 \\ X_4 \\ Y_4 \end{bmatrix}$$

nodal loads on the beam elem | nodal forces on original quadrilateral

Beam stiffness matrix:

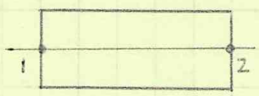
$\underset{\sim}{P} = \underset{\sim}{K} \cdot \underset{\sim}{U}$ of original element

$\underset{\sim}{R}^T \underset{\sim}{P} = \underset{\sim}{R}^T \cdot \underset{\sim}{K} \cdot \underset{\sim}{U}$

$\uparrow \underset{\sim}{P}_x = \underset{\sim}{R}^T \underset{\sim}{K} \underset{\sim}{R} \underset{\sim}{U}_x$, so, $\underline{\underline{\underset{\sim}{K}_x = \underset{\sim}{R}^T \underset{\sim}{K} \underset{\sim}{R}}}$ with $\underset{\sim}{K}$ being original element's stiffness matrix

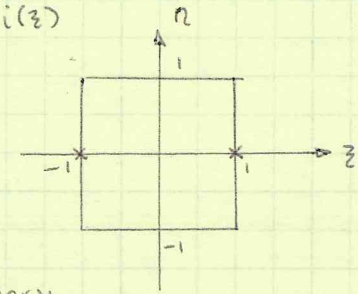
BEAMS, PLATES, SHELLS

Two-node beam finite element



$$u = \sum_{i=1} \left(u_i - \theta_i \eta \, h/2\right) N_i(\xi)$$

$$v = \sum_{i=1} (v_i) N_i(\xi)$$



Next step:

obtain the $\underset{\sim}{B}, \underset{\sim}{D}$ matrices.

Apply assumption of engineering beam theory

$\underline{\sigma_y = 0}$

in general, the stress component normal to any lamina of the beam is set to zero

↳ any surface of constant η

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} E/1-\nu^2 & \nu E/1-\nu^2 & 0 \\ & E/1-\nu^2 & 0 \\ & & G \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

now, insert $\sigma_y = 0$

$$\frac{\nu E}{1-\nu^2}\varepsilon_x + \frac{E}{1-\nu^2}\varepsilon_y = 0$$

$$\varepsilon_y = -\nu\varepsilon_x$$

$$\sigma_x = E\varepsilon_x$$

$$\begin{bmatrix} \sigma_x \\ \tau_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}}_{\underset{\sim}{D}} \begin{bmatrix} \varepsilon_x \\ \gamma_{xy} \end{bmatrix}$$

$\underset{\sim}{\sigma} = \underset{\sim}{D}\underset{\sim}{\varepsilon}, \ \underset{\sim}{\varepsilon} = \underset{\sim}{B}\underset{\sim}{u} \longleftarrow \begin{bmatrix} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{bmatrix}^T$

↳ matrix of derivatives of interpolation functions

$$B = \begin{bmatrix} \frac{dN_1}{dx} & 0 & -\eta\frac{h}{2}\frac{dN_1}{dx} & \frac{dN_2}{dx} & 0 & -\eta\frac{h}{2}\frac{dN_2}{dx} \\ 0 & \frac{dN_1}{dx} & -\frac{d\eta}{dy}\frac{h}{2}N_1 & 0 & \frac{dN_2}{dx} & -\frac{d\eta}{dy}\frac{h}{2}N_2 \end{bmatrix}$$

$$\underline{\underline{\frac{\partial x}{\partial \xi} = \frac{L}{2}, \quad \frac{\partial y}{\partial \eta} = \frac{h}{2}}}$$

$$\frac{\partial N_1}{\partial x} = \frac{\partial N_1}{\partial \xi}\cdot\underbrace{\frac{\partial \xi}{\partial x}}_{2/L}$$

$2/L$ - only true because the Jacobian is a diagonal matrix

BEAMS, PLATES, SHELLS

Two-node beam element

$$\underset{\sim}{K} = d \int_{\Omega} \underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} \, d\Omega$$

$$\text{or, } \underset{\sim}{K} = d \int_{\Omega} \underset{\sim}{B}_{\varepsilon}^T [E] \underset{\sim}{B}_{\varepsilon} \, d\Omega + d \int_{\Omega} \underset{\sim}{B}_s^T [G] \underset{\sim}{B}_s \, d\Omega$$

first row of $\underset{\sim}{B}$ | second row of $\underset{\sim}{B}$

$$K_{\varepsilon} = d \int_{\Omega} \underset{\sim}{B}_{\varepsilon}^T \underset{\sim}{B}_{\varepsilon} \, d\Omega$$

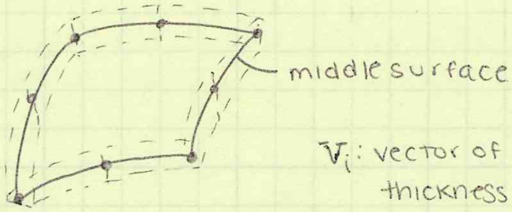
$$K_{\gamma} = d \int_{\Omega} \underset{\sim}{B}_s^T \underset{\sim}{B}_s \, d\Omega$$

can be evaluated exactly using 2×2 gauss, or reduced with 1×1 gauss

combined,

$$\underset{\sim}{K} = E \underset{\sim}{K}_{\varepsilon} + G \underset{\sim}{K}_{\gamma}$$

