University of Texas
Dept. of Civil, Arch. & Env. Eng.

Instructor: E. B. Williamson
CE 381P

CE 381P COMPUTER METHODS IN STRUCTURAL ANALYSIS

Spring 2007

Course Purpose:

CE 381P focuses on computing the response of structural frames and trusses. This class will extend many of the concepts treated in courses on matrix methods of analysis by developing a general framework for analyzing complicated structural systems. A significant portion of the course will focus on computing the response of structures accounting for nonlinear effects.

Course Objectives:

By the end of the course, you should be able to do the following:

- Derive stiffness/flexibility relations for structural members using virtual work principles.
- Use and/or develop structural analysis software to analyze complicated structural systems.
- Interpret the output from computer-based analyses for the purpose of structural design.
- Determine the critical load of structural systems.
- Compute the load-deformation behavior of a structure accounting for nonlinear effects.

Topics:

Introduction: Review of basic concepts and matrix methods of structural analysis

Virtual Work Principles in Structural Analysis

Review of the Principle of Virtual Displacements and Virtual Forces
Stiffness Relationships for Prismatic and Non-prismatic Members
Matrix Form of Compatibility and Equilibrium
Stiffness Method of Analysis
Flexibility Method of Analysis
Flexibility/Stiffness Transformations

Large Displacement Behavior of Structures

Overview of Structural Stability
Snap-through and Bifurcation
Linearized Buckling and Eigenvalue Problems
Nonlinear Solution Methods: Newton-Raphson, Euler, and Arc-Length Methods (Time Permitting)

Fundamentals of Plasticity Theory (Time Permitting)

Stiffness Matrix Formulation Including Inelasticity
Computational aspects of plastic hinge analysis for frame members

Text (recommended):

McGuire, W., Gallagher, R. H., and Ziemian, R. D. (2000). *Matrix Structural Analysis: Second Edition.* John Wiley & Sons, Inc., New York.

Other References

Kassimali, A. (1999). *Matrix Analysis of Structures*. Brooks/Cole Publishing Company, Pacific Grove, CA.

Hibbeler, R. C. (2006). *Structural Analysis, Sixth Edition*. Prentice Hall, Upper Saddle River, NJ.

Sennet, R. E. (1994). *Matrix Analysis of Structures*. Waveland Press, Inc. Prospect Heights, IL.

Crisfield, M. A. (1991). *Non-Linear Finite Element Analysis of Solids and Structures, Vol. 1*. John Wiley & Sons, Inc., New York.

Levy, R. and Spillers, W. R. (1995). *Analysis of Geometrically Nonlinear Structures*. Chapman & Hall, New York.

Przemienieki, J. S. (1985). *Theory of Matrix Structural Analysis*. Dover Publications, Inc., New York. (Originally published by McGraw Hill Book Company in 1968).

Strang, W. G. (1980). *Linear Algebra and its Applications, Second Edition*. Academic Press, New York.

Course Websites:

A website for the course can be found at the following URL:
http://www.ce.utexas.edu/prof/williamson/ce381p/

This site contains information about reading assignments from the recommended text, a student version of SAP2000 that you can download to your own computer, and other useful information. In addition, copies of handouts, MATHCAD worksheets, and information will be maintained using Blackboard. You are encouraged to check these websites regularly for updates and announcements.

Office Hours:

MWF 9:15 - 10:30 AM

Office: ECJ 4.722
Phone: 475-6175
email: ewilliamson@mail.utexas.edu

**Note:* I maintain an "open door" policy outside of regularly scheduled office hours. If the door to my office is open, please feel free to stop in.

Prerequisites:

Students enrolled in the course are expected to have had an advanced undergraduate course in structural analysis, be able to compute the response of frame and truss structures using the stiffness method and the flexibility method, be familiar with basic concepts in linear algebra and differential equations, and be comfortable with computer programming.

Conduct of Course:

The course consists primarily of lectures, homework and group assignments, a midterm exam, and a final examination. Attendance is essential. Homework problems are subject to the due dates stated when distributed. *Late homework (any that come in after the beginning of the period on the due date) will receive a maximum grade of 50%.* Late work will not be accepted from any student more than two times over the course of the semester.

Grades will be computed using the following distribution: Homework and Group Assignments (30%), Midterm exam (35%), and Final Examination (35%). The date for the midterm exam will be announced by the instructor. The final exam will be given at the time designated in the University timetable (11 May 2007, 2:00 - 5:00 PM).

Course Evaluation:

The students will evaluate the course and the instructor on forms provided by the Measurement and Evaluation Center.

Course Drop Dates:

From the 1st through the 4th class day, graduate students can drop or add a course on Rose or TEX. Beginning with the 5th class day, graduate students must initiate any adds or drops in their department. Graduate students can drop a class until the last class day with permission from the departmental Graduate Advisor and the Dean. Graduate students with GRA/TA/Grader positions or with Fellowships may not drop below 9 hours in a long session.

Academic Integrity:

As engineers you will be responsible for upholding the canons of ethics for the profession. A test of your ability to do so is to uphold the University's Academic Honesty Policy. While I do not anticipate problems of this nature, any instances of academic dishonesty will be dealt with immediately and severely in accordance with published procedures. Students who violate University rules on scholastic dishonesty are subject to disciplinary penalties, including the possibility of failure in the course and/or dismissal from the University. Because such dishonesty harms the individual, all students, and the integrity of the University, policies on scholastic dishonesty will be strictly enforced. For further information, visit the Student Judicial Services web site http://deanofstudents.utexas.edu/sjs/.

Additional Information:

Web-based, password-protected class sites will be associated with all academic courses taught at the University. Syllabi, handouts, assignments and other resources are types of information that may be available within these sites. Site activities could include exchanging email, engaging in class discussions and chats, and exchanging files. In addition, electronic class rosters will be a component of the sites. Students who do not want their names included in these electronic class rosters must restrict their directory information in the Office of the Registrar, Main Building, Room 1. For information on restricting directory information, see the Undergraduate Catalog or go to: http://www.utexas.edu/student/registrar/catalogs/gi00-01/app/appc09.html.

Conclusion:

Enjoy Computer Methods in Structural Analysis! This class will be as challenging as any you will have at U.T. To perform well, you must study diligently as the material will build from the first lesson through the remainder of the semester. Get off to a good start in your graduate studies - it can't possibly get any better than this!

The University of Texas at Austin provides, upon request, appropriate academic adjustments for qualified students with disabilities. Any student with a documented disability (physical or cognitive) who requires academic accommodations should contact the Services for Students with Disabilities area of the Office of the Dean of Students at 471-6259 as soon as possible to request an official letter outlining authorized accommodations. For more information, contact that Office, or TTY at 471-4641, or the College of Engineering Director of Students with Disabilities at 471-4321.

Two kinds of boundaries:

1. Natural – deals with force conditions
2. Essential – deals with displacement restrictions

Boundary Value Problem

$$\frac{d}{dx}(E(x)A(x)u'(x)) = -p(x)$$

STIFFNESS-BASED ANALYSIS

Principle of Virtual Work (PVW) (or displacements)

B = strain/displacement matrix (for axial members; = **H** for flexural)

Relates nodal displacements to strain within the member

Conservation of virtual work

$$\delta W_{\text{int}} = \int EA(s)u'(x)\overline{u}'(x)dx = \delta W_{ext} = P\overline{u}(app)$$

Using various equations derived,

$$\underline{k} = \int_0^L \underline{B}_x^T(x)E(x)A(x)\underline{B}_x dx \text{ – axial member}$$

$$\underline{k} = \int_0^L (\underline{H}'')^T E(x)I(x)\underline{H}''dx \text{ – flexural member}$$

Shape functions

Cubic, for beams:

$$H_1 = 1-3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3, H_1'' = \frac{-6}{L^2} + 12\frac{x}{L^3}$$

$$H_2 = x - 2\frac{x^2}{L} + 2\frac{x^3}{L^2}, H_2'' = \frac{-4}{L} + 6\frac{x}{L^2}$$

$$H_3 = 3\frac{x^2}{L} - 2\left(\frac{x}{L}\right)^3, H_3'' = \frac{6}{L^2} - 12\frac{x}{L^3}$$

$$H_4 = -\frac{x^2}{L} + \frac{x^3}{L^2}, H_4'' = \frac{-2}{L} + 6\frac{x}{L^2}$$

Axial forces:

$$N(x) = E \cdot A(x) \cdot u'(x)$$

FLEXIBILITY-BASED ANALYSIS

Principle of Virtual Forces

1. Choose independent and dependent variables
   Must result in statically determinate, stable structure
   Example: pinned structure, **moments** are independent
2. Establish equilibrium, writing equations for dependents using independents
   Write equation for moment, force at any point along beam
   $M(x) = D\,F_F$, where F_F are independent forces, D relates them to location
3. Declare virtual system to match in geometry
4. Set complimentary internal virtual work equal to complimentary external virtual work

   $$f = \int_0^L D^T \frac{1}{E(x)\beta(x)} D$$

5. Transform to k; find Φ: relates Q_1 (dependent variables) to Q_0 (independent variables)

   $$Q_1 = \Phi Q_0$$

   Use Φ to get:

   $$k_{00} = f^{-1}$$

   $$k_{10} = \Phi f^{-1}, k_{01} = f^{-1}\Phi^T$$

   $$k_{11} = \Phi f^{-1}\Phi^T$$

6. Make force vector, solve

Equivalent load vectors

Generalized equation using the **stiffness** method:

$$F = \int_V B^T E\varepsilon_o dV - \int_V B^T \sigma_o dV + \int_V H^T \Lambda() dV + \int_\Omega H^T p() d\Omega + \int_S H^T w(x) dS + \sum H^T P_i + \sum (H'(x_i))^T M_i + R$$

Generalized equation using the **flexibility** method:

$$q_{f0} = \int_0^L D^T \left[\frac{1}{E(x)\beta(x)} Q_P(x) + \varepsilon_T(x) \right] dx$$

$$Q_{FE} = -f^{-1} q_{f0}$$

In this equation, β is either I or A

Q is the distribution of bending moment over the length

ε_T is the thermal strain resultant

In flexure, thermal curvature

$$\kappa(x) = \frac{\alpha(x)}{I(x)} \int_{-h(x)/2}^{h(x)/2} \Delta T \cdot y \cdot b dy = \frac{\alpha \cdot \Delta T}{d} \text{ (for linear } \Delta\text{T)}$$

In a truss, thermal axial strain

Forces found are fixed-end, so apply negatives

Find dependent forces using equations developed previously and equilibrium (Φ matrix)

Fixed-end forces for load and thermal effects on attached sheet

Numerical integration

Newton-Cotes

Evenly spaced

N = 1, $c_0 = c_1 = 1$

N = 2, $c_0 = 1/3$, $c_1 = 4/3$, $c_2 = 1/3$

Gaussian

Integration Order	Station ξ_i	Weight w_i
1	0	2
2	$\sqrt{1/3}$	1
	$-\sqrt{1/3}$	1
3	$-\sqrt{3/5}$	5/9
	0	8/9
	$\sqrt{3/5}$	5/9

Will evaluate exactly to order 2n (order = number of terms)

$$\int_0^L f(x) = \frac{b-a}{2} \left[f(\sqrt{x(\xi)}) + f(-\sqrt{x(\xi)}) \right]$$

$$x(\xi) = \frac{1}{2} \left[a(1-\xi) + b(1+\xi) \right]$$

NONLINEAR ANALYSIS

- material behavior (yielding)
- kinematics (geometry nonlinearity)
- change of constraints / contact

Establish equilibrium in the deformed position

Graphical representations

- Solid line is stable; dashed is unstable
- Bifurcation point: intersection of two equilibrium equations
- Limit point/load: location and load at which equilibrium switches between stable and unstable

Virtual Work approach

1. Deform member
2. Add additional rotation $\delta\theta$
3. External work = change in height of the location of the load between original deformation and $\delta\theta$ position; negative for common signage
4. Internal work = moment in original deformed position times $\delta\theta$
   Original moment is equal to $k\theta$, similar to $F = kx$ for a linear spring
5. Set internal and external virtual works equal to each other; solve for P
   Don't cancel $\theta = 0$ terms, or sines; figure out what equilibrium condition they apply to
6. For stability:
   $$\frac{d}{d\theta}\delta W_{\text{int}} > \frac{d}{d\theta}\delta W_{ext}$$
   Equation will be in terms of P
7. Substitute various equilibrium equations for P into inequality to determine stability

Energy-based approach

1. Select a datum from which to consider loads
2. Potential energy of the load, V = height of the load above the datum times the load (in deformed position); positive if load is above datum
3. Internal strain energy of the spring, $U = 0.5kx^2$, or area under M-θ curve
4. Solve for equilibrium using the derivative of the total energy (rate of change of energy)
   $$\frac{d}{d\theta}\Pi = \frac{d}{d\theta}(U+V) = 0$$
   Solve for P, equilibrium equations
5. For stability: take the second derivative of Π
   $$\frac{d^2}{d\theta^2}\Pi > 0$$

Imperfections

Reference rotation to the vertical

$$V = PL\cos\theta$$

$$U_{rot} = \frac{1}{2}k(\theta - \theta_o)^2$$

$$U_{trans} = \frac{1}{2}kL^2(\sin\theta - \sin\theta_o)^2$$

Follow the same steps as above for equilibrium and stability calculations

Stability in Multi-DOF systems

The partial derivative of the total energy with respect to each variable must equal zero

Hessian matrix, $\mathbf{H}(\Pi(x))$ – matrix of partial second derivatives

$$\begin{bmatrix} \frac{\partial^2}{\partial x_1^2} & \frac{\partial^2}{\partial x_1 \partial x_n} \\ \frac{\partial^2}{\partial x_1 \partial x_n} & \frac{\partial^2}{\partial x_n^2} \end{bmatrix}$$ For stability, all eigenvalues or principal minors must be > 0

Principal minors: determinant of increasing number of terms from the top left to the bottom right

$$PM1 = \left| \frac{\partial^2}{\partial x_1^2} \right|$$

$$PM2 = \begin{vmatrix} \frac{\partial^2}{\partial x_1^2} & \frac{\partial^2}{\partial x_1 \partial x_n} \\ \frac{\partial^2}{\partial x_1 \partial x_n} & \frac{\partial^2}{\partial x_n^2} \end{vmatrix}$$, etc. Number of principal minors = number of row/columns

Smallest value for P controls; PM group holding that value gives buckling mode shapes

Buckling mode shapes

1. Evaluate derivative of Π at P_{cr} values
   Linearized approximations are generally acceptable at this point
2. Only shape, not magnitude, can be determined
   Select a value for θ_1 (such as 1), find θ_2 in terms of it

CONTINUOUS SYSTEMS

Non-generalized systems

- Pay attention to geometry, do not assume linear response from before
- Use U, V equations as such:

$$U = \frac{1}{2}kx^2$$, with x being rotation/displacement, h is the height of the load above datum

$$V = P(h_o + \Delta h)$$

- Equation for U assumes engineering strain, prismatic member, and constant E
- General equation:

$$U = \frac{1}{2}E \cdot \varepsilon^2 \cdot A \cdot L$$

Engineering strain says $\varepsilon = \Delta / L$, other strains result in different equations

Assumptions include:

- Constant E
- Prismatic member (constant A, L)
- Use original length and area in integrations
- Nonlinear geometry, but not inelasticity

Tangent Stiffness

$$k_T = \frac{dP(w)}{dw}$$, where w is the displacement variable

Types of strain

Engineering: $\varepsilon = \dfrac{L_f - L_o}{L_o} = \dfrac{\partial u}{\partial x}$ Natural: $\varepsilon = \dfrac{L_f - L_o}{L_f}$ Green's: $\varepsilon = \dfrac{L_f^2 - L_o^2}{2L_o^2} = \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left[\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right.\right.$

Logarithmic: $\varepsilon = \ln\dfrac{L_f}{L_o}$ Almansi: $\varepsilon = \dfrac{L_f^2 - L_o^2}{2L_f^2}$

Only makes a difference in strain calculation when displacements are large, L_f does not ~ L_o

$$\frac{1}{2}EAL \cdot \left(\frac{L_f^2 + L^2}{2L^2}\right)^2$$

$$\frac{1}{2}EA\frac{K}{4L^{\cancel{4}3}}(L_f^2 + L^2)^2$$

$$\frac{1}{8}K\frac{1}{L^2}(L_f^2 + L^2)^2$$

Nonlinear Stiffness Matrices

Basic principle: $\delta W_{\text{int}} = \int_{vol} \delta\varepsilon \cdot \sigma \cdot dVol$

Stiffness matrix is composed of a linear part and a geometric (nonlinear) part

$\left[1 \quad 0 \quad -\frac{1}{2} \quad -\frac{1}{2} \quad 0 \right.$

$$k_L = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, k_G = \frac{F}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

For a beam,

$$k_G = \frac{F}{L}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 6/5 & L/10 & 0 & -6/5 & L/10 \\ 0 & L/10 & 2L^2/15 & 0 & -L/10 & -L^2/30 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -6/5 & -L/10 & 0 & 6/5 & -L/10 \\ 0 & L/10 & -L^2/30 & 0 & -L/10 & 2L^2/30 \end{bmatrix}$$

Evaluate for individual elements, transform and map to global DOFs

Equilibrium equation has P terms on both sides

First solving method includes iterating for P; annoying and difficult

Second solving method:

1. Apply small load – P = 1, or P = P/λ
2. Compute linear displacements using k_L (no k_G!)
3. Calculate axial forces in linear case as done in 363
4. Use calculated axial forces in k_G formulations

   $k_G = \frac{P}{L}[\ ]$, where P is axial force from 3.
5. Increase loads by λ – now P = λ or P = P
6. Assume axial forces scale by λ

   $k_G' = \lambda k_G$
7. Use principal minor analysis to solve for λ or P

General Info and Equations

Map to global coordinates

$$K = T^T \cdot k \cdot T$$

$$T = \begin{bmatrix} \cos & \sin & 0 \\ -\sin & \cos & 0 \\ 0 & 0 & 1 \end{bmatrix}$$, angles measured from the horizontal; matrix repeated to be 6x6

Partitioned Matrix

$$\begin{bmatrix} k_{00} & k_{01} \\ k_{10} & k_{11} \end{bmatrix} \cdot \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} = \begin{bmatrix} Q_0 \\ Q_1 \end{bmatrix} - \begin{bmatrix} Q_{0FE} \\ Q_{1FE} \end{bmatrix}$$

$k_{00} q_0 + k_{01} q_1 = Q_0 - Q_{0FE}$

$k_{10} q_0 + k_{11} q_1 = \cancel{Q_1} - Q_{1FE}$

Q_{FE}

$q_1 = -k_{11}^{-1} k_{10} q_0$

$k_{00} q_0 + k_{01}(-k_{11}^{-1} k_{10} q_0) = Q_0$

$k' = k_{00} - k_{01} k_{11}^{-1} k_{10}$ ✓

Modified k:

$$k_{00} - k_{01} k_{11}^{-1} k_{10}$$

Modified F:

$$R_0 - k_{01} k_{11}^{-1} R_1$$

Release mechanisms

Used in situations such as the formation of a hinge

R_1, above, equals zero

Only tracks non-released DOF terms directly; get 3x3 stiffness matrix

Then, add row of zeros back in, repermute, map to global coordinates

$$r_1 = k_{11}^{-1} \left[R_1 - k_{10} r_0 \right]$$

Constraints

Relate one DOF to another

Use Γ matrix to relate unconstrained, constrained DOFs

$u_{unconstrain} = \Gamma u_{constrain}$

$k_{constrain} = \Gamma^T k_{unconstrain} \Gamma$

Symmetry

With respect to a line or a point

Symmetric loading: same values, opposite direction

Use weird roller as boundary

Antisymmetric loading: same values, same direction

Use normal roller for boundary

Extra interpolation functions

2-node axial element

$$N_1 = 1 - \frac{x}{L}, N_2 = \frac{x}{L}$$

$$B_1 = -\frac{1}{L}, B_2 = \frac{1}{L}$$

3-node axial element

$$N_1 = 1 - 3\frac{x}{L} + 2\left(\frac{x}{L}\right)^2, B_1 = -\frac{3}{L} + 4\frac{x}{L^2}$$

$$N_2 = 4\left(\frac{x}{L}\right)\left(1 - \frac{x}{L}\right), B_2 = \frac{4}{L}\left(1 - 2\frac{x}{L}\right)$$

$$N_3 = -\frac{x}{L} + 2\left(\frac{x}{L}\right)^2, B_3 = -\frac{1}{L} + 4\frac{x}{L^2}$$

Quadratic Equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Taylor Series Expansion

$$f(x+\Delta x) = f(x) + \Delta x \cdot \frac{df}{dx}_x + \frac{1}{2}\Delta x^2 \cdot \frac{d^2 f}{dx^2}_x \ldots$$

With multiple values (x is a vector), take partial derivatives for each

Binomial Expansion

$$\sqrt{1+x^2} \sim 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 \ldots$$

Derivatives by Parts

$$\frac{d}{dx}\big(f(x)g(x)\big) = f(x)g'(x) + f'(x)g(x)$$

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CE 381P Computer Methods in Structural Analysis

Midterm Exam – Spring 2007

Instructions: There are four (4) questions. Attempt to answer all of them. **Turn in the exam sheet with your exam.**

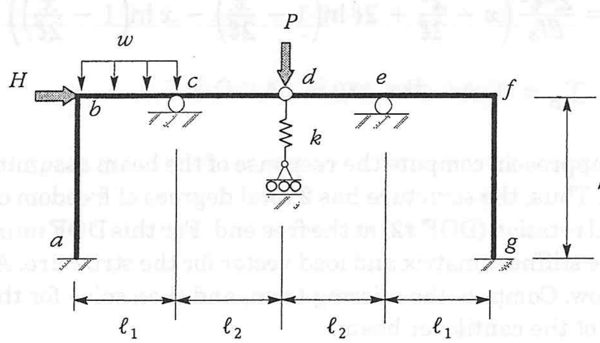
1. (20%) Shown in the sketch below is a truss element of constant cross-section with one node at each end of the member. The element has elastic modulus E and cross-sectional area A.

$$u(x) = \left(\cos\frac{\pi x}{2\ell}\right)u_1 + \left(\sin\frac{\pi x}{2\ell}\right)u_2$$

(a). Using the expression for the displaced shape shown in the figure and a similar expression to describe the virtual displacements, develop the 1,1 term in the stiffness matrix for the element.

(b). Based on your knowledge of the exact solution for a truss bar of constant cross-section, comment on the appropriateness of the assumed shape functions for developing the stiffness matrix of this element (do they violate any of the required properties for shape functions?). Indicate whether or not you would expect accurate solutions for the response of a truss structure using the stiffness matrix you developed.

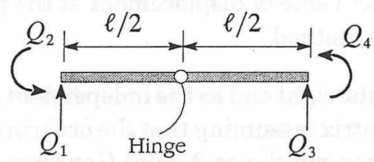
2. (30%) The structure shown in the figure below is to be analyzed using symmetry. All members have the same EI and are assumed to be axially rigid (i.e., $EA \to \infty$). The structure is hinged at d, and translation of the hinge is resisted by a linear spring with stiffness k.



(a). Considering the left half of the symmetric structure, establish the appropriate loads and support conditions for the <u>**symmetric response**</u>. Develop the stiffness matrix and load vector needed to compute the response. Clearly show your degree of freedom numbering convention.

(b) Repeat part *(a)* for the <u>**antisymmetric response**</u>.

3. (20%) Consider the flexural element with a hinge in the middle (shown in the figure below). Assume that EI is uniform for the element and axial effects can be ignored.



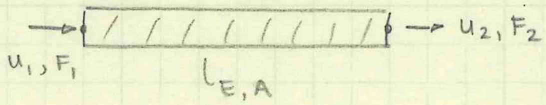
CE 381P: ANALYSIS

CATHERINE HOVELL
11 April 2007

80

MIDTERM

1.



$$u(x) = \cos\frac{\pi x}{2L} \cdot u_1 + \sin\frac{\pi x}{2L} u_2$$

$$N(x) = \left[\cos\frac{\pi x}{2L} \quad \sin\frac{\pi x}{2L}\right]$$

$$u(x) = \underset{\sim}{N}(x) \cdot \underset{\sim}{u}$$

$B(x) \equiv$ strain/disp. matrix

$$= N'(x)$$

$$= \left[-\pi/2L \sin\frac{\pi x}{2L} \quad \frac{\pi}{2L}\cos\frac{\pi x}{2L}\right]$$

$$K_{axial} = \int_0^L \underset{\sim}{B}^T(x)\, EA\, \underset{\sim}{B}(x)\, dx$$

$$K_{11} = \int_0^L B_1^2(x)\, EA\, dx = EA \int_0^L \frac{\pi^2}{4L^2}\left(\sin\frac{\pi x}{2L}\right)^2 dx$$

16.5 / 20

since I can't remember the exact form of $\int \sin^2$, going to use Gauss to solve numerically

$$\sin = x + \frac{x^3}{3!} \cdots$$

squared → 3rd order

values explained in prob. 4d

$$= \frac{EA\,\pi^2}{4L^2}\left[\sin\left(\frac{\pi}{2L}0.211L\right)^2 + \sin\left(\frac{\pi}{2L}0.789L\right)^2\right] \cdot \frac{L}{2}$$

NEED TO DIVIDE INTERVAL OF INTEGRATION BY 2

$L \to 1$

$$K_{11} = \frac{EA\pi^2}{4L} \cdot \frac{1}{2} = \frac{\pi^2}{8}\frac{EA}{L}$$

b) The shape functions, while not ~~in violation~~ of any rules (continuous, =1 at node, =0 at other node, not equal to zero between nodes...) are too complex for such a simple beam. While the end displacements may come close to predicting the actual, between the nodes the assumed deflected shape will not match the true shape well.

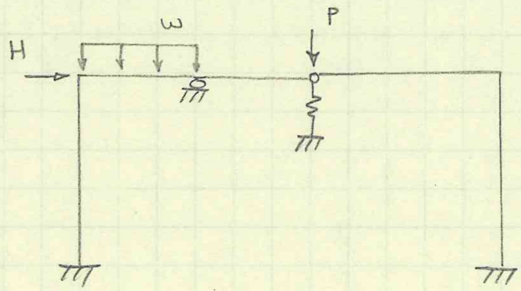
$\Sigma N_i \neq 1 \;\forall x$

K_{11} should $= \frac{EA}{L}$, not $2.47\frac{EA}{L}$

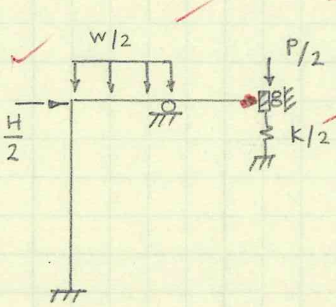
$1.24\frac{EA}{l}$

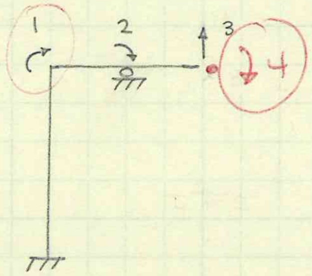
MIDTERM

2.



a.)





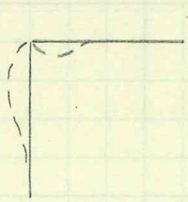
ROTATION CAN TAKE PLACE @ HINGE AND MUST BE INCLUDED IN DOF

21/30

Loads

DOFS

(1)

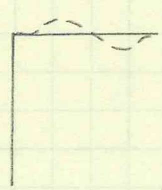


$$k_{11} = \frac{4EI}{h} + \frac{4EI}{L_1}$$

$$k_{12} = \frac{2EI}{L_1}$$

$$k_{13} = 0$$

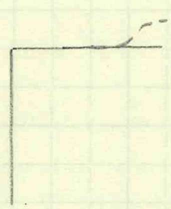
(2)



$$k_{22} = \frac{4EI}{L_1} + \frac{4EI}{L_2}$$

$$k_{32} = \frac{6EI}{L_2^2}$$

(3)



$$k_{33} = \frac{12EI}{L_2^3} + \frac{k}{2}$$

MIDTERM

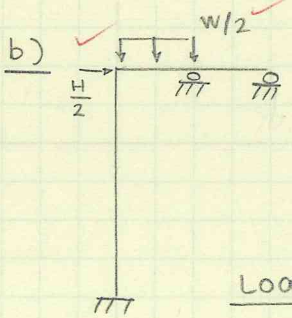
2. compiling terms,

$$
\underset{\sim}{K} = EI \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{4}{h} + \frac{4}{L_1} & \frac{2}{L_1} & 0 \\ & \frac{4}{L_1} + \frac{4}{L_2} & \frac{6}{L_2^2} \\ & & \frac{12}{L_2^3} + \frac{k}{2} \end{bmatrix} \end{array}
$$

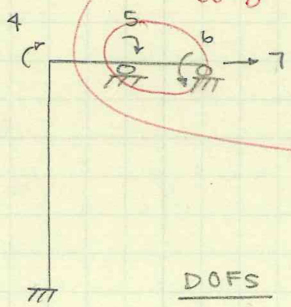
$$
\underset{\sim}{F} = \begin{bmatrix} \frac{wL_1}{4} \\ -\frac{wL_1}{4} \\ -P/2 \end{bmatrix}
$$

THESE ARE ~~???~~ FORCES, NOT MOMENTS THAT CORRESPOND TO DOF

b)



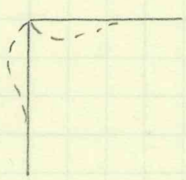
Loads



DOFs

why are some DOF ↻ WHILE OTHERS ARE ↺ ?

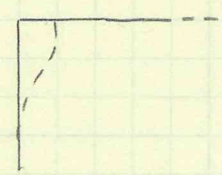
(4)



same as DOF 1

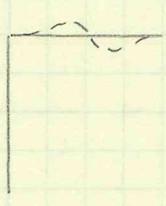
$k_{47} = \frac{-6EI}{h^2}$

(7)



$k_{77} = \frac{12EI}{h^3}$

(5)

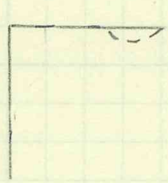


$k_{55} = k_{22}$

$k_{56} = \frac{-2EI}{L_2^2}$

$k_{57} = 0$

(6)



$k_{66} = \frac{4EI}{L_2}$

$k_{67} = 0$

CE381P: ANALYSIS | C.HOVELL 11Apr1107 | 4/

MIDTERM

2. compiling,

$$
\begin{array}{ccccc}
 & 4=1 & 5=2 & 6 & 7 \\
K_{ant1} = EI & \multicolumn{4}{c}{\left[\begin{array}{cccc}
4/h + 4/L_1 & 2/L_1 & 0 & -6/h^2 \\
 & 4/L_1 + 4/L_2 & -2/L_2^{\cancel{2}} & 0 \\
 & & 4/L_2 & 0 \\
 & & & 12/h^3
\end{array}\right]}
\end{array}
$$

NOT L^2

$$
F_{ant1} = \begin{bmatrix} \frac{wL_1}{4} \\ \frac{-wL_1}{4} \\ 0 \\ H/2 \end{bmatrix}
$$

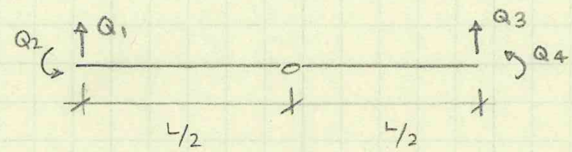
WRONG TERMS FOR MOMENT

- 5 total DOFs for system
- must be taken into account during assembly / addition of ~~influences~~

EACH CASE IS ANALYZED INDIVIDUALLY AND FINAL ANSWER OBTAINED BY APPROPRIATELY COMBINING TERMS

MIDTERM

3.



I'm not really sure how to approach this, so I tried a few things. Method 3 (next page, but similar to M1 ←) is probably my best guess, but it has problems.