Now, beam $\longrightarrow$ shell



$\vec{V}_i$: vector of length equal to half the thickness of the shell at node i, directed through the thickness

At each node, DOFS: $u_i, v_i, w_i, \underline{\theta_i, \phi_i}$

infinitesimal rotations of $\vec{V}_i$ about axes normal to it

Torsional – Axisymmetric Analysis

In this type of analysis, the only nonzero displacement component is the azimuthal one, in general, a function of the radial and axial coordinates (not of the azimuthal coordinate):

$$u = 0$$

$$v : \text{a function of } r, z$$

$$w = 0$$

The only* non-zero strain components resulting from this displacement field are:

$$\gamma_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r}$$

$$\gamma_{\theta z} = \frac{\partial v}{\partial z}$$

Assuming isotropic elastic behavior, the only* non-zero stress components are:

$$\tau_{r\theta} = G\gamma_{r\theta}$$

$$\tau_{\theta z} = G\gamma_{\theta z}$$

Thus, for this type of analysis we set:

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \gamma_{r\theta} \\ \gamma_{\theta z} \end{bmatrix}, \quad \underset{\sim}{\sigma} = \begin{bmatrix} \tau_{r\theta} \\ \tau_{\theta z} \end{bmatrix}, \quad \underset{\sim}{D} = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}$$

$$\underset{\sim}{B} = \begin{bmatrix} \frac{\partial N_1}{\partial r} - \frac{N_1}{r} & \cdots \\ \frac{\partial N_1}{\partial z} & \end{bmatrix}, \quad \underset{\sim}{N} = \begin{bmatrix} N_1 & \cdots \end{bmatrix}$$

and calculate the element stiffness matrix:

$$\underset{\sim}{K} = 2\pi \int_{\Omega_{2D}} r\, \underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}\, d\Omega_{2D}$$

load vector due to traction (in the azimuthal direction):

$$2\pi \int_{B_{2D}} r\, \underset{\sim}{N}^T [T_\theta]\, dB_{2D}$$

etc.

* nonzero

Stress and Strain Transformations

Consider two alternative, rectangular, Cartesian systems of reference. For the purposes of this discussion, we will refer to one of these systems as the "standard," or "global" system, with base vectors:

$$\underset{\sim}{e1}\,,\ \underset{\sim}{e2}\,,\ \underset{\sim}{e3}$$

in the x, y and z directions, respectively. The other system will be referred to as the "special" or "local" system, with base vectors:

$$\underset{\sim}{a}\,,\ \underset{\sim}{b}\,,\ \underset{\sim}{n}$$

(in the x', y' and z' directions, respectively). These unit, mutually orthogonal, vectors can be written in terms of their components with respect to the global base vectors ($\underset{\sim}{e1}, \underset{\sim}{e2}, \underset{\sim}{e3}$) as:

$$\underset{\sim}{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, \quad \underset{\sim}{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}, \quad \underset{\sim}{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

(Note that these components are the dot products of the corresponding base vectors. For example, $b_z = \underset{\sim}{b} \cdot \underset{\sim}{e3}$, etc.)

Stress Components:

- ★ With respect to the global system:
  $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
- ★ With respect to the local system:
  $\sigma_a, \sigma_b, \sigma_n, \tau_{ab}, \tau_{bn}, \tau_{na}$

Strain Components:

- ★ With respect to the global system:
  $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
- ★ With respect to the local system:
  $\varepsilon_a, \varepsilon_b, \varepsilon_n, \gamma_{ab}, \gamma_{bn}, \gamma_{na}$

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Transformation of Stress Components:

The traction (vector) on a plane oriented in the direction $\underset{\sim}{a}$ is given by:

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} \sigma_x a_x + \tau_{xy} a_y + \tau_{zx} a_z \\ \tau_{xy} a_x + \sigma_y a_y + \tau_{yz} a_z \\ \tau_{zx} a_x + \tau_{yz} a_y + \sigma_z a_z \end{bmatrix}$$

Taking the component of this traction in the direction $\underset{\sim}{a}$, we calculate σ_a:

$$\begin{aligned} \sigma_a &= (\sigma_x a_x + \tau_{xy} a_y + \tau_{zx} a_z)\, a_x \\ &\quad + (\tau_{xy} a_x + \sigma_y a_y + \tau_{yz} a_z)\, a_y \\ &\quad + (\tau_{zx} a_x + \tau_{yz} a_y + \sigma_z a_z)\, a_z \\ &= \sigma_x \cdot (a_x)^2 + \sigma_y \cdot (a_y)^2 + \sigma_z \cdot (a_z)^2 \\ &\quad + \tau_{xy} \cdot (2 a_x a_y) + \tau_{yz} \cdot (2 a_y a_z) + \tau_{zx} \cdot (2 a_z a_x) \end{aligned}$$

Similarly, taking the component of the traction in the direction $\underset{\sim}{b}$, we calculate σ_{ab}:

$$\begin{aligned} \sigma_{ab} &= (\sigma_x a_x + \tau_{xy} a_y + \tau_{xz} a_z)\, b_x \\ &\quad + (\tau_{xy} a_x + \sigma_y a_y + \tau_{yz} a_z)\, b_y \\ &\quad + (\tau_{zx} a_x + \tau_{yz} a_y + \sigma_z a_z)\, b_z \\ &= \sigma_x \cdot (a_x b_x) + \sigma_y \cdot (a_y b_y) + \sigma_z \cdot (a_z b_z) \\ &\quad + \tau_{xy} \cdot (a_x b_y + a_y b_x) + \tau_{yz} \cdot (a_y b_z + a_z b_y) + \tau_{zx} \cdot (a_z b_x + a_x b_z) \end{aligned}$$