15/20

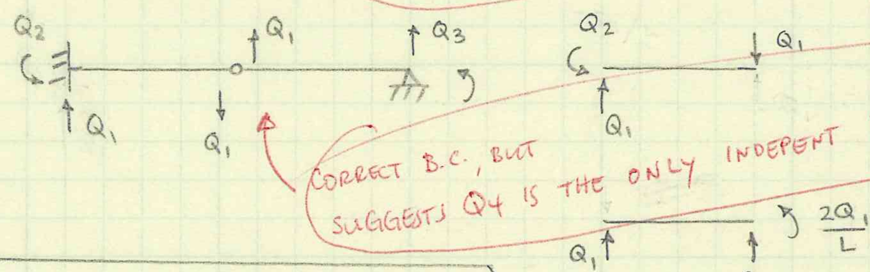
independent variables:

Q_1, Q_3 X No!

ONLY 1 INDEPENDENT FORCE QUANTITY

dependents:

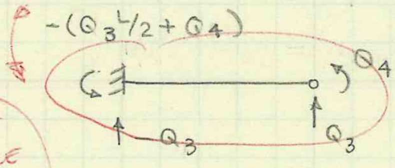
Q_2, Q_4



CORRECT B.C., BUT SUGGESTS Q4 IS THE ONLY INDEPENT FORCE QUANTITY

mathematically, you can write all forces in terms of one. ✓
However, that does not leave a stable structure.
YES IT DOES

Right side: Q_3, Q_4 ~~independent~~
No!



NOT SURE HOW YOU GOT THIS FBD

$$\curvearrowleft M(x) = -Q_3 x + Q_3 \frac{L}{2} + Q_4$$

$$\underset{\sim}{D} = \left[\, L/2 - x \quad 1 \,\right]$$

$$\underset{\sim}{f} = \frac{1}{EI} \int_0^L \underset{\sim}{D}^T \underset{\sim}{D}\, dx$$

WHY IS LEFT END FIXED FOR FBD OF RIGHT SIDE?

11: $(L/2)^2 - 2x(L/2) + x^2$

$$\frac{L^2}{4}x - \frac{L}{2}x^2 + \frac{x^3}{3}\Big|_0^L$$

$$\underset{\sim}{f} = \begin{bmatrix} L^3/12 & 0 \\ 0 & L \end{bmatrix} \frac{1}{EI}$$

12: $(L/2 - x) \rightarrow \frac{L}{2}x - \frac{x^2}{2}\Big|_0^L$

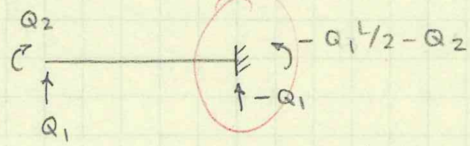
22: $x\Big|_0^L$

MIDTERM

3.

Left side:

?

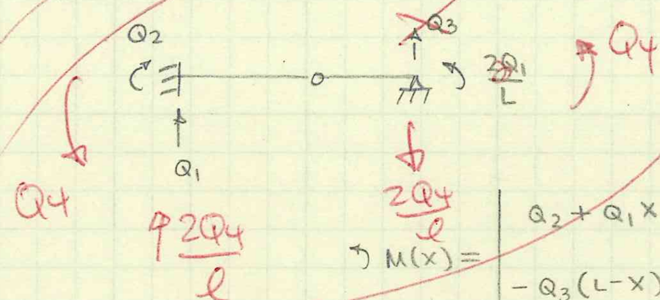


everything is the same, just backwards

$$f = \frac{1}{EI}\begin{bmatrix} L^3/12 & 0 \\ 0 & L \end{bmatrix}$$

can't combine, as they're modeled differently

Return to:



independent: 1, 2, 3

dependent: 4

everything in terms of Q_4, the independent QUANTITY

$$M(x) = \begin{cases} Q_2 + Q_1 x & x < L/2 \\ -Q_3(L-x) - \frac{2Q_1}{L} & x > L/2 \end{cases}$$

$$\underset{\sim}{D}_1 = \begin{bmatrix} x & 1 & 0 \end{bmatrix}$$

$$\underset{\sim}{D}_2 = \begin{bmatrix} -2/L & 0 & x-L \end{bmatrix}$$

I really need one equation, but moment eqs aren't continuous across hinges - SHOULD BE IF NO INTERMEDIATE LOAD!

$$\underset{\sim}{f} = \int_0^L \underset{\sim}{D}^T \underset{\sim}{D}\, dx \cdot \frac{1}{EI}$$

$$= \begin{bmatrix} L^3/3 & L^2/2 & 0 \\ & L & 0 \\ & & 0 \end{bmatrix}\frac{1}{EI}, \quad \begin{bmatrix} 4 & 0 & L \\ & 0 & 0 \\ & & L^3/3 \end{bmatrix}\frac{1}{EI}$$

using $\underset{\sim}{D}_1$ using $\underset{\sim}{D}_2$

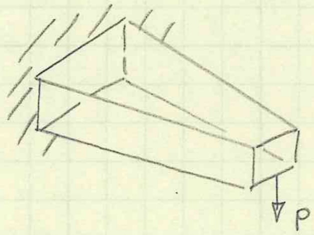
zeros on the diagonals generally aren't good. ✓

$$\phi = \begin{bmatrix} 2/L & 0 & 0 \end{bmatrix}, \text{ as } Q_4 = \frac{2Q_1}{L}$$

MIDTERM

27.5/30

4.



$b(x) = b_0(1 - x/2L)$
$h = h$

$$v(x)_{exact} = \frac{2PL^2}{EI_0}\left(x - \frac{x^2}{2L} + 2L\ln\left(1-\frac{x}{2L}\right) - x\ln\left(1-\frac{x}{2L}\right)\right)$$

$\underset{\sim}{k}$ formulation for a beam:

$$\underset{\sim}{k} = \int_0^L (\underset{\sim}{H}'')^T E(x) I(x) \underset{\sim}{H}''\, dx$$

$$H_3'' = 6/L^2 - 12x/L^3$$

$$H_4'' = -2/L + 6x/L^2$$

$$k_{11} = \int_0^L (H_3'')^2 E I(x)\, dx$$

$\quad \llcorner\ 1/12\, b_0 (1 - x/2L) h^3$

I_0

$$= \frac{E b_0 h^3}{12}\int_0^L (6/L^2 - 12x/L^3)^2 (1 - x/2L)\, dx$$

$$\left(\frac{36}{L^4} - \frac{72x}{L^5}\cdot 2 + \frac{144x^2}{L^6}\right)(1 - x/2L)$$

$$36/L^4 - \frac{144x}{L^5} + \frac{144x^2}{L^6} - \frac{18x}{L^5} + \frac{72x^2}{L^6} - \frac{72x^3}{L^7}$$

$$\frac{36x}{L^4} - \frac{72x^2}{L^5} + \frac{48x^3}{L^6} - \frac{9x^2}{L^5} + \frac{24x^3}{L^6} - \frac{18x^4}{L^7}\Big|_0^L$$

$$= \frac{9}{L^3}$$

$$k_{11} = \frac{3}{4}\frac{E b_0 h^3}{L^3} \quad \text{in terms of } I_0 = 1/12\, b h_0^3,$$

$$= \frac{9EI_0}{L^3}$$

$$\text{a. } k_{11} = \frac{9EI_0}{L^3}$$

MIDTERM

4.

$$\underset{\sim}{k} = EI_o \begin{bmatrix} 9/L^3 & -4/L^2 \\ -4/L^2 & 5/2L \end{bmatrix} \qquad \underset{\sim}{F} = \begin{bmatrix} -P \\ 0 \end{bmatrix}$$

using a calculator to solve,

$$\underset{\sim}{q} = \underset{\sim}{k}^{-1} \cdot \underset{\sim}{F}$$

$$q = \frac{P}{EI_o} \begin{bmatrix} -0.385L^3 \\ -0.615L^2 \end{bmatrix}$$

b)

$$v(L)_{exact} = \frac{2PL^2}{EI_o}\left[L - \frac{L}{2} + 2L\ln\frac{1}{2} - L\ln\frac{1}{2}\right]$$

$$= -0.386\frac{PL^3}{EI_o}$$

$$\frac{V_{exact} - V_{approx}}{V_{exact}} \cdot 100 = \frac{-0.386 - (-0.385)}{-0.386} \times 100 = 0.43\%$$

Error = 0.43%

c) Approximate midspan deflection:

$$u(x) = H_1\cancel{u_1} + H_2\cancel{u_2} + H_3u_3 + H_4u_4$$

$$u(L/2) = \left[\frac{3}{L^2}\cdot\left(\frac{L}{2}\right)^2 - 2\left(\frac{L/2}{L}\right)^3\right]u_3 + \left[-\frac{(L/2)^2}{L} + \frac{(L/2)^3}{L^2}\right]u_4$$

u_1 above $\qquad$ u_2 above

$$= \frac{1}{2}u_3 + (-0.125L)u_4$$

$$= \left[\frac{1}{2}(-0.385L^3) - 0.125(-0.615L^3)\right]\frac{P}{EI_o} = -0.1156\frac{PL^3}{EI_o}$$

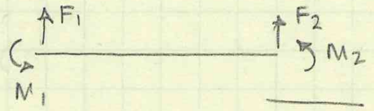
Exact: $\frac{PL^3}{EI_o}\cdot 2 \cdot \left[\frac{1}{2} - \frac{1}{8} + 2\ln(3/4) - \frac{1}{2}\ln(3/4)\right] = -0.113\frac{PL^3}{EI_o}$

Error = 2.3%

P.S. I hate my ~~calculator.~~

MIDTERM

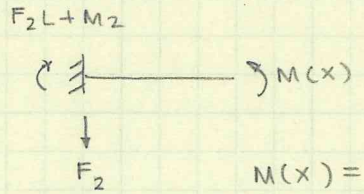
4.



independents

$F_1 = -F_2$

$M_1 = -(F_2 L + M_2)$



$M(x) = (F_2 L + M_2) - F_2 x$

$= M_2 + F_2 (L - x)$

$\underset{\sim}{D} = [\, L-x \quad 1 \,]$

$\underset{\sim}{f} = \int_0^L \underset{\sim}{D}^T \dfrac{\underset{\sim}{D}}{E\beta(x)}\, dx$

$I(x) = \frac{1}{12} h^3 b_0 (1 - x/2L)$

$f_{11} = \dfrac{12}{Eh^3 b_0} \, I_0 \int^L (L-x)^2 (1 - x/2L)^{-1} dx$

2.nd order Gauss:

$\int_{-1}^{1} fxn(x)\, dx = fxn(\sqrt{1/3}) + fxn(-\sqrt{1/3})\, dx$

LIMITS ARE 0, L

values are on a $[-1, 1]$ interval

from $[0; L]$, use $0.211L$ and $0.789L$

$(1/2 \pm \sqrt{1/3} \cdot \frac{1}{2})$

$fxn(x) = \dfrac{(L-x)^2}{1 - x/2L}$, $fxn(0.211L) = 0.696$

$(0.789L) = 0.074$

$\frac{L}{2}$ $\sum$ add, x 12

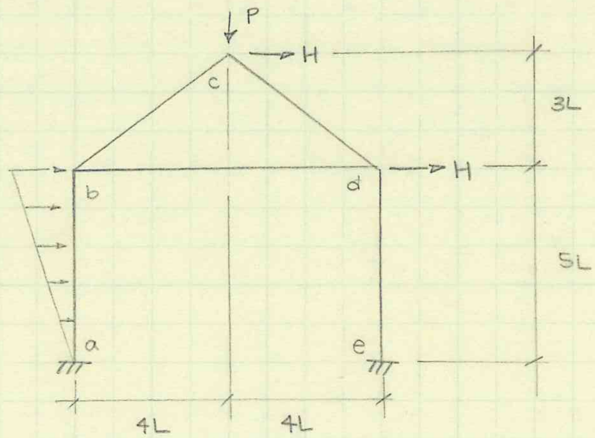
d) $f_{11} = \dfrac{9.23}{Eh^3 b_0}$

OFF BY A FACTOR OF 2, AND YOUR UNITS ARE NOT CORRECT

$f_{11} = .3848 \dfrac{\ell^3}{EI_0}$

HOMEWORK #1

10/10



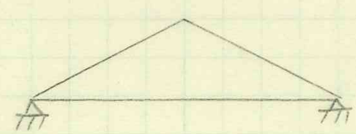
1. a) Statically indeterminate to the first degree.

$$\text{degree} = \frac{\#\text{ of reaction}}{\text{forces}} - \frac{\#\text{ of eq. of}}{\text{motion (3)}} - \#\text{ of hinges}$$

degree = 6 − 3 − 2

↳ one at the top of each column.

Alternatively, consider just the truss:



Both supports allow for shear and axial force – 4 reactions, 3 equations, 1 degree indeterminate.

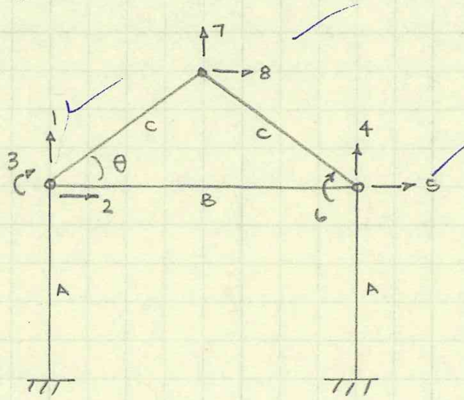
The columns with load are statically determinate.

b) Kinematically determinate to the 8th degree.

There are eight unknown displacements / rotations, as marked on the next page.

HOMEWORK #1

Numbering scheme:

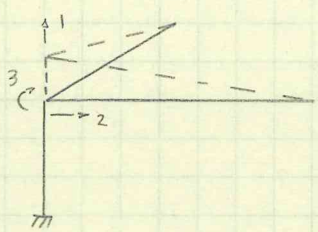


letters refer to lengths

A, E constant

consider each degree of freedom individually:

1)



$$K_{11} = \frac{EA}{L_A} + \frac{EA}{L_c}\sin\theta\cdot\sin\theta$$

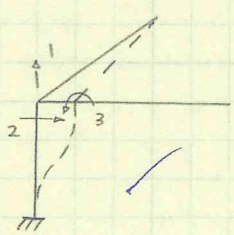
$$K_{12} = \frac{EA}{L_c}\sin\theta\cdot\cos\theta$$

$$K_{13} = K_{14} = K_{15} = K_{16} = 0$$

$$K_{17} = -\frac{EA}{L_c}\sin\theta\cdot\sin\theta$$

$$K_{18} = -\frac{EA}{L_c}\sin\theta\cdot\cos\theta$$

2)



$$K_{22} = \frac{12EI}{L_A^3} + \frac{EA}{L_B} + \frac{EA}{L_c}\cos\theta\cdot\cos\theta$$

$$K_{23} = \frac{-6EI}{L_A^2}$$

$$K_{24} = 0$$

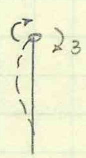
$$K_{25} = \frac{-EA}{L_B}$$

$$K_{26} = 0$$

$$K_{27} = \frac{-EA}{L_c}\cos\theta\cdot\sin\theta$$

$$K_{28} = -\frac{EA}{L_c}\cos\theta\cdot\cos\theta$$

3)



$$K_{33} = \frac{4EI}{L_A}$$

$K_{34} \ldots = 0$ (all zero)

HOMEWORK #1

Stiffness matrix

DOF 4 is very similar to 1

4)

$$k_{44} = \frac{EA}{L_A} + \frac{EA}{L_c} \sin\theta \cdot \sin\theta$$

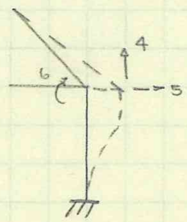
$$k_{45} = \frac{-EA}{L_c} \sin\theta \cdot \cos\theta$$

$$k_{46} = 0$$

$$k_{47} = -\frac{EA}{L_c} \sin\theta \cdot \sin\theta$$

$$k_{48} = \frac{EA}{L_c} \sin\theta \cdot \cos\theta$$

5)



$$k_{55} = \frac{12EI}{L_A^3} + \frac{EA}{L_B} + \frac{EA}{L_c} \cos\theta \cdot \cos\theta$$

$$k_{56} = \frac{-6EI}{L_A^2}$$

$$k_{57} = \frac{EA}{L_c} \cos\theta \cdot \sin\theta$$

$$k_{58} = \frac{-EA}{L_c} \cos\theta \cdot \cos\theta$$

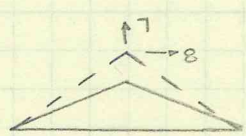
6)

$$k_{66} = \frac{4EI}{L_A}$$

$$k_{67} = 0$$

$$k_{68} = 0$$

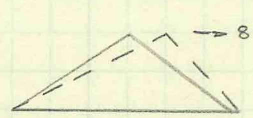
7)



$$k_{77} = \frac{EA}{L_c} \sin\theta \cdot \sin\theta \cdot 2$$

$k_{78} = 0$ – forces cancel

8)



$$k_{88} = \frac{EA}{L_c} \cos\theta \cdot \cos\theta \cdot 2$$

HOMEWORK #1

Assembly of the stiffness matrix, considering:

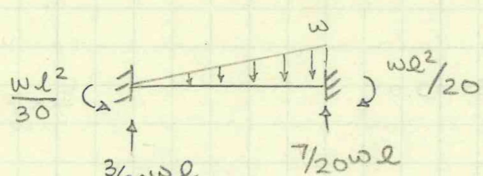
$L_A = L_c = 5\ell,\ L_B = 8\ell$ (in following, $L = \ell$)

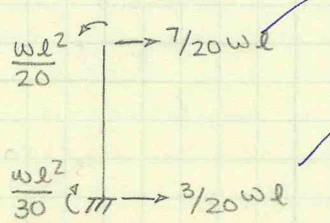
$\sin\theta = 3/5,\ \cos\theta = 4/5$

2)

$$\underset{\sim}{K} = \begin{bmatrix} \frac{9EA}{125L} & \frac{12EA}{125L} & 0 & 0 & 0 & 0 & \frac{-9EA}{125L} & \frac{-12EA}{125L} \\ \frac{12EA}{125L} & \frac{12EI}{125L^3}+\frac{EA}{8L}+\frac{16EA}{125L} & \frac{-6EI}{25L^2} & 0 & \frac{-EA}{8L} & 0 & \frac{-12EA}{125L} & \frac{-16EA}{125L} \\ 0 & \frac{-6EI}{25L^2} & \frac{4EI}{5L} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{9EA}{125L} & \frac{-12EA}{125L} & 0 & \frac{-9EA}{125L} & \frac{12EA}{125L} \\ 0 & \frac{-EA}{8L} & 0 & \frac{-12EA}{125L} & \frac{12EI}{125L^3}+\frac{EA}{8L}+\frac{16EA}{125L} & \frac{-6EI}{25L^2} & \frac{12EA}{125L} & \frac{-16EA}{125L} \\ 0 & 0 & 0 & 0 & \frac{-6EI}{25L^2} & \frac{4EI}{5L} & 0 & 0 \\ \frac{-9EA}{125L} & \frac{-12EA}{125L} & 0 & \frac{-9EA}{125L} & \frac{12EA}{125L} & 0 & \frac{18EA}{125L} & 0 \\ \frac{-12EA}{125L} & \frac{-16EA}{125L} & 0 & \frac{12EA}{125L} & \frac{-16EA}{125L} & 0 & 0 & \frac{32EA}{125L} \end{bmatrix}$$

Distributed load.





add to load vector

2)

$$\underset{\sim}{F} = \begin{bmatrix} 0 \\ 7/20\,w(5L) \\ -w(5L)^2/20 \\ 0 \\ H \\ 0 \\ -P \\ H \end{bmatrix}$$

HOMEWORK #1

Given:

$E = 30{,}000\ ksi$
$A = 9\ in^2$
$I = 432\ in^4$
$L = 4ft$ or $48in$

$w = 3k/ft$ or $0.25\ k/in$
$P = 50K$
$H = 20K$

Numerically,

K=

1530	540	0	0	0	0	-405	-540
540	1434.375	-1350	0	-703.125	0	-540	-720
0	-1350	216000	0	0	0	0	0
0	0	0	1530	-540	0	-405	540
0	-703.125	0	-540	1434.375	-1350	540	-720
0	0	0	0	-1350	216000	0	0
-405	-540	0	-405	540	0	810	0
-540	-720	0	540	-720	0	0	1440

units vary

F=

0	K
21	K
-720	K·in
0	K
20	K
0	K·in
-50	K
20	K

u=

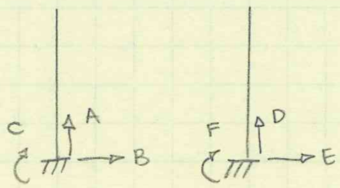
-0.015556
10.01955
0.059289
-0.028889
10.06934
0.062933
-0.117148
10.06333

in

$F = K \cdot u$

HOMEWORK #1

Now consider supports:



$$R_A = \frac{-EA}{5L} \cdot u_1 = 17.5\ K$$

$$R_B = \frac{-12EI}{(5L)^3} u_2 + \frac{6EI}{(5L)^2} u_3 - \tfrac{3}{20} w(5L) = -41.68\ K$$

from distributed load

$$M_c = \frac{-6EI}{(5L)^2} u_2 + \frac{2EI}{5L} \cdot u_3 - \tfrac{1}{30} \cdot w(5L)^2 = -633.6\ k \cdot ft$$

$$R_D = \frac{-EA}{5L} \cdot u_4 = 32.5\ K$$

$$R_E = \frac{-12EI}{(5L)^3} u_5 + \frac{6EI}{(5L)^2} u_6 = -28.32\ K$$

$$M_F = \frac{-6EI}{(5L)^2} u_5 + \frac{2EI}{5L} u_6 = -566.4\ k \cdot ft$$

check: $\Sigma F_y = 50$
$\Sigma F_x = 70$
$\Sigma M = 0$

$$RR = \begin{vmatrix} -1125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -11.25 & 1350 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1350 & 108000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1125 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -11.25 & 1350 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1350 & 108000 & 0 & 0 \end{vmatrix}$$

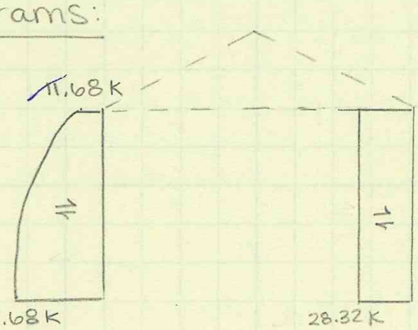
$$RFs = \begin{vmatrix} 17.5 \\ -41.67997 \\ -633.5995 \\ 32.5 \\ -28.32003 \\ -566.4005 \end{vmatrix} \begin{matrix} k \\ k \\ k\text{-}ft \\ k \\ k \\ k\text{-}ft \end{matrix}$$

$$\underbrace{RFs}_{\text{reaction forces}} = RR \cdot \underbrace{u}_{\text{reaction responses (eqs. above)}}$$

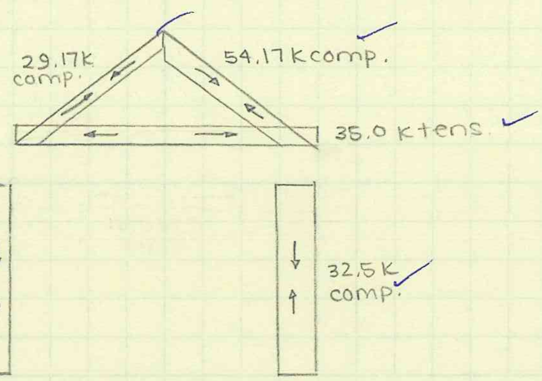
HOMEWORK #1

Using reaction values,

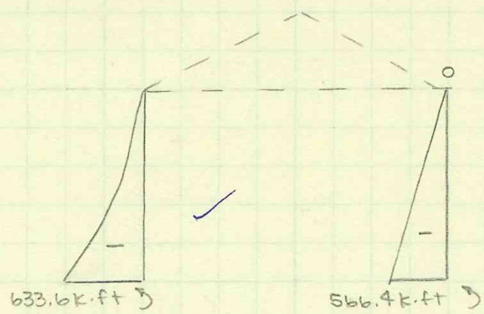
Shear Diagrams:



Axial Diagrams



Bending Moments

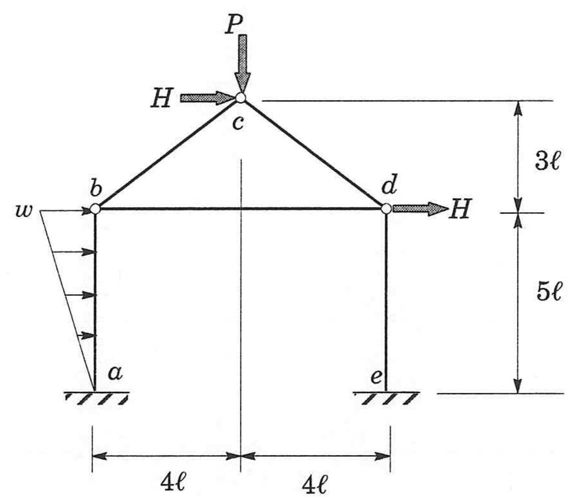


University of Texas
CE 381P

Department of Civil Engineering
Instructor: E. B. Williamson

Homework 1 **Due: 26 Jan 2007**

The structural frame shown in the figure below consists of a truss that is supported on two columns. All members have modulus E, cross-sectional area A, and moment of inertia I. The frame is subjected to point loads P and H as well as a distributed load that varies linearly over the height of the column on the left. All dimensions are shown on the sketch.



1. Answer the following questions about the structure shown in the figure:

    (a). Is the structure statically determinate or indeterminate? If indeterminate, indicate the degree of static indeterminacy.

    (b). Is the structure kinematically determinate or indeterminate? If indeterminate, indicate the degree of kinematic indeterminacy (i.e., the number of degrees of freedom). Account for axial and flexural deformations in computing your answer.

2. Develop the stiffness matrix and load vector for the structure. Be sure to show your degree of freedom (DOF) and element numbering scheme on a sketch of the structure. Shear deformations in the columns can be ignored, but you should account for axial and flexural deformations.

3. Draw the shear force, axial force, and bending moment diagrams for the columns, and indicate the value of the axial force in all members of the truss. Use the following values for the system parameters:

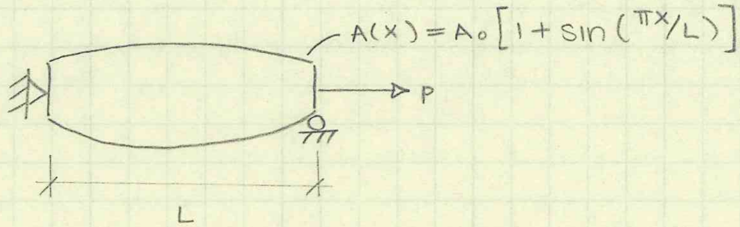
$E = 30,000\ ksi \quad I = 432\ in^4 \quad w = 3\ k/ft \quad H = 20\ kips$

$A = 9\ in^2 \quad \ell = 4\ ft \quad P = 50\ kips$

In order to solve the governing system of equations, you are encouraged to utilize software such as EXCEL or MATHCAD rather than using hand-based methods.

HOMEWORK #2

10/10



1. Boundary value problem

$$\frac{d}{dx}\left[E \cdot A(x) \cdot \frac{du}{dx}\right] = -p(x)$$

$$\frac{d}{dx}\left[E \cdot A_0\left[1 + \sin(\pi \cdot x/L)\right] \cdot \frac{du}{dx}\right] = 0$$

$$E \cdot A_0 \int d\left[1 + \sin(\pi x/L)\right] \frac{du}{dx} = \int 0 \, dx$$

$$E \cdot A_0 \left[1 + \sin(\pi x/L)\right] \frac{du}{dx} = c$$

$$du = \frac{c \cdot dx}{E \cdot A_0 \left[1 + \sin(\pi x/L)\right]}$$

$$B_0 = \frac{c}{EA_0}$$

$$du = \frac{B_0 \, dx}{1 + \sin(\pi x/L)}$$

Integrating,

$$u(x) = -B_0 \frac{L}{\pi} \tan\left(\pi/4 - \frac{\pi x}{2L}\right) + B_1$$

considering $u(0) = 0$,

$$B_1 = \frac{L}{\pi} B_0$$

$$N(L) = P, \quad N(x) = E \cdot A(x) \cdot u'(x)$$

$$u'(L) = B_0$$
$$A(L) = A_0$$

$$B_0 = \frac{P}{EA_0}, \quad B_1 = \frac{LP}{EA_0 \pi}$$

$$\boxed{u(x) = \frac{L}{\pi} \frac{P}{EA_0}\left[1 - \tan\left(\pi/4 - \frac{\pi}{2L} x\right)\right]}$$

HOMEWORK #2

coefficients:

2. Flexibility coefficient

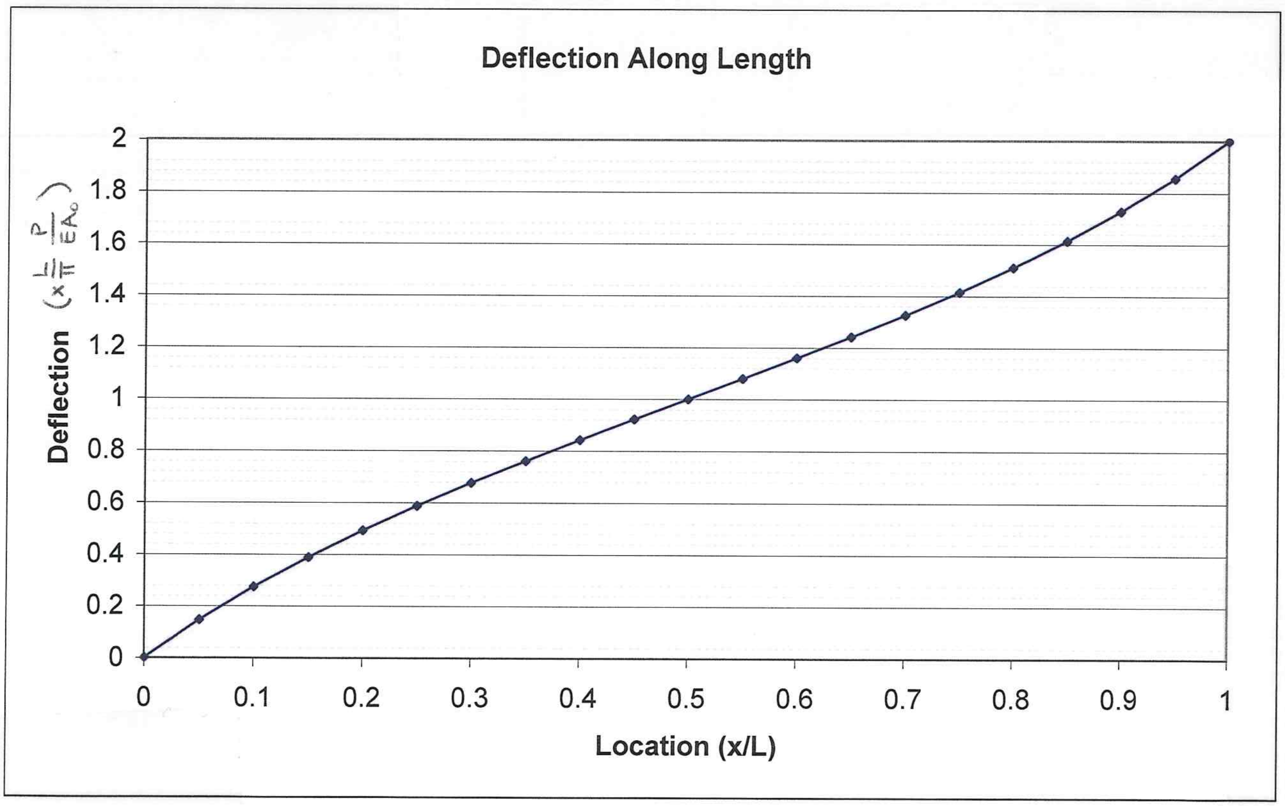
$$u(L) = P \cdot \frac{L/\pi}{E \cdot A_o}(2)$$