(Note that we are taking the symmetry of the stress tensor into consideration. The rest of the stress components with respect to the local system can be calculated in a similar manner.) The inverse relationships can be obtained similarly using the components of the global base vectors with respect to the local base vectors:

$$\underset{\sim}{e1} = \begin{bmatrix} a_x \\ b_x \\ n_x \end{bmatrix}, \quad \underset{\sim}{e2} = \begin{bmatrix} a_y \\ b_y \\ n_y \end{bmatrix}, \quad \underset{\sim}{e3} = \begin{bmatrix} a_z \\ b_z \\ n_z \end{bmatrix}$$

In matrix forms, the direct and inverse relationships are given on the next two pages.

06 December 06

$$\begin{bmatrix} \sigma_a \\ \sigma_b \\ \sigma_n \\ \tau_{ab} \\ \tau_{bn} \\ \tau_{na} \end{bmatrix} = \underbrace{\begin{bmatrix} (a_x)^2 & (a_y)^2 & (a_z)^2 & 2a_x a_y & 2a_y a_z & 2a_z a_x \\ (b_x)^2 & (b_y)^2 & (b_z)^2 & 2b_x b_y & 2b_y b_z & 2b_z b_x \\ (n_x)^2 & (n_y)^2 & (n_z)^2 & 2n_x n_y & 2n_y n_z & 2n_z n_x \\ a_x b_x & a_y b_y & a_z b_z & a_x b_y + a_y b_x & a_y b_z + a_z b_y & a_z b_x + a_x b_z \\ b_x n_x & b_y n_y & b_z n_z & b_x n_y + b_y n_x & b_y n_z + b_z n_y & b_z n_x + b_x n_z \\ n_x a_x & n_y a_y & n_z a_z & n_x a_y + n_y a_x & n_y a_z + n_z a_y & n_z a_x + n_x a_z \end{bmatrix}}_{\underset{\sim}{R}} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}$$

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$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} \underbrace{\begin{bmatrix} (a_x)^2 & (b_x)^2 & (n_x)^2 & 2a_x b_x & 2b_x n_x & 2n_x a_x \\ (a_y)^2 & (b_y)^2 & (n_y)^2 & 2a_y b_y & 2b_y n_y & 2n_y a_y \\ (a_z)^2 & (b_z)^2 & (n_z)^2 & 2a_z b_z & 2b_z n_z & 2n_z a_z \\ a_x a_y & b_x b_y & n_x n_y & a_x b_y + a_y b_x & b_x n_y + b_y n_x & n_x a_y + n_y a_x \\ a_y a_z & b_y b_z & n_y n_z & a_y b_z + a_z b_y & b_y n_z + b_z n_y & n_y a_z + n_z a_y \\ a_z a_x & b_z b_x & n_z n_x & a_z b_x + a_x b_z & b_z n_x + b_x n_z & n_z a_x + n_x a_z \end{bmatrix}}_{\underset{\sim}{R}^{-1}} \begin{bmatrix} \sigma_a \\ \sigma_b \\ \sigma_n \\ \tau_{ab} \\ \tau_{bn} \\ \tau_{na} \end{bmatrix}$$

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Transformation of Strain Components

We proceed along lines similar to the transformation of stress components. However, we must keep in mind that the strain-tensor components are

$$\varepsilon_x, \varepsilon_y, \varepsilon_z, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}$$

instead of

$$\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$$

Thus, the relationships derived for the stress components can be used for the strain components as well, provided we consider $\varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}$ instead of $\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ and $\varepsilon_{ab}, \varepsilon_{bn}, \varepsilon_{na}$ instead of $\gamma_{ab}, \gamma_{bn}, \gamma_{na}$.

Recalling that $\gamma_{xy} = 2\varepsilon_{xy}$, etc., we can write the direct and inverse relationships of strain components given on the next two pages. Note the correspondence between the transformation matrices for stress and strain components!

$$
\begin{bmatrix} \varepsilon_a \\ \varepsilon_b \\ \varepsilon_n \\ \gamma_{ab} \\ \gamma_{bn} \\ \gamma_{na} \end{bmatrix}
=
\underbrace{\begin{bmatrix}
(a_x)^2 & (a_y)^2 & (a_z)^2 & a_x a_y & a_y a_z & a_z a_x \\
(b_x)^2 & (b_y)^2 & (b_z)^2 & b_x b_y & b_y b_z & b_z b_x \\
(n_x)^2 & (n_y)^2 & (n_z)^2 & n_x n_y & n_y n_z & n_z n_x \\
2a_x b_x & 2a_y b_y & 2a_z b_z & a_x b_y + a_y b_x & a_y b_z + a_z b_y & a_z b_x + a_x b_z \\
2b_x n_x & 2b_y n_y & 2b_z n_z & b_x n_y + b_y n_x & b_y n_z + b_z n_y & b_z n_x + b_x n_z \\
2n_x a_x & 2n_y a_y & 2n_z a_z & n_x a_y + n_y a_x & n_y a_z + n_z a_y & n_z a_x + n_x a_z
\end{bmatrix}}_{(R^{-1})^T}
\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}
$$

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$$
\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}
=
\underbrace{\begin{bmatrix}
(a_x)^2 & (b_x)^2 & (n_x)^2 & a_x b_x & b_x n_x & n_x a_x \\
(a_y)^2 & (b_y)^2 & (n_y)^2 & a_y b_y & b_y n_y & n_y a_y \\
(a_z)^2 & (b_z)^2 & (n_z)^2 & a_z b_z & b_z n_z & n_z a_z \\
2a_x a_y & 2b_x b_y & 2n_x n_y & a_x b_y + a_y b_x & b_x n_y + b_y n_x & n_x a_y + n_y a_x \\
2a_y a_z & 2b_y b_z & 2n_y n_z & a_y b_z + a_z b_y & b_y n_z + b_z n_y & n_y a_z + n_z a_y \\
2a_z a_x & 2b_z b_x & 2n_z n_x & a_z b_x + a_x b_z & b_z n_x + b_x n_z & n_z a_x + n_x a_z
\end{bmatrix}}_{\underset{\sim}{R}^T}
\begin{bmatrix} \varepsilon_a \\ \varepsilon_b \\ \varepsilon_n \\ \gamma_{ab} \\ \gamma_{bn} \\ \gamma_{na} \end{bmatrix}
$$

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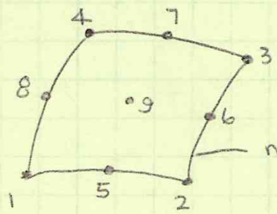
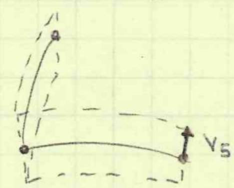
SHELL ELEMENTS



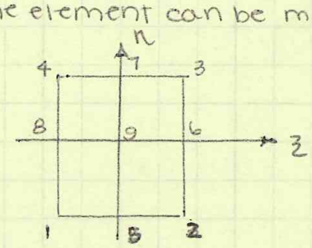
middle surface, or lamina

has some defined thickness

vectors $V_i \dots V_n$ define line from node to surface of shell



The element can be mapped onto $\xi \cdot \eta \cdot \zeta$ space



$\zeta = 0$: middle lamina
$\zeta = -1$: "inner" lamina
$\zeta = 1$: "outer" lamina

u_i, v_i, w_i – displacements of node i
θ_i, ϕ_i – (infinitesimal) rotations of V_i about two orthogonal directions

$\underset{\sim}{A1}, \underset{\sim}{A2}$

↳ orthogonal to each other and to V_i

$A1 \cdot A2 = 0, \; A1 \cdot V = 0, \; A2 \cdot V = 0$

length of A1, A2 are equal to the length of V_i (half the thickness)

decide: θ_i rotation about $A1_i$
ϕ_i rotation about $A2_i$

Displacement vector at node i, through thickness (as a function of ζ)

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix}_i = \underset{\sim}{u}_i = \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} - \zeta \theta_i \underset{\sim}{A2}_i + \zeta \phi_i \underset{\sim}{A1}_i$$

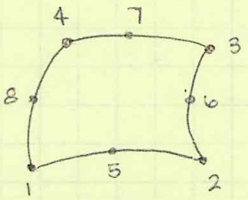
Now interpolate:

$$\underset{\sim}{u}(\xi, \eta, \zeta) = \sum \left(\begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} - \zeta \theta_i \underset{\sim}{A2}_i + \zeta \phi_i \underset{\sim}{A1}_i \right) N_i(\xi, \eta)$$

↳ Lagrange interpolation function for node i on 9-node element

SHELL ELEMENTS

Theory and calculations



$V_i, A1_i, A2_i,$ ~~[illegible]~~ – orthonomal triplet at node i

$$u = \sum \left[\begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} - \theta_i \zeta \underset{\sim}{A2_i} + \phi_i \zeta A1_i \right] N_i(\xi, \eta)$$

$u_i, v_i, w_i, \theta_i, \phi_i$ – degrees of freedom at each node

Element is mapped to $\xi \cdot \eta \cdot \zeta$

$$\underset{\sim}{x}(\xi,\eta,\zeta) = (\hat{x}_i + \zeta \hat{V}_i) N_i(\xi,\eta)$$

Impose engineering shell theory

$\sigma_n = 0$ – the normal stress for any lamina (ζ is a constant)

- consider any point in the element
- obtain a set of three unit vectors, mutually orthogonal $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{n}$ where $\underset{\sim}{n}$ is normal to the lamina through the point in consideration
- calculate:

$\dfrac{\partial \underset{\sim}{x}}{\partial \xi}$ – tangent to the lamina

$\dfrac{\partial \underset{\sim}{x}}{\partial \eta}$ – also tangent

$$\frac{\dfrac{\partial \underset{\sim}{x}}{\partial \xi} \times \dfrac{\partial \underset{\sim}{x}}{\partial \eta}}{\left| \dfrac{\partial \underset{\sim}{x}}{\partial \xi} \times \dfrac{\partial \underset{\sim}{x}}{\partial \eta} \right|} = \underset{\sim}{n}$$ – unit vector normal to the lamina

$$\underset{\sim}{a} = \frac{\partial \underset{\sim}{x} / \partial \xi}{|\partial \underset{\sim}{x} / \partial \xi|}$$ – unit vector tangent

$\underset{\sim}{b} = \underset{\sim}{n} \times \underset{\sim}{a}$ – unit vector orthogonal to $\underset{\sim}{n}$, (and therefore tangent to the lamina) and orthogonal to $\underset{\sim}{a}$