$$\underline{\underline{\frac{2L/\pi}{E \cdot A_o}}}$$

3. Stiffness coefficient

$$K = \frac{1}{\text{flex}} = EA_o \frac{\pi}{2L}$$

$$\underline{\underline{K = EA_o \, \pi/2L}}$$



HOMEWORK #2

4. Using P.V.W.

$$A(x) = A_0\left[1+\sin(\pi x/L)\right]$$

$$u(x) = a_0 + a_1 x$$
$$\bar{u}(x) = \bar{a}_0 + \bar{a}_1 x$$

considering essential boundary condition

$u(0)=0$,

$$u(x) = a_1 x \;,\; u'(x) = a_1$$
$$\bar{u}(x) = \bar{a}_1 x \;,\; \bar{u}'(x) = \bar{a}_1$$

$$\delta W_{int} = \int E\cdot A(x)\cdot u'(x)\cdot \bar{u}'(x)$$

$$= E\cdot A_0\cdot a_1\cdot \bar{a}_1 \int_0^L 1+\sin(\pi x/L)\,dx$$

$$\hookrightarrow x - \frac{L}{\pi}\cos(\pi x/L)\Big|_0^L = L + \frac{L}{\pi} + \frac{L}{\pi}$$

$$= E\cdot A_0\cdot a_1\cdot \bar{a}_1\cdot L\,(1+2/\pi)$$

$$\delta W_{ext} = P\cdot \bar{u}(L)$$

$$= P\cdot \bar{a}_1\cdot L$$

Establishing equilibrium,

$$EA_0 a_1 \bar{a}_1 L\,(1+2/\pi) = P\bar{a}_1 L$$

$$\left[EA_0 a_1 L(1+2/\pi) - PL\right]\bar{a}_1 = 0$$

$$a_1 = \frac{P}{EA_0(1+2/\pi)} \rightarrow \frac{2+\pi}{\pi}$$

$$\boxed{u(x) = \frac{\pi}{2+\pi}\,\frac{P}{EA_0}\cdot x}$$

compare BVP to approximate

$$\frac{2L}{\pi}\cdot\frac{P}{EA_0} = 0.637 \qquad \frac{\pi}{2+\pi}\cdot L = 0.611$$

$$\frac{0.637-0.611}{0.637} = 0.0408$$

$$\boxed{\text{Error} = 4.1\%}$$

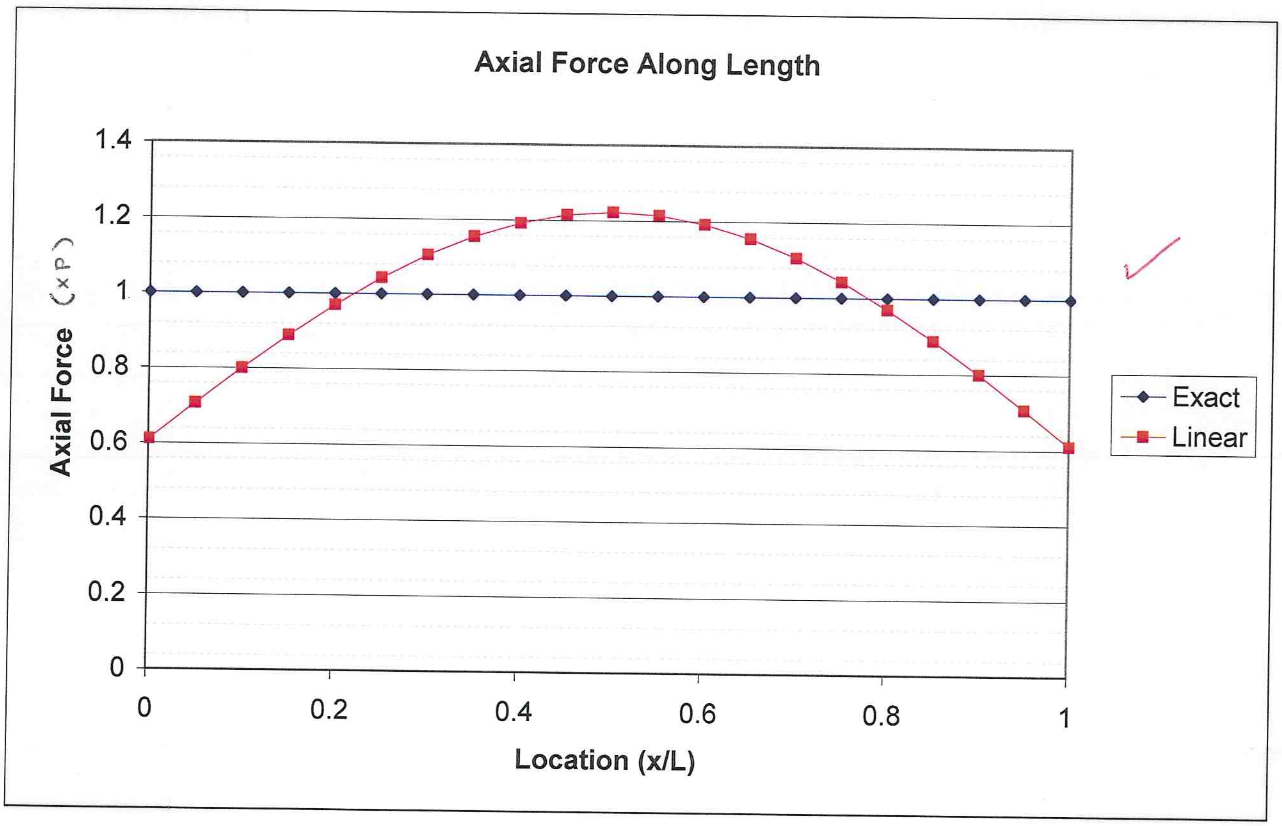
HOMEWORK #2

4. Axial force variation

$$N(x) = E \cdot A(x) \cdot u'(x)$$

$$u'_{exact} = \frac{1}{2} \frac{P}{EA_o} \left[1 + \tan\left(\pi/4 - \pi/2L \cdot x\right)^2 \right]$$

$$u'_{lin} = \frac{\pi}{2+\pi} \frac{P}{EA_o}$$



$$N_{lin} = P \cdot \frac{\pi}{2+\pi} \left[1 + \sin\left(\pi x/L\right) \right]$$

$$N_{exact} = \frac{P}{2} \left[1 + \tan\left(\pi/4 - \pi/2L \cdot x\right)^2 \right] \left[1 + \sin\left(\pi x/L\right) \right]$$

HOMEWORK #2

5. P.V.W., using quadratic approximations

$$u(x) = a_0 + a_1 x + a_2 x^2$$
$$\bar{u}(x) = \bar{a}_0 + \bar{a}_1 x + \bar{a}_2 x^2$$

using essential boundary, $a_0, \bar{a}_0$ again $= 0$

$$\delta W_{int} = E \cdot A_0 \int_0^L \left[1 + \sin(\pi x/L)\right] \underbrace{u'}_{a_1 + 2a_2 x} \cdot \underbrace{\bar{u}'}_{\bar{a}_1 + 2\bar{a}_2 x} dx$$

$$= E \cdot A_0 \int_0^L \left[1 + \sin(\pi x/L)\right](a_1 + 2a_2 x)(\bar{a}_1 + 2\bar{a}_2 x)\, dx$$

using mathcad to solve,

$$= EA_0 \frac{L}{\pi}\Big[a_1\bar{a}_1\pi + 4a_2\bar{a}_2 L^2 - 8a_2\bar{a}_2\frac{L^2}{\pi^2} + (a_1\bar{a}_2 L + a_2\bar{a}_1 L + \tfrac{4}{3}a_2\bar{a}_2 L^2)\pi$$
$$+ a_1\bar{a}_1 + 2a_1\bar{a}_2 L + 2a_2\bar{a}_1 L + a_1\bar{a}_1 - 8a_2\bar{a}_2\frac{L^2}{\pi^2}\Big]$$

$$= EA_0\left[\bar{a}_1(a_1 L + a_2 L^2)(1 + 2/\pi) + \bar{a}_2\left[a_1 L^2(1 + 2/\pi) + a_2 L^3(4/3 + 4/\pi - 16/\pi^3)\right]\right]$$

$$\delta W_{ext} = P\bar{u}(L) = P \cdot \bar{a}_1 L + P\bar{a}_2 L^2$$

Applying equilibrium, either $\bar{a}_1 = \bar{a}_2 = 0$ or

$$(a_1 L + a_2 L^2)(1 + 2/\pi) - PL = 0$$

$$a_1 L^2(1 + 2/\pi) + a_2 L^3(\text{const.}) - PL^2 = 0$$

$$EA_0 \begin{bmatrix} L + 2/\pi L & L^2 + 2/\pi L^2 \\ L^2(1 + 2/\pi) & \underbrace{\text{const.}}_{2.091} \cdot L^3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} PL \\ PL^2 \end{bmatrix}$$

$a_1 = 0.611$
$a_2 = 0$

HOMEWORK #2

5. (cont'd)

$a_1 = 0.611 = \pi/(2+\pi)$

$a_2 = 0$

> quadratic approximation matches linear

$$u(x) = \frac{\pi}{2+\pi} \frac{P}{EA_0} \cdot x$$

Error, axial force variation are the same.

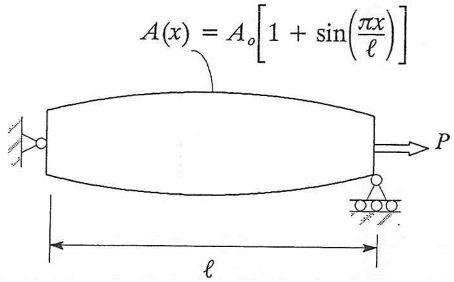
University of Texas
Dept. of Civil, Arch. & Env. Eng.

Instructor: E. B. Williamson
CE 381P

CE 381P Computer Methods in Structural Analysis

Homework 2 **Due: 12 Feb 2007**

The axial force member shown in the figure below has a cross-sectional area that varies trigonometrically over the length of the member. For the following problems, assume that the material response is linear and elastic.



1. Set up and solve the Boundary Value Problem (BVP) for the truss member shown in the figure (i.e., determine the displacement as a function of x). The solution to the resulting differential equation will require the integration of a complicated function. Thus, you are encouraged to make use of published solutions that are widely available. A good website that shows the solution to a large number of indefinite integrals can be found at `http://www.sosmath.com/tables/tables.html`.

2. Compute the *flexibility coefficient*.

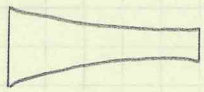
3. Compute the *stiffness coefficient*.

4. Using the Principle of Virtual Displacements, approximate a solution to the problem by assuming that both the real displacements and virtual displacements vary linearly over the length of the member. Compute the percentage error between your approximate solution and the exact solution determined by solving the BVP for the displacement at the end of the bar. In addition, graph the axial force variation in the bar and compare it to the exact solution.

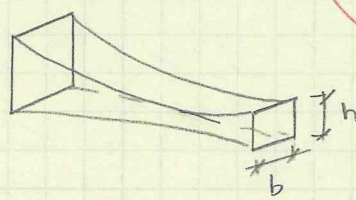
5. Repeat Problem 4. but assume that both the real displacement and virtual displacement vary quadratically (be sure to use a *complete* quadratic expression that includes a constant, linear, and quadratic term) over the length of the member.

18.5/20

HOMEWORK #3



$h(x) = h_o(1 - x/3L)^2$



$A(x) = h(x)\,b(x)$

$b(x) = b_o(1 - \sin \pi x/6L)^2$

a) Linear elements

$L_1 = 1 - x/L$ $\quad B_1 = -1/L$ $\quad u(x) = a_o + a_1 x$

$L_2 = x/L$ $\quad B_2 = 1/L$ $\quad \bar{u}(x) = \bar{a}_o + \bar{a}_1 x$

$$k = \int_0^L \underset{\sim}{B}^T E A(x) \underset{\sim}{B}\, dx$$

$\hookrightarrow A(x) = b_o h_o (1 - \sin \pi x/6L)^2 (1 - x/3L)^2$

$$k = \int_0^L \begin{bmatrix} -1/L \\ 1/L \end{bmatrix} E b_o h_o \underbrace{(1-\sin \pi x/6L)^2 (1 - x/3L)^2}_{\text{only part that varies with } x,\ \text{same value for all } k \text{ terms}} \begin{bmatrix} -1/L & 1/L \end{bmatrix} dx$$

$$\mathbb{A} = \int_0^L (1 - \sin \pi x/6L)^2 (1 - x/3L)^2\, dx$$

$$k = E b_o h_o \mathbb{A} \begin{bmatrix} 1/L^2 & -1/L^2 \\ -1/L^2 & 1/L^2 \end{bmatrix} = \frac{E b_o h_o \mathbb{A}}{L^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Now: solve for $\mathbb{A}$

HOMEWORK #3

Solving for A using Gaussian Quadrature

$$A = \int_0^L (1-\sin{\pi x/6L})^2 (1-x/3L)^2 \, dx$$

using the first term of the sine series (x), equation order would be 5 (x^4)

Need 3rd order Gauss (2N = 6 > 5)

$$\int f(x)\,dx = 5/9 f(-\sqrt{3/5}) + 8/9 f(0) + 5/9 f(\sqrt{3/5})$$

From [-1, 1] to [0, L]:

$$x(\xi) = \tfrac{1}{2}\left[a(1-\xi) + b(1+\xi)\right]$$

$$x(-\sqrt{3/5}) = \tfrac{1}{2}\left[0 + L(1-\sqrt{3/5})\right] = 0.113L$$

$$x(\sqrt{3/5}) = 0.887L$$

$$A = \left[5/9(0.82) + 8/9(0.38) + 5/9(0.151)\right]\frac{L}{2}$$

$$A = 0.439L$$

$$k = \frac{EA_0}{L}(0.439)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

since $u_1 = 0$, use k_{22} only to find u_2:

$$k_{22} u_2 = F_2$$

$$u_2 = \frac{P}{k_{22}} \qquad \underline{u_2 = 2.276 \frac{PL}{EA_0}}$$

virtual displacement matches real

HOMEWORK #3

b) use quadratic element
- base equation is the same
- $\underset{\sim}{B}$ changes:

$$N_1 = 1 - 3^x/_L + 2(^x/_L)^2 \qquad B_1 = -^3/_L + {^{4x}}/_{L^2}$$
$$N_2 = 4\left[{^x}/_L - ({^x}/_L)^2\right] \qquad B_2 = {^4}/_L - {^{8x}}/_{L^2}$$
$$N_3 = -{^x}/_L + 2({^x}/_L)^2 \qquad B_3 = -{^1}/_L + {^{4x}}/_{L^2}$$

$\underset{\sim}{B}$ now varies with x, cannot betaken out of integral

$$K_{11} = E b_o h_o \int_0^L (-3/L + 4x/L^2)^2 (1 - \sin{}^{\pi x}/_{6L})^2 (1 - {}^x/_{3L})^2 \, dx$$

again considering only first term of sine, order becomes 7 (x^6)

Need 4th order Gauss (2N = 8)

ξ value	weight w_i
$[(3+2\sqrt{6/5})/7]^{1/2}$	$(3-\sqrt{5/6})/6$
$[(3-2\sqrt{6/5})/7]^{1/2}$	$(3+\sqrt{5/6})/6$
$-[(3-2\sqrt{6/5})/7]^{1/2}$	$(3+\sqrt{5/6})/6$
$-[(3+2\sqrt{6/5})/7]^{1/2}$	$(3-\sqrt{5/6})/6$

solving in mathcad,

$$k = \frac{L}{2}\left[w_1 f(\xi_1) + w_2 f(\xi_2) + w_3 f(\xi_3) + w_4 f(\xi_4) \right]$$

$$k = \frac{EA_o}{L}\begin{bmatrix} 1.663 & -1.871 & 0.208 \\ -1.871 & 2.591 & -0.719 \\ 0.208 & -0.719 & 0.512 \end{bmatrix}$$

$\underset{\sim}{k}\underset{\sim}{u} = \underset{\sim}{F}$, ignore first row, column, $F' = \begin{bmatrix} 0 \\ P \end{bmatrix}$

$$\underset{\sim}{u}' = \underset{\sim}{k}'^{-1} \cdot \underset{\sim}{F}'$$

$$\underset{\sim}{u} = \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.892 \\ 3.212 \end{bmatrix} \frac{PL}{EA_o}$$

virtual disp again same as real

HOMEWORK #3

c) Approximation using h-type refinement

$$k = \frac{EA_e}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{for each element}$$