SHELL ELEMENTS

Theory and calculations, cont'd

Using engineering shell theory

considering $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{n}$,

$$\underbrace{\begin{bmatrix} \sigma_a \\ \sigma_b \\ \sigma_{ab} \\ \sigma_{bn} \\ \sigma_{na} \end{bmatrix}}_{\hat{\underset{\sim}{\sigma}}} = \underbrace{\begin{bmatrix} E/1-\nu^2 & \nu E/1-\nu^2 & 0 & 0 & 0 \\ & E/1-\nu^2 & 0 & 0 & 0 \\ & & G & 0 & 0 \\ & & & G & 0 \\ & & & & G \end{bmatrix}}_{\hat{\underset{\sim}{D}}} \underbrace{\begin{bmatrix} \varepsilon_a \\ \varepsilon_b \\ \gamma_{ab} \\ \gamma_{bn} \\ \gamma_{na} \end{bmatrix}}_{\hat{\underset{\sim}{\varepsilon}}}$$

At the global level,

$$\underset{\sim}{\sigma} = \underset{\sim}{R}^{-1} \hat{\underset{\sim}{\sigma}}$$

$\uparrow$ 6×5 matrix (given in handout, except omit column 3)

$$\hat{\underset{\sim}{\varepsilon}} = \underset{\sim}{R}^{-T} \underset{\sim}{\varepsilon}$$

$\uparrow$ 5×6 matrix (omit row 3)

−1, −T signs not inverse or transpose, they're just the names of the matrix

Other equations,

$$\hat{\underset{\sim}{\sigma}} = \hat{\underset{\sim}{D}} \hat{\underset{\sim}{\varepsilon}}, \quad \underset{\sim}{D} = \underset{\sim}{R}^{-1} \hat{\underset{\sim}{D}} \underset{\sim}{R}^{-T}$$

Element stiffness matrix:

$$K = \iiint B^T D B \det(J) \, d\xi \, d\eta \, d\zeta$$

B matrix is a 6×45 matrix

Out-of-plane shear (distortion)

considered separately from extensional deformation

$$\hat{\underset{\sim}{D}} = \hat{\underset{\sim}{D}}_1 + \hat{\underset{\sim}{D}}_2$$

$\hat{D}_1$: all parts of the $\hat{D}$ matrix excluding the single Gs in the corner — fill with zeros

$\hat{D}_2$: $\hat{D}$ matrix of just Gs in corner, all else zero

Forms: $\underset{\sim}{K} = \underset{\sim}{K}_1 + \underset{\sim}{K}_2$

K_1: full integration

K_2: reduced integration

INTRODUCTIONS

$$A \int_0^L \delta \underset{\sim}{\varepsilon}\, \underset{\sim}{\sigma}\, dx = \delta \underset{\sim}{u}\, \underset{\sim}{P} + A \int_0^L \delta \underset{\sim}{u}\, \gamma\, dx$$ – one dimensional rod w/ dead load

$$N_1 = \frac{x_2 - x}{x_2 - x_1}, \quad N_2 = \frac{x - x_1}{x_2 - x_1}$$ – pg 15

$$B = \frac{\partial N}{\partial x}, \frac{\partial N}{\partial y} \text{ etc.}$$

3-node elements – pg 19

$$\int_\Omega \delta \varepsilon^T \sigma\, d\Omega = \int_B \delta u^T T\, dB + \int_\Omega \delta u^T b\, d\Omega$$

$F = K \cdot u$

TWO DIMENSIONS – pg 30

$$u = \sum N_i u_i$$

interpolation functions for Δ on pg 31

Map elements to ξ-η space

$$\det(J) = \frac{\partial x}{\partial \xi} \cdot \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \cdot \frac{\partial y}{\partial \xi}$$

or, ratio of areas in physical, ξ-η space

$$\frac{\partial x}{\partial \xi} = \sum \frac{\partial N_i}{\partial \xi} x_i$$

$$x = \sum N_i(\xi, \eta) x_i, \quad y = \sum N_i(\xi, \eta) y_i$$

N functions for mapped $\square$ on pg 34d
non-mapped in handout

$$K_e = d \int_{-1}^{1} \int_{-1}^{1} B^T D B \cdot \det(J)\, d\xi\, d\eta$$

GAUSS

one point:

$$\int_{-1}^{1} F(\xi)\, d\xi = 2F(0)$$

two point:

$$= 1 \cdot F(-1/\sqrt{3}) + 1 \cdot F(1/\sqrt{3})$$

three point:

$$= \frac{5}{9} F\left(-\sqrt{\tfrac{3}{5}}\right) + \frac{8}{9} F(0) + \frac{5}{9} F\left(\sqrt{\tfrac{3}{5}}\right)$$

Accuracy to a level
$2n-1$

Power of ξ, η must be less
for it to be good
(except in reduced integration)

1-D

Principle of Virtual Work

$$\underbrace{A\int_0^L \delta\varepsilon \cdot \sigma\, dx}_{\text{internal}} = \underbrace{\delta u_0 \cdot P_0 + \delta u_L P_L + A\int_0^L \delta u \cdot \gamma\, dx}_{\text{external}}$$

where $\delta\varepsilon = \dfrac{d(\delta u)}{dx}$

$\delta u_0 = \delta u(0)$

$\delta u_L = \delta u(L)$

Integration by Parts

$$\frac{d}{dx}(f\cdot g) = \frac{df}{dx}\cdot g + f\cdot\frac{dg}{dx} \Rightarrow f\cdot\frac{dg}{dx} = \frac{d}{dx}(f\cdot g) - \frac{df}{dx}\cdot g$$