Try 4 elements
- calculate h, b at midpoint of element
- K dependent on A, Le

$$\left(\frac{A}{L}\right)_1 = 3.208,\ \left(\frac{A}{L}\right)_2 = 1.984,\ \left(\frac{A}{L}\right)_3 = 1.154,\ \left(\frac{A}{L}\right)_4 = 0.624$$

Assembling K:

$$K = E\begin{bmatrix} 3.208 & -3.208 & 0 & 0 & 0 \\ & 5.193 & -1.984 & 0 & 0 \\ & & 3.1384 & -1.154 & 0 \\ & & & 1.779 & -0.624 \\ & & & & 0.624 \end{bmatrix}$$

Ignoring row, col 1, F=P at end,

$$u = \begin{bmatrix} 0.3116 \\ 0.8156 \\ 1.682 \\ 3.284 \end{bmatrix}\frac{PL}{EA_0}$$

---

try 6, 8, 10 elements to confirm accuracy
excel-calculated matrices, graph on following pages

HOMEWORK #3

Six Elements L= 0.1667

k6=

5.18733	-5.18733	0	0	0	0	0
-5.18733	8.998753	-3.81142	0	0	0	0
0	-3.81142	6.543007	-2.73158	0	0	0
0	0	-2.73158	4.635563	-1.90398	0	0
0	0	0	-1.90398	3.190123	-1.28614	0
0	0	0	0	-1.28614	2.124439	-0.83829
0	0	0	0	0	-0.83829	0.838294

Eight Elements L= 0.125

k8=

7.17643	-7.17643	0	0	0	0	0	0	0
-7.17643	12.89687	-5.72044	0	0	0	0	0	0
0	-5.72044	10.21931	-4.49887	0	0	0	0	0
0	0	-4.49887	7.985846	-3.48697	0	0	0	0
0	0	0	-3.48697	6.147116	-2.66014	0	0	0
0	0	0	0	-2.66014	4.654606	-1.99446	0	0
0	0	0	0	0	-1.99446	3.46154	-1.46708	0
0	0	0	0	0	0	-1.46708	2.52364	-1.05656
0	0	0	0	0	0	0	-1.05656	1.056562

Ten Elements L= 0.10

k10=

0.916984	-0.91698	0	0	0	0	0	0	0	0	0
-0.91698	1.683421	-0.76644	0	0	0	0	0	0	0	0
0	-0.76644	1.401674	-0.63524	0	0	0	0	0	0	0
0	0	-0.63524	1.157039	-0.5218	0	0	0	0	0	0
0	0	0	-0.5218	0.946347	-0.42455	0	0	0	0	0
0	0	0	0	-0.42455	0.766444	-0.3419	0	0	0	0
0	0	0	0	0	-0.3419	0.614227	-0.27233	0	0	0
0	0	0	0	0	0	-0.27233	0.486686	-0.21436	0	0
0	0	0	0	0	0	0	-0.21436	0.38093	-0.16657	0
0	0	0	0	0	0	0	0	-0.16657	0.294218	-0.12765
0	0	0	0	0	0	0	0	0	-0.12765	0.127645

Deflections

u6=

0
0.1928
0.4551
0.8212
1.3465
2.1240
3.3169

1% increase

u8=

0
0.1393
0.3142
0.5364
0.8232
1.1991
1.7005
2.3822
3.3286

0.35% increase

u10=

0
0.1091
0.2395
0.3969
0.5886
0.8241
1.1166
1.4838
1.9503
2.5507
3.3341

0.16% increase

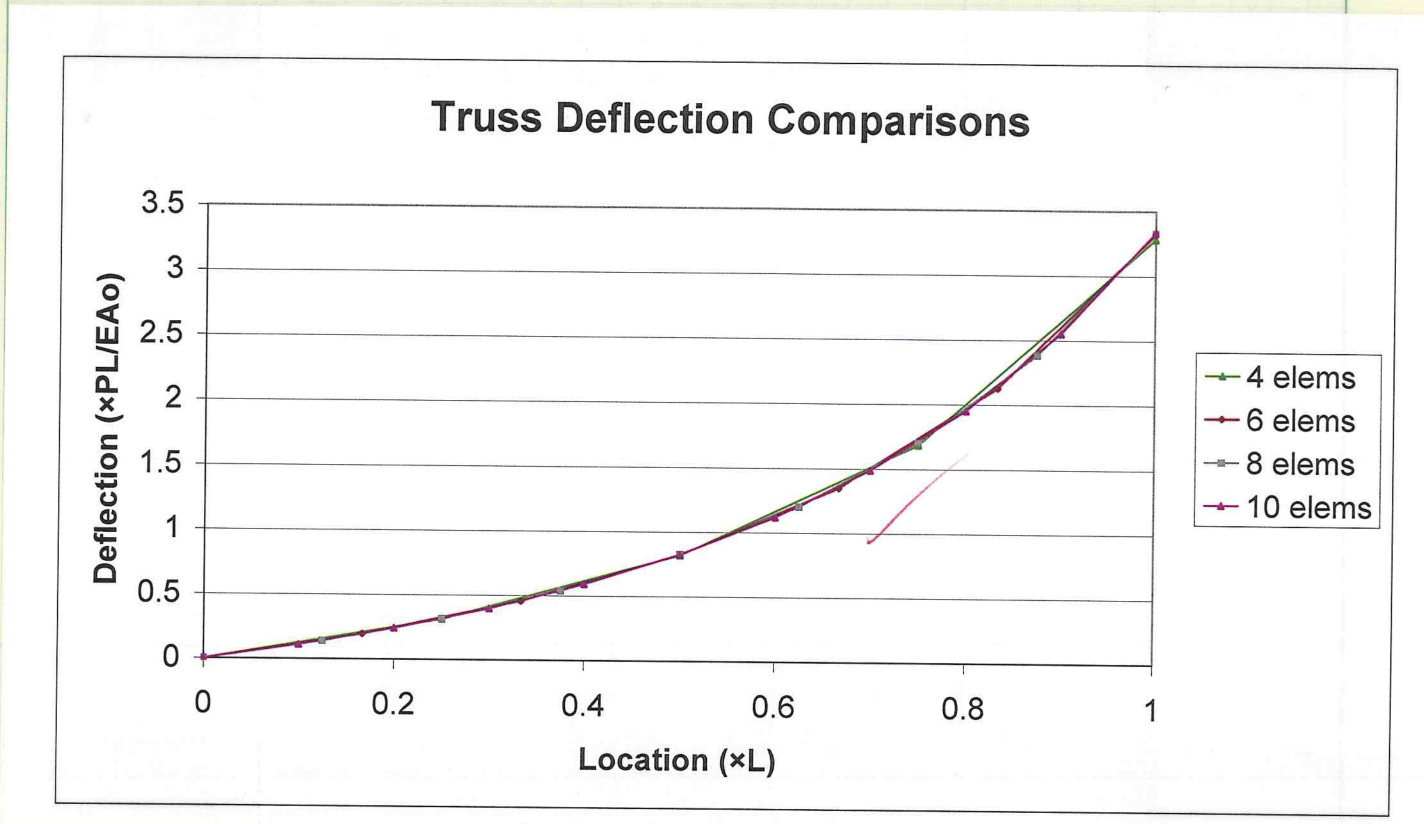
CE 381P: ANALYSIS

C. HOVELL
21 Feb 07

6/10

HOMEWORK #3

comparison of truss versions:



c) Four elements would generally be okay for estimation.

CE 381P: ANALYSIS

C. HOVELL
21 Feb 07

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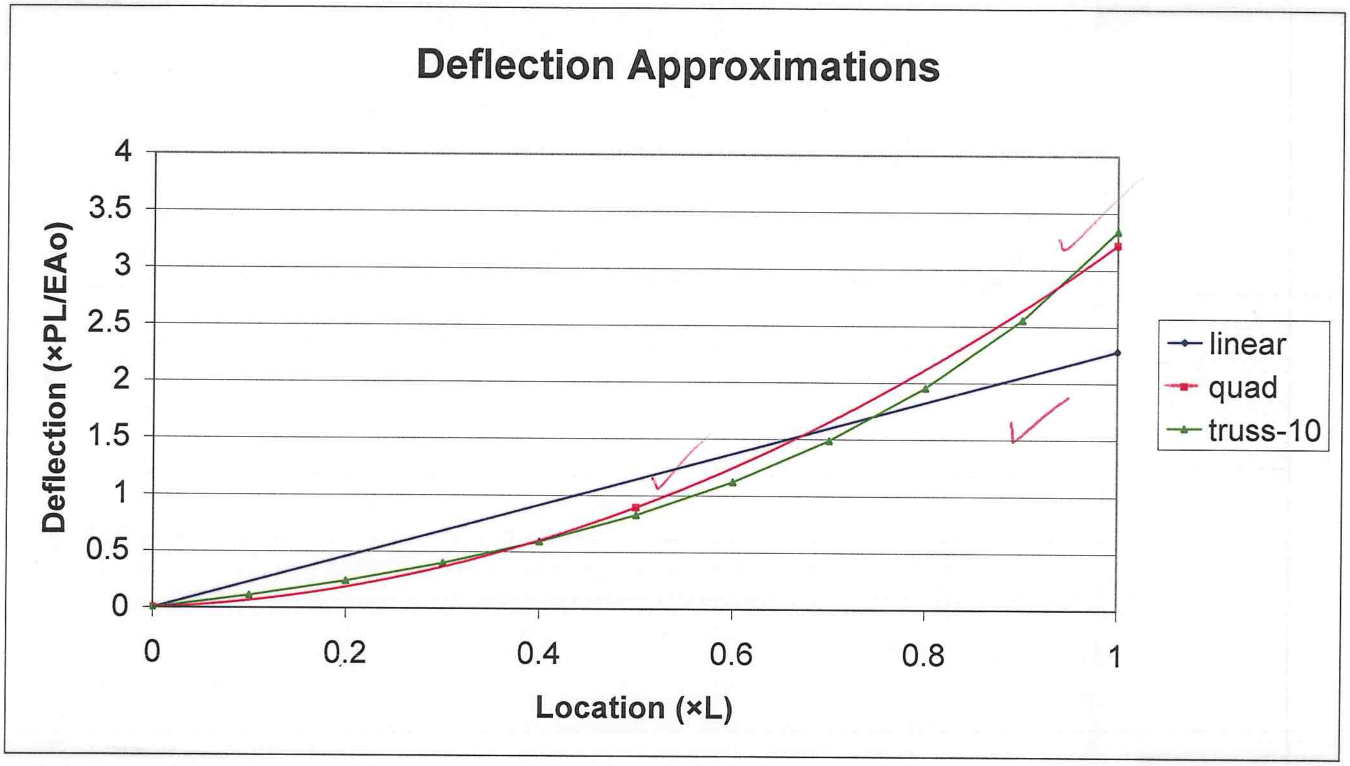
HOMEWORK #3

d) Graphical comparison of estimation methods

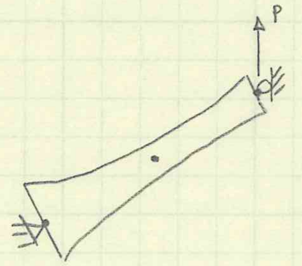


HOMEWORK #3

e)

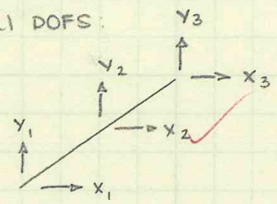
$$K_b = \frac{EA_o}{L}\begin{bmatrix} 1.663 & -1.871 & 0.208 \\ & 2.591 & -0.719 \\ & & 0.511 \end{bmatrix}$$

stiffness matrix for quadratic element in part b

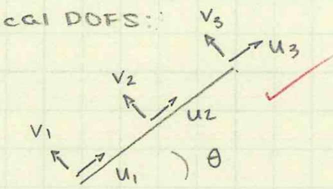


$A(x) = b(x)h(x)$

global DOFS:



local DOFS:



Need to establish relationship

$$u_1\cos\theta - v_1\sin\theta = x_1$$
$$u_1\sin\theta + v_1\cos\theta = y_1$$

or,

$$\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ & & c & -s & 0 & 0 \\ & & s & c & 0 & 0 \\ & & & & c & -s \\ & & & & s & c \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

global $\underset{\sim}{x}$ = transform $\underset{\sim}{T}^{-1}$ · local $\underset{\sim}{u}$

$$T = \text{inverse} \uparrow = \begin{bmatrix} c & s & & & & \\ -s & c & & & & \\ & & c & s & & \\ & & -s & c & & \\ & & & & c & s \\ & & & & -s & c \end{bmatrix}$$

CE381P: ANALYSIS | | C.HOVELL 21Feb07 | 9/10

HOMEWORK #3

e) transform k using transformation matrix T

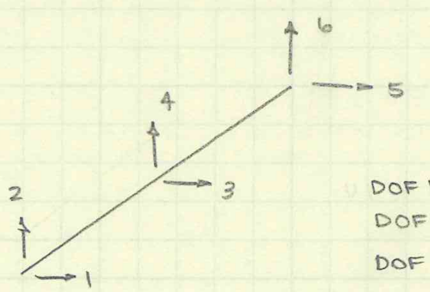
$k_{global} = T^T \cancel{k_{local}}\, T$ SHOULD BE $\underset{\sim}{T}\, \underset{\sim}{k}_{LOCAL}\, \underset{\sim}{T}^T$ FOR YOUR $\underset{\sim}{T}$

k_{local} must include non-axial considerations =

$$\begin{bmatrix} 1.663 & 0 & -1.871 & 0 & 0.208 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 2.591 & 0 & -0.719 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0.511 & 0 \\ & & & & & 0 \end{bmatrix} \frac{EA_o}{L}$$

$$k_{global} \rightarrow \begin{bmatrix} 1.663\cdot\cos^2 & 1.663\cdot\cos\cdot\sin & (-1.871)\cdot\cos^2 & (-1.871)\cdot\cos\cdot\sin & .208\cdot\cos^2 & .208\cdot\cos\cdot\sin \\ 1.663\cdot\cos\cdot\sin & 1.663\cdot\sin^2 & (-1.871)\cdot\cos\cdot\sin & (-1.871)\cdot\sin^2 & .208\cdot\cos\cdot\sin & .208\cdot\sin^2 \\ (-1.871)\cdot\cos^2 & (-1.871)\cdot\cos\cdot\sin & 2.591\cdot\cos^2 & 2.591\cdot\cos\cdot\sin & (-.719)\cdot\cos^2 & (-.719)\cdot\cos\cdot\sin \\ (-1.871)\cdot\cos\cdot\sin & (-1.871)\cdot\sin^2 & 2.591\cdot\cos\cdot\sin & 2.591\cdot\sin^2 & (-.719)\cdot\cos\cdot\sin & (-.719)\cdot\sin^2 \\ .208\cdot\cos^2 & .208\cdot\cos\cdot\sin & (-.719)\cdot\cos^2 & (-.719)\cdot\cos\cdot\sin & .511\cdot\cos^2 & .511\cdot\cos\cdot\sin \\ .208\cdot\cos\cdot\sin & .208\cdot\sin^2 & (-.719)\cdot\cos\cdot\sin & (-.719)\cdot\sin^2 & .511\cdot\cos\cdot\sin & .511\cdot\sin^2 \end{bmatrix}$$

Now, must consider constraints
- structure only has 2 DOF

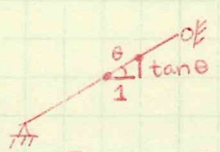


DOF 1 = 0
DOF 2 = 0
DOF 5 = 0

DOF 4 = 1/2 DOF 6
(~~slightly off, but 1/2~~ >> movement)

MUST ALSO ACCOUNT FOR CONSTRAINT BETWEEN DOF 3 & DOF 4

constraint matrix Γ



$\underset{\sim}{u}_{unconstrain} = \underset{\sim}{\Gamma}\, \underset{\sim}{u}_{constrain}$

$$\Gamma = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ \tan\theta & 1/2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{\Gamma} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \cancel{0} & 0 \\ 0 & 1/2 & \cancel{0} & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

HOMEWORK #3

e) constrain K

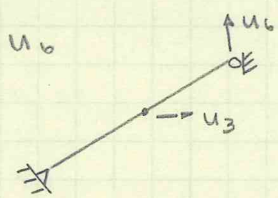
$$K_{const} = \Gamma^T K_{glob} \Gamma$$

consider K_{glob} values not equal to zero

$$K_{glob} = \frac{EA_0}{L} \begin{bmatrix} 2.591\cos^2\theta & \cancel{0.5765\cos\theta\sin\theta} \\ 0.5765\cos\theta\sin\theta & 0.4398\sin^2\theta \end{bmatrix}$$

to solve, use $\underset{\sim}{u}$ considering

u_3, u_6



University of Texas
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CE 381P

CE 381P Computer Methods in Structural Analysis

Homework 3 **Due: 21 Feb 2007**

The axial member shown in Fig. 1 has a cross-sectional area that varies over its length. The description of the cross-sectional dimensions as a function of position relative to the left end of the member are given by the following relationships:

$$A(x) = b(x)\,h(x)$$

$$h(x) = h_o\left(1 - \frac{x}{3\ell}\right)^2$$

$$b(x) = b_o\left(1 - \sin\left(\frac{\pi x}{6\ell}\right)\right)^2$$

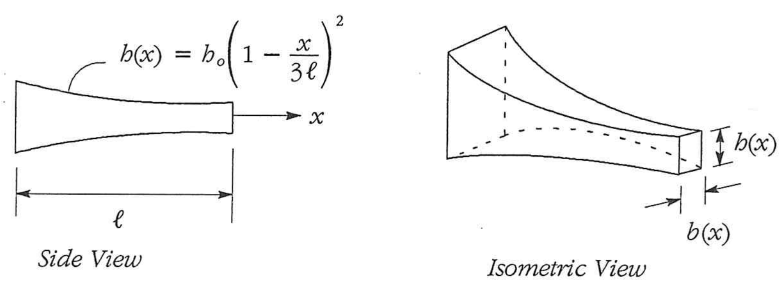


Fig. 1 Description of flared axial member

For this assignment, you are to compute the response of a flared truss member using virtual work principles.