Galerkin Method

linear: approximation defined over $[0, L]$ $u(x) \approx ax$ into pr. of v.w. :

where: $\varepsilon = \frac{du}{dx} = a$, $\sigma = E\varepsilon = E\cdot a$, $\delta u = \delta a\cdot x$, $\delta\varepsilon = \delta a$, $\delta u_0 = 0$, $\delta u_L = \delta a\cdot L$

$$A\cdot\int_0^L \delta a\cdot E\cdot a\cdot dx = 0\cdot P_0 + \delta a\cdot L\cdot P_L + A\int_0^L \delta a\cdot x\cdot\gamma\cdot dx$$

$$u \approx \frac{Qx}{EA} + \frac{\gamma L x}{2E}$$

quadratic: approximation defined over $[0, L]$ $u(x) \approx ax + bx^2$ into pr. of v.w.:

where: $u = [x \ \ x^2]\begin{bmatrix} a \\ b\end{bmatrix}$ $\quad \delta u = \delta a\cdot x + \delta b\cdot x^2 = [x \ \ x^2]\begin{bmatrix}\delta a\\ \delta b\end{bmatrix}$

$\varepsilon = \frac{du}{dx} = a + 2b\cdot x = [1 \ \ 2x]\begin{bmatrix} a \\ b\end{bmatrix}$ $\quad \delta\varepsilon = [1 \ \ 2x]\begin{bmatrix}\delta a\\ \delta b\end{bmatrix}$

$\sigma = E\cdot\varepsilon = [E][1 \ \ 2x]\begin{bmatrix} a \\ b\end{bmatrix}$ $\quad \delta u_0 = 0$, $\delta u_L = [L \ \ L^2]\begin{bmatrix}\delta a\\ \delta b\end{bmatrix}$

$$A\int_0^L [1 \ \ 2x]\begin{bmatrix}\delta a\\ \delta b\end{bmatrix}[E][1 \ \ 2x]\begin{bmatrix} a \\ b\end{bmatrix}dx = 0\cdot P_0 + [L \ \ L^2]\begin{bmatrix}\delta a\\ \delta b\end{bmatrix}P_L + A\cdot\int_0^L [x \ \ x^2]\begin{bmatrix}\delta a\\ \delta b\end{bmatrix}\gamma\, dx$$

$$\therefore \left(\int_0^L \begin{bmatrix} EA & EA\cdot 2x \\ EA\cdot 2x & EA\cdot 4x\end{bmatrix}dx\right)\begin{bmatrix} a\\ b\end{bmatrix} = \begin{bmatrix} Q\cdot L \\ Q\cdot L^2\end{bmatrix} + \int_0^L \begin{bmatrix} x \\ x^2\end{bmatrix}[\gamma A]\, dx$$

Alternative Approach

don't approximate over entire $[0, L]$ w/ one equation, approximate by elements

linear $\quad 1 \overset{(e)}{\text{———}} 2$ $\quad u = a^{(e)} + b^{(e)}\cdot x$

$u(x_1) = u_1 = a^{(e)} + b^{(e)}\cdot x_1$

$u(x_2) = u_2 = a^{(e)} + b^{(e)}\cdot x_2$

$$u(x) = \underbrace{\left(\frac{x_2 - x}{x_2 - x_1}\right)}_{N_1(x)}u_1 + \underbrace{\left(\frac{x - x_1}{x_2 - x_1}\right)}_{N_2(x)}u_2$$

interpolation functions

apply pr. of v.w. to element:

$u = [N_1 \ \ N_2]\begin{bmatrix} u_1\\ u_2\end{bmatrix}$ $\quad \delta u = [N_1 \ \ N_2]\begin{bmatrix}\delta u_1\\ \delta u_2\end{bmatrix}$

$\varepsilon = \frac{du}{dx} = \left[\frac{dN_1}{dx} \ \ \frac{dN_2}{dx}\right]\begin{bmatrix} u_1\\ u_2\end{bmatrix}$ $\quad \delta\varepsilon = \left[\frac{dN_1}{dx} \ \ \frac{dN_2}{dx}\right]\begin{bmatrix}\delta u_1\\ \delta u_2\end{bmatrix}$

$\sigma = E\cdot\varepsilon = [E]\left[\frac{dN_1}{dx} \ \ \frac{dN_2}{dx}\right]\begin{bmatrix} u_1\\ u_2\end{bmatrix}$

$$[\delta u_1 \ \delta u_2]\left(\int_{x_1}^{x_2}\begin{bmatrix}\frac{dN_1}{dx}\\ \frac{dN_2}{dx}\end{bmatrix}[EA]\left[\frac{dN_1}{dx} \ \ \frac{dN_2}{dx}\right]dx\right)\begin{bmatrix} u_1\\ u_2\end{bmatrix} = [\delta u_1 \ \delta u_2]\begin{bmatrix} P_1\\ P_2\end{bmatrix} + [\delta u_1 \ \delta u_2]\left(\int_{x_1}^{x_2}\begin{bmatrix} N_1\\ N_2\end{bmatrix}[\gamma A]\, dx\right)$$

$$\therefore \begin{bmatrix} EA/L_e & -EA/L_e \\ -EA/L_e & EA/L_e\end{bmatrix}\begin{bmatrix} u_1\\ u_2\end{bmatrix} = \begin{bmatrix} P_1\\ P_2\end{bmatrix} + \begin{bmatrix}\frac{1}{2}\gamma A L_e\\ \frac{1}{2}\gamma A L_e\end{bmatrix}$$

HW1

$\underset{\sim}{B} = \left[\frac{dN_1}{dx} \quad \frac{dN_2}{dx} \right]$

$\underset{\sim}{D} = [E]$ "rigidity matrix"

$\underset{\sim}{K} = A\int_{x_1}^{x_2} \underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B}\, dx$ "stiffness matrix"

$\underset{\sim}{\Gamma} = A\int_{x_1}^{x_2} \underset{\sim}{N}^T [\gamma]\, dx$ "element load vector"

$\underset{\sim}{K}\,\underset{\sim}{U} = \underset{\sim}{P} + \underset{\sim}{\Gamma}$

Simpson's Rule to Integrate Small polynomial:

$\int_a^b f(x)\,dx \cong (b-a)\left[\frac{1}{6} f(a) + \frac{2}{3} f\left(\frac{a+b}{2}\right) + \frac{1}{6} f\right.$

*exact if f is polynomial of degree at most 3

Quadratic

$N_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$ $\quad N_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$ $\quad N_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$

$$\underset{\sim}{K} = \int_{x_1}^{x_3} \underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}\, dx = \frac{EA}{L}\begin{bmatrix} 7/3 & -8/3 & 1/3 \\ -8/3 & 16/3 & -8/3 \\ 1/3 & -8/3 & 7/3 \end{bmatrix}$$

Mapping - Handout (1-D)

element domain onto standard interval $[-1, 1]$

$N(x) \to N(\xi)$

3-D

Stress Tensor

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}, \qquad \underset{\sim}{T} = \underset{\sim}{\sigma}\,\underset{\sim}{n}$$

pr. of v.w.:

$$\int_\Omega \delta \underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma}\, d\Omega = \int_B \delta \underset{\sim}{u}^T \underset{\sim}{T}\, dB + \int_\Omega \delta \underset{\sim}{u}^T (\underset{\sim}{b} - \rho \ddot{\underset{\sim}{u}})$$

Body Force Density

body force per unit volume, $\underset{\sim}{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\rho g \end{bmatrix}$ when gravity in neg. z-direction

ρ = mass density

apply pr. of v.w.: (x-direction)

$$\delta u\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x\right) = \delta u \cdot \rho \frac{\partial u^2}{\partial t^2}$$

(y + z-directions similar to get 3 equations of force equilibrium)

Ω: Domain

B: Boundary

Integration by Parts using Green's Thm

$$\int_\Omega \frac{df}{dx}\, d\Omega = \int_B f \cdot n_x\, dB$$

Axisymmetric - pg 47

r, θ, z
$\hookrightarrow$ azimuthal, hoopstress (dir 33)

integrals include r

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \\ \frac{N_1}{r} & 0 & \\ & \frac{\partial N_1}{\partial z} & \cdots \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \end{bmatrix}, \quad D = \begin{bmatrix} \lambda+2G & \lambda & \lambda & 0 \\ \lambda & \lambda+2G & \lambda & 0 \\ \lambda & \lambda & \lambda+2G & 0 \\ 0 & 0 & 0 & G \end{bmatrix}$$

$$F = 2\pi \int_B N^T T r \, dB$$

$$N_1 = \frac{r-R}{\Delta R} + 1 = \frac{r_2 - r}{r_2 - r_1}$$

$$N_2 = \frac{r-R}{\Delta R} = \frac{r-r_1}{r_2-r_1}$$

4 Node Tetrahedron - pg 52

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & 0 & \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \\ 0 & 0 & \frac{\partial N_1}{\partial z} & \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & 0 & \cdots \\ 0 & \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial y} & \\ \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_1}{\partial x} & \end{bmatrix}$$

$N_1 = \xi$
$N_2 = 1 - \xi - \zeta - \eta$
$N_3 = \zeta$
$N_4 = \eta$

$$K = \int_\Omega B^T D B \, d\Omega$$

$$\det(J) = \frac{\text{vol. in physical space}}{\text{vol. in parent space}} = \frac{\partial x}{\partial \xi} \times \frac{\partial x}{\partial \eta} \cdot \frac{\partial x}{\partial \zeta}$$

Nodal forces under pressure (traction):

$$F = \int_B N^T T \, dB$$
$\hookrightarrow$ $-pn$, with n = unit vector normal to B

Incompressibility - pg 56

D matrix has ν in denominator

(or, $1-2\nu$)

as $\nu \to 0.5$, value $\to 0$

$$\sigma_m = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = \frac{1}{3}(3\lambda + G\cdot 2)\underbrace{(\varepsilon_x + \varepsilon_y + \varepsilon_z)}_{\text{volumetric strain}}$$