(a). Use linear shape functions (i.e., use a 2-noded element) to approximate the real and virtual displacements. The integral for the stiffness matrix should be solved numerically using Gaussian Quadrature. You must select an appropriate order scheme to ensure a solution with sufficient accuracy. Assume that the elastic modulus E is constant over the volume of the member.

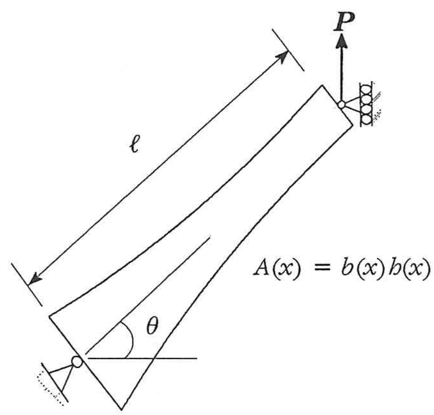
(b). Repeat part *(a)* using quadratic shape functions (i.e., use a 3-noded element) to approximate the real and virtual displacements.

(c). Using truss elements that have a constant cross-section, determine how many elements would be needed to model the member shown in Fig. 1 for acceptably accurate results.

(d). Assuming the left end is pinned and the right end of the bar in Fig. 1 is loaded by a force P, plot (on the same graph) the variation in displacement over the mem-

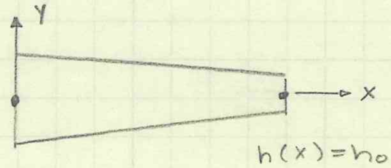
ber length using the solutions obtained above. Thus, you should have three curves, one corresponding to each case above.

(e). Using the stiffness matrix you compute in part *(b)*, develop the global structural stiffness matrix for the structure shown in the sketch below. To complete this exercise, you will need to determine how to represent the local element stiffness matrix in global coordinates. Doing so will require you to constrain the motion correctly so that only axial deformations can occur.

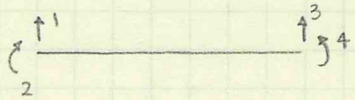


HOMEWORK #4

7/10



$h(x) = h_0 \left[1 - x/2L \right]$



$$\underset{\sim}{K}_{beam} = \int_0^L (H'')^T E(x)\, I(x)\, H''\, dx$$

$L \rightarrow$ const.

$$I(x) = 1/12\, b\, h(x)^3$$
$$= 1/12\, b h_0^3 \left[1 - x/2L \right]^3$$

$$H = \begin{bmatrix} 1 - 3x^2/L^2 + 2x^3/L^3 \\ x - 2x^2/L + x^3/L^2 \\ 3x^2/L^2 - 2x^3/L^3 \\ -x^2/L + x^3/L^2 \end{bmatrix} \qquad H'' = \begin{bmatrix} -6/L^2 + 12x/L^3 \\ -4/L + 6x/L^2 \\ 6/L^2 - 12x/L^3 \\ -2/L + 6x/L^2 \end{bmatrix}$$

$$\underset{\sim}{K}_{beam} = \int_0^L (H'')^T E H'' \cdot 1/12\, b h_0^3 \left[1 - x/2L \right]^3 dx$$

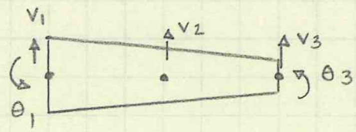
using MathCAD, — where is your mathcad output?

a)

$$K = E b h_0^3 \begin{bmatrix} \frac{81}{160L^3} & \frac{13}{40L^2} & \frac{-81}{160L^3} & \frac{29}{160L^2} \\ \frac{13}{40L^2} & \frac{19}{80L} & \frac{-13}{40L^2} & \frac{7}{80L} \\ \frac{-81}{160L^3} & \frac{-13}{40L^2} & \frac{81}{160L^3} & \frac{-29}{160L^2} \\ \frac{29}{160L^2} & \frac{7}{80L} & \frac{-29}{160L^2} & \frac{3}{32L} \end{bmatrix}$$

HOMEWORK #4

b)



$$\underset{\sim}{q}^T = [\, v_1 \;\; \theta_1 \;\; v_2 \;\; v_3 \;\; \theta_3 \,]$$

$$v(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$v(0) = v_1$$
$$v(L/2) = v_2$$
$$v(L) = v_3$$

$$v'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$$

$$v'(0) = \theta_1$$
$$v'(L) = \theta_3$$

$$\left.\begin{aligned} v_1 &= a_0 \\ \theta_1 &= a_1 \\ v_2 &= a_0 + \tfrac{L}{2} a_1 + \tfrac{L^2}{4} a_2 + \tfrac{L^3}{8} a_3 + \tfrac{L^4}{16} a_4 \\ v_3 &= a_0 + L a_1 + L^2 a_2 + L^3 a_3 + L^4 a_4 \\ \theta_3 &= a_1 + 2a_2 L + 3a_3 L^2 + 4a_4 L^3 \end{aligned}\right|$$

5 equations,
5 unknowns
$(a_0 - a_4)$

$$\underset{\sim}{q} = \underset{\sim}{coef} \cdot \underset{\sim}{a}, \quad \underset{\sim}{a}^T = [\, a_0 \;\; a_1 \;\; a_2 \;\; a_3 \;\; a_4 \,]$$

$$\underset{\sim}{coef} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & L/2 & L^2/4 & L^3/8 & \frac{L^4}{16} \\ 1 & L & L^2 & L^3 & L^4 \\ 0 & 1 & 2L & 3L^2 & 4L^3 \end{bmatrix}$$

$$\underset{\sim}{a} = \underset{\sim}{coef}^{-1} \cdot \underset{\sim}{q}$$

$$\underset{\sim}{a} = \begin{bmatrix} v_1 \\ \theta_1 \\ -\frac{11}{L^2} v_1 - \frac{4}{L}\theta_1 + \frac{16}{L^2} v_2 - \frac{5}{L^2} v_3 + \frac{\theta_3}{L} \\ \frac{18}{L^3} v_1 + \frac{5}{L^2}\theta_1 - \frac{32}{L^3} v_2 + \frac{14}{L^3} v_3 - \frac{3}{L^2}\theta_3 \\ -\frac{8}{L^4} v_1 - \frac{2}{L^3}\theta_1 + \frac{16}{L^4} v_2 - \frac{8}{L^4} v_3 + \frac{2}{L^3}\theta_3 \end{bmatrix}$$

HOMEWORK #4

b) therefore,

$$v(x) = v_1 + \theta_1 x + \left[-^{11}/_{L^2} v_1 - ^{4}/_{L}\theta_1 + ^{16}/_{L^2} v_2 - ^{5}/_{L^2} v_3 + ^{\theta_3}/_{L} \right] x^2$$
$$+ \left[^{18}/_{L^3} v_1 + ^{5}/_{L^2}\theta_1 - ^{32}/_{L^3} v_2 + ^{14}/_{L^3} v_3 - ^{3}/_{L^2}\theta_3 \right] x^3$$
$$+ \left[-^{8}/_{L^4} v_1 - ^{2}/_{L^3}\theta_1 + ^{16}/_{L^4} v_2 - ^{8}/_{L^4} v_3 + ^{2}/_{L^3}\theta_3 \right] x^4$$

collect terms by DOF $(v_1, v_2 \ldots)$

$$H_1(x) = 1 - ^{11}/_{L^2} x^2 + ^{18}/_{L^3} x^3 - ^{8}/_{L^4} x^4 \quad - v_1$$
$$H_2(x) = x - ^{4}/_{L} x^2 + ^{5}/_{L^2} x^3 - ^{2}/_{L^3} x^4 \quad - \theta_1$$
$$H_3(x) = ^{16}/_{L^2} x^2 - ^{32}/_{L^3} x^3 + ^{16}/_{L^4} x^4 \quad - v_2$$
$$H_4(x) = -^{5}/_{L^2} x^2 + ^{14}/_{L^3} x^3 - ^{8}/_{L^4} x^4 \quad - v_3$$
$$H_5(x) = ^{x^2}/_{L} - ^{3}/_{L^2} x^3 + ^{2}/_{L^3} x^4 \quad - \theta_3$$

Differentiate 2x:

$$H''(x) = \begin{bmatrix} -^{22}/_{L^2} + ^{108}/_{L^3} \cdot x - ^{96}/_{L^4} \cdot x^2 \\ -^{8}/_{L} + ^{30}/_{L^2} \cdot x - ^{24}/_{L^3} \cdot x^2 \\ ^{32}/_{L^2} - ^{192}/_{L^3} \cdot x + ^{192}/_{L^4} \cdot x^2 \\ -^{10}/_{L^2} + ^{84}/_{L^3} \cdot x - ^{96}/_{L^4} \cdot x^2 \\ ^{2}/_{L} - ^{18}/_{L^2} \cdot x + ^{24}/_{L^3} \cdot x^2 \end{bmatrix}$$

$$K = ^{1}/_{12}\, b h_o^3 E \int_0^L (H'')^T (H'') (1 - ^{x}/_{2L})^3 \, dx$$

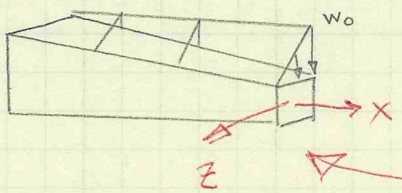
using mathCAD, → include print out w/ HW solution

b)

$$K = E b h_o^3 \begin{bmatrix} \frac{795}{224L^3} & \frac{687}{560L^2} & \frac{-361}{70L^3} & \frac{1801}{1120L^3} & \frac{-41}{160L^2} \\ \frac{687}{560L^2} & \frac{279}{560L} & \frac{-11}{7L^2} & \frac{193}{560L^2} & \frac{-2}{35L} \\ \frac{-361}{70L^3} & \frac{-11}{7L^2} & \frac{296}{35L^3} & \frac{-33}{10L^3} & \frac{9}{14L^2} \\ \frac{1801}{1120L^3} & \frac{193}{560L^2} & \frac{-33}{10L^2} & \frac{379}{224L^3} & \frac{-433}{1120L^2} \\ \frac{-41}{160L^2} & \frac{-2}{35L} & \frac{9}{14L^2} & \frac{-433}{1120L^2} & \frac{137}{1120L} \end{bmatrix}$$

HOMEWORK #4

c)



$$-w(x) = \frac{w_o}{L}x$$

$$-w(z) = \frac{w_o}{b}z + \frac{w_o}{2}$$

@ $z = \frac{b}{2}$
$w(z) = 0$

to combine,

$$w(x,z) = \frac{-1}{w_o}\left(\frac{w_o}{L}x\right)\left(\frac{w_o}{b}z + \frac{w_o}{2}\right)$$

$$\boxed{\text{c1)}\quad w(x,z) = \frac{-w_o}{L}x\left(\frac{z}{b}+\frac{1}{2}\right)}$$

$$w(x,z) = w_o\left(\frac{x}{l}\right)\left(-\frac{z}{b}+\frac{1}{2}\right)$$

$$F(x) = \int_S H^T w(x,z)\,dS$$

$$F_{1_{2\cdot node}} = \int_{-\frac{b}{2}}^{\frac{b}{2}}\int_0^L -\left[{}^{-6}/_{L^2} + {}^{12x}/_{L^3}\right]\left[\frac{w_o}{L}x\left({}^{z}/_{b} + {}^{1}/_{2}\right)\right]dx\,dz$$

H" ?

$$= -\frac{1}{2}\frac{w_o b}{L}$$ – found using mathcad

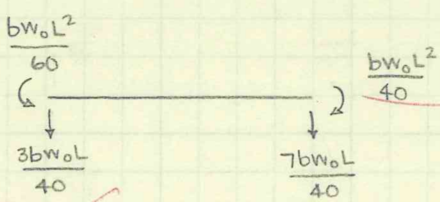
NOT SURE WHAT THIS CALCULATION IS

?

Two·noded Element:

c2)

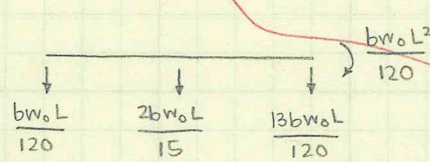
$$F = bw_o\begin{bmatrix} -3L/40 \\ -L^2/60 \\ -7L/40 \\ L^2/40 \end{bmatrix}$$



NOT SURE HOW YOU GOT THESE VALUES – IT IS NOT CONSISTENT WITH YOUR PREVIOUS COMPUTATION & YOU HAVE NOT GIVEN THE MATHCAD COMPUTATIONS

Three·noded Element

$$F = bw_o\begin{bmatrix} -L/120 \\ 0 \\ -2L/15 \\ -13L/120 \\ L^2/120 \end{bmatrix}$$



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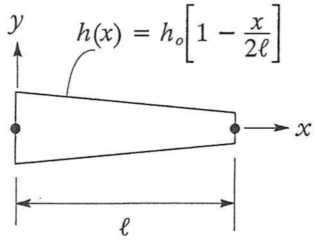
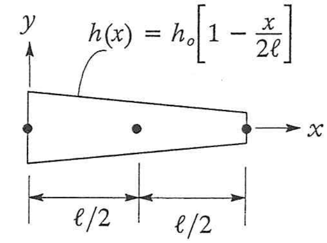
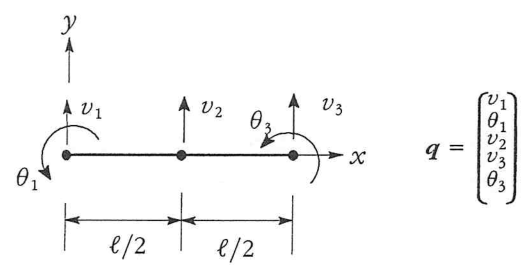
CE 381P

Homework 4 **Due: 2 MAR 2007**

The flexural member shown in the figure below has a depth that varies linearly over the length of the member. The width of the member b is constant. Assume that the material response is linear and elastic.