σ_m → mean normal stress

Deviatoric stress, strain:

$$s = \sigma - \sigma_m \cdot I \qquad \sigma_m = K\cdot\varepsilon = \underbrace{(\lambda + 2/3 G)}_{K,\ \text{bulk modulus}}(\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

$$e = \varepsilon - 1/3\,\varepsilon\cdot I \qquad s = 2G\cdot e$$

B_b - bulk

$$B_b = \Big[\underbrace{\frac{\partial N_1}{\partial x}\ \frac{\partial N_1}{\partial y}\ \left(\frac{\partial N_1}{\partial z}\right)}_{\text{contributions associated with node 1}}\ \frac{\partial N_2}{\partial x}\ \frac{\partial N_2}{\partial y}\ \left(\frac{\partial N_2}{\partial z}\right)\ \frac{\partial N_3}{\partial x}\ \frac{\partial N_3}{\partial y}\ \left(\frac{\partial N_3}{\partial z}\right)\Big]$$

$$B_s = \begin{bmatrix} \frac{2}{3}\frac{\partial N_1}{\partial x} & -\frac{1}{3}\frac{\partial N_1}{\partial y} & -\frac{1}{3}\frac{\partial N_1}{\partial z} & \\ -\frac{1}{3}\frac{\partial N_1}{\partial x} & \frac{2}{3}\frac{\partial N_1}{\partial y} & -\frac{1}{3}\frac{\partial N_1}{\partial z} & \\ -\frac{1}{3}\frac{\partial N_1}{\partial x} & -\frac{1}{3}\frac{\partial N_1}{\partial y} & \frac{2}{3}\frac{\partial N_1}{\partial z} & \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & 0 & \cdots \\ 0 & \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial y} & \\ \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_1}{\partial x} & \end{bmatrix}$$

$K_e = K_b + K_s$

bulk modulus (D_b) (pg 59)

$$K_b = \int_\Omega B_b^T [K] B_b \, d\Omega$$

$$K_s = \int_\Omega B_s^T D_s B_s \, d\Omega$$

use selectively reduced integration

- reduced on bulk half
- full on shear half

Hybrid Formulation - pg 60

For incompressibility

$$K_s = \int_\Omega B_s^T D_s B_s \, d\Omega$$

$$D_s = \begin{bmatrix} 2G & 0 & 0 & 0 & 0 & 0 \\ 0 & 2G & 0 & 0 & 0 & 0 \\ 0 & 0 & 2G & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix}$$

$$G = \int_\Omega B_b^T N_p \, d\Omega$$

↳ interpolation functions (for ~~integration points~~ pressure nodes)

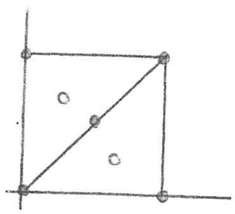
$$F = \int_\Omega N_p^T \frac{1}{K} N_p \, d\Omega$$ — is zero when incompressible

Augmented matrix

$$\begin{bmatrix} \underset{\sim}{K_s} & -\underset{\sim}{G} \\ -\underset{\sim}{G}^T & -\underset{\sim}{F} \end{bmatrix}$$

(−) signs if equations are as above

$\sigma_m = -\underset{\sim}{N_p}\,\underset{\sim}{p}$, $\underset{\sim}{p}$: nodal values of pressure



2 disp nodes = 4 eqs

2 pressure nodes = 2 eqs

$$r = \frac{4}{2} = 2 \checkmark$$

less than 2 — very bad

more than 2 — underconstrained

Map to $\xi \cdot \eta$ space

$$N_1 = \frac{1}{2}(1-\xi)$$

$$N_2 = \frac{1}{2}(1+\xi)$$

$$u = \sum (u_i - \eta\theta_i \tfrac{h}{2}) N_i(\xi)$$

$$V = \sum v_i N_i(\xi)$$

$K_x = R^T K R$, K = stiffness matrix of quadrilateral

R = transformation matrix (pg 68)

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & -\eta\frac{h}{2}\frac{\partial N_1}{\partial x} & \\ 0 & \frac{\partial N_1}{\partial x} & -\frac{\partial \eta}{\partial y}\frac{h}{2}N_1 & \cdots \end{bmatrix} \qquad \frac{\partial x}{\partial \xi} = \frac{L}{2}, \quad \frac{\partial y}{\partial \eta} = \frac{h}{2}$$

can be split into ε, γ - first, second rows of B

$$\underset{\sim}{K} = E \cdot d \int_\Omega B_\varepsilon^T B_\varepsilon \, d\Omega + G \cdot d \int_\Omega B_\gamma^T B_\gamma \, d\Omega$$

$$\underset{\sim}{u}(\xi,\eta,\zeta) = \sum \left[\begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} - \zeta\theta_i \underset{\sim}{A_{2i}} + \zeta\phi_i A_{1i}\right] N_i(\xi,\eta)$$

$$\underset{\sim}{n} = \frac{\partial \underset{\sim}{x}/\partial\xi \times \partial \underset{\sim}{x}/\partial\eta}{|\partial \underset{\sim}{x}/\partial\xi \times \partial \underset{\sim}{x}/\partial\eta|}$$ – unit normal to lamina

$$\underset{\sim}{a} = \frac{\partial \underset{\sim}{x}/\partial\xi}{|\partial \underset{\sim}{x}/\partial\xi|}$$ – unit vector tangent to lamina

$$\underset{\sim}{b} = \hat{n} \times \underset{\sim}{a}$$

$$K = \iiint \underbrace{B^T D B}_{6\times 45 \text{ matrix}} \det(J)\, d\xi\, d\eta\, d\zeta, \quad \underset{\sim}{D} = \underset{\sim}{R}^{-1} \cdot \underbrace{\hat{\underset{\sim}{D}}}_{\text{on pg 73}} \hat{R}^{-T}$$ (defined in handout)