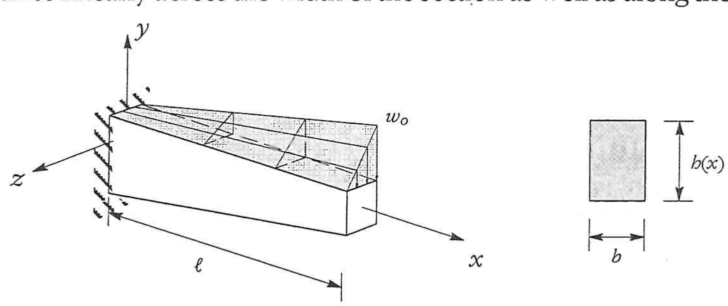




(a). Use cubic shape functions (i.e., use a 2-noded element) to approximate the real and virtual displacements, and develop the element stiffness matrix. The governing integrals for each term in the stiffness matrix should be solved numerically. You may use an appropriate order Gaussian Quadrature scheme or the built-in evaluation tool in MATHCAD.

(b). Repeat part *(a)* using a 3-noded beam element, with only vertical translation as a degree of freedom at the middle node, to approximate the real and virtual displacements.

(c). The beam shown in the figure below is loaded on its top surface by a force distribution that varies linearly across the width of the section as well as along the length of the member.

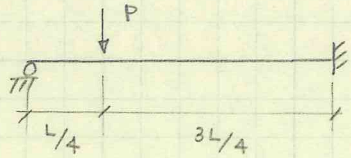


(c.1) Find an expression for the applied load $w(x, z)$.

(c.2) Find the equivalent nodal forces/moments (assume the beam is restrained against torsion and responds as a planar member) to use for analysis by the stiffness method. Consider the case in which the beam is modeled with two nodes and with three nodes as indicated in part *(b)*.

HOMEWORK #5

20/20 good job!

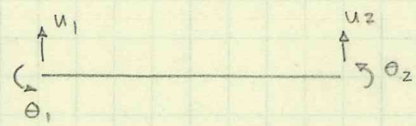


single beam, calculate midspan deflection

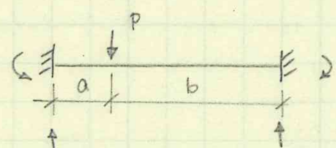
$$\underset{\sim}{K}\,\underset{\sim}{q} = \underset{\sim}{P}$$

$$\underset{\sim}{K} = EI\begin{bmatrix} 12/L^3 & 6/L^2 & -12/L^3 & 6/L^2 \\ 6/L^2 & 4/L & -6/L^2 & 2/L \\ -12/L^3 & -6/L^2 & 12/L^3 & -6/L^2 \\ 6/L^2 & 2/L & -6/L^2 & 4/L \end{bmatrix}$$

$$\underset{\sim}{q} = \begin{bmatrix} u_1 \\ \theta_1 \\ u_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \theta_1 \\ 0 \\ 0 \end{bmatrix}$$



$\underset{\sim}{P}$ from fixed end forces:

$\frac{Pab^2}{L^2}$  $\frac{Pa^2b}{L^2}$

$$P\left(\frac{b}{L} + \frac{ab^2}{L^3} - \frac{a^2b}{L^3}\right) \qquad P\left(\frac{a}{L} + \frac{a^2b}{L^3} - \frac{ab^2}{L^3}\right)$$

$$a = \frac{L}{4},\ b = \frac{3L}{4}$$

$$\underset{\sim}{P} = \begin{bmatrix} -27/32\,P \\ -9/64\,PL \\ -5/32\,P \\ 3/64\,PL \end{bmatrix} + \text{Rxns}$$

Apply constraints:

- remove rows, columns 1,3,4

$$\left[4EI/L\right]\theta_1 = \frac{-9}{64}PL$$

$$\theta_1 = \frac{-9PL^2}{256EI}$$

HOMEWORK #5

calculate midspan deflection using shape functions

$$u(x) = H_1(x)\,u_1 + H_2(x)\,\theta_1 + H_3(x)\,u_2 + H_4(x)\,\theta_2$$

$$u_1 = u_2 = \theta_2 = 0$$

$$u(x) = H_2(x)\,\theta_1$$

$$H_2(x) = x - 2\,{x^2}/{L} + {x^3}/{L^2}$$

$$H_2(L/2) = \frac{L}{2} - 2\,\frac{L^2/4}{L} + \frac{L^3/8}{L^2}$$

$$= \frac{L}{2} - \frac{L}{2} + \frac{L}{8} = \frac{L}{8}$$

$$u\left(\frac{L}{2}\right) = \frac{L}{8}\,\theta_1 = \frac{L}{8}\,\frac{9}{256}\,\frac{-PL^2}{EI}$$

$$u\left(\frac{L}{2}\right) = \frac{-9PL^3}{2048EI}$$

considering $L = 160$, $EI = 1\times10^6$,

$$u(L/2) = \frac{-9\,(160)^3}{2048\,(1\times10^6)}\,P$$

$u(L/2) = 0.018P$ (down) until plastic hinge forms

compare to exact case (from AISC manual):

$$u(x) = \frac{Pa}{12EI\,L^3}\,(L-x)^2\,(3L^2x - a^2x - 2a^2L)$$

with $x = L/2$, $a = L/4$,

$u(L/2) = \frac{43}{6144}\,\frac{PL^3}{EI}$ — over 50% larger than stiffness method answer

$$u = 0.0287P$$

check rotation

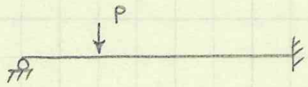
$$\theta(x) = \frac{Pb^2}{12EIL^3}\left[3aL^2 - 6Lx^2 - 3ax^2\right]$$

$$a = L/4,\; b = 3L/4,\; x = 0$$

$\theta(0) = \frac{9PL^2}{256EI}$ — matches exactly

HOMEWORK #5

Find hinge location



two likely points: under load
at fixed end

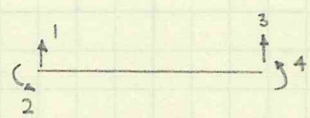
$\underset{\sim}{K}\,\underset{\sim}{u} = \underset{\sim}{F}, \quad \underset{\sim}{F} = \underset{\sim}{\text{reactions}} + \underset{\sim}{F}_{\text{fixed end}}$

$\underset{\sim}{Rxn} = \underset{\sim}{K}\,\underset{\sim}{u} - \underset{\sim}{F}_{fix} = \begin{bmatrix} -\frac{27}{32}P \\ -\frac{9}{64}PL \\ -5/32\,P \\ 3/64\,PL \end{bmatrix}$

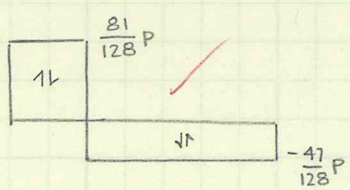
on pg 1

$\underset{\sim}{u} = \begin{bmatrix} 0 \\ -9PL^2/256EI \\ 0 \\ 0 \end{bmatrix}$

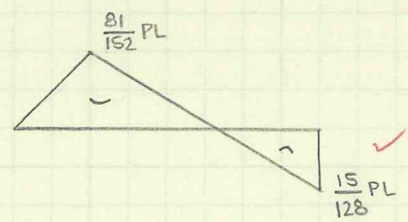
$\underset{\sim}{Rxn} = \begin{bmatrix} 81/128\,P \\ 0 \\ 47/128\,P \\ -15/128\,PL \end{bmatrix}$



Shear:



Moment:



$M_{load} = 0.158PL$
$M_{end} = 0.117PL$

controls: hinge will occur here first

a) It is possible to predict the location of first hinging using a single element. However, determining the deflection at midspan after this point is difficult – a discontinuity in slope occurs within the 2-noded beam. A comparison of the calculated deflection to the exact solution is on the next page.

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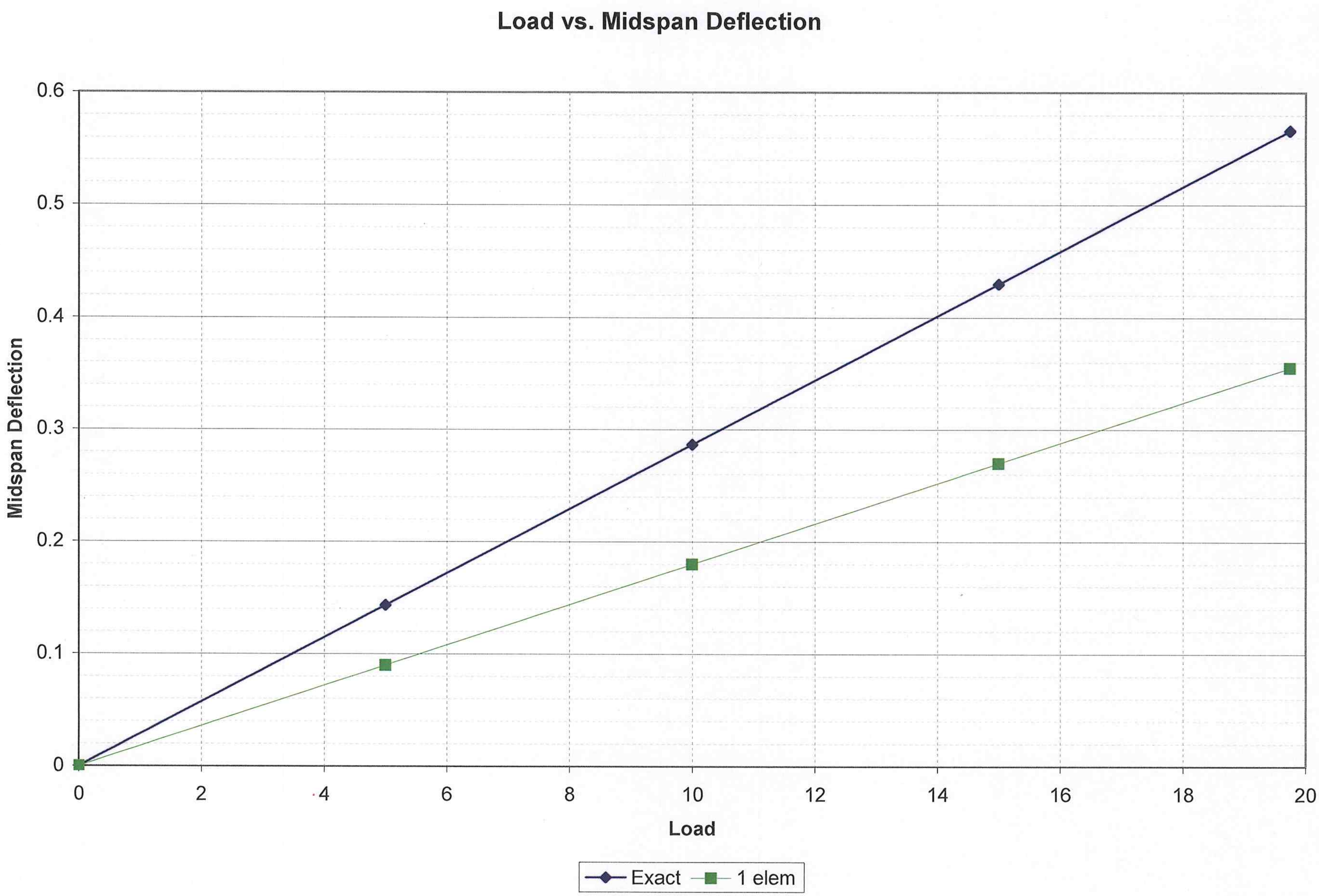

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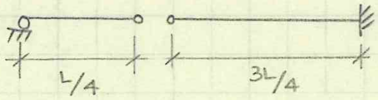
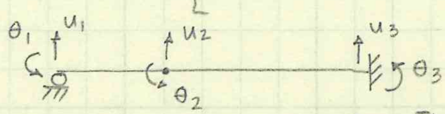


$$\underset{\sim}{k} = EI\begin{bmatrix} 12/L_1^3 & 6/L_1^2 & -12/L_1^3 & 6/L_1^2 & 0 & 0 \\ & 4/L_1 & -6/L_1^2 & 2/L_1 & 0 & 0 \\ & & \frac{12}{L_1^3}+\frac{12}{L_2^3} & -\frac{6}{L_1^2}+\frac{6}{L_2^2} & -\frac{12}{L_2^3} & 6/L_2^2 \\ & & & 4/L_1+4/L_2 & -6/L_2^2 & 2/L_2 \\ & & & & 12/L_2^3 & -6/L_2^2 \\ & & & & & 4/L_2 \end{bmatrix}$$



$$\underset{\sim}{P} = \begin{bmatrix} R_1 \\ 0 \\ -P \\ 0 \\ R_3 \\ M_3 \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ \theta_1 \\ u_2 \\ \theta_2 \\ 0 \\ 0 \end{bmatrix}$$

delete rows/columns 1, 5, 6

solve $\underset{\sim}{k}' \underset{\sim}{q}' = \underset{\sim}{P}'$

$$k_{mod}(EI, L_o) \rightarrow \begin{pmatrix} \frac{16}{L_o}\cdot EI & \frac{-96}{L_o^2}\cdot EI & \frac{8}{L_o}\cdot EI \\ \frac{-96}{L_o^2}\cdot EI & \frac{7168}{9\cdot L_o^3}\cdot EI & \frac{-256}{3\cdot L_o^2}\cdot EI \\ \frac{8}{L_o}\cdot EI & \frac{-256}{3\cdot L_o^2}\cdot EI & \frac{64}{3\cdot L_o}\cdot EI \end{pmatrix}$$

$$P_{mod} := \begin{pmatrix} 0 \\ -P \\ 0 \end{pmatrix}$$

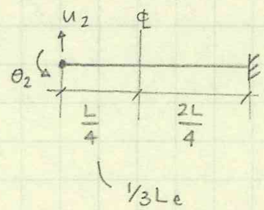
$$q_{mod} := k_{mod}(EI, L_o)^{-1}\cdot P_{mod}$$

$$q_{mod} \rightarrow \begin{array}{r} \theta_1 = \\ u_2 = \\ \theta_2 = \end{array}\begin{pmatrix} \frac{-9}{256}\cdot\frac{L_o^2}{EI}\cdot P \\ \frac{-117}{16384}\cdot\frac{L_o^3}{EI}\cdot P \\ \frac{-63}{4096}\cdot\frac{L_o^2}{EI}\cdot P \end{pmatrix}$$

– matches value from single element approx.

HOMEWORK #5

Use shape functions to get midspan Δ



$$u(x) = H_1(x)\,u_2 + H_2(x)\,\theta_2$$

$$H_1(L_e/3) = 1 - \frac{3}{L_e^2}\left(\frac{L_e^2}{9}\right) + \frac{2}{L_e^3}\left(\frac{L_e^3}{27}\right)$$

$$= 1 - \frac{1}{3} + 2/27 = 20/27$$

$$H_2(L_e/3) = \frac{L_e}{3} - \frac{2}{L_e}\frac{L_e^2}{9} + \frac{1}{L_e}\frac{L_e^3}{27}$$

$$= L_e\left(1/3 - 2/9 + 1/27\right)$$

$$= 4L_e/27 = 1/9\,L \qquad (L_e = 3L/4)$$

$$u(L/2) = 20/27\,u_2 + L/9\,\theta_2$$

$$u_{mid} = \frac{-43}{6144}\frac{PL^3}{EI}$$

using $L = 160$, $EI = 1\times10^6$,

$u_{mid} = 0.0287P$ – matches exact solution

Now, consider hinge – at load point first

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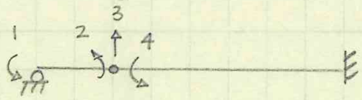
once a hinge forms,



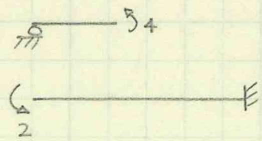
2 · two · noded beams — similar to handout

$$\underset{\sim}{K} = EI \begin{bmatrix} 4/L_1 & 2/L_1 & -6/L_1^2 & 0 \\ 2/L_1 & 4/L_1 & -6/L_1^2 & 0 \\ -6/L_1^2 & -6/L_1^2 & \frac{12}{L_1^3}+\frac{12}{L_2^3} & 6/L_2^2 \\ 0 & 0 & 6/L_2^2 & 4/L_2 \end{bmatrix}, \quad \underset{\sim}{R} = \begin{bmatrix} 0 \\ M_p \\ P \\ -M_p \end{bmatrix} \qquad M_p = 500$$

Now, consider a partitioned matrix:

$$\begin{Bmatrix} Q_{A\,FE} \\ Q_{B\,FE} \end{Bmatrix} + \left[\begin{array}{c|c} K_{AA} & K_{AB} \\ \hline K_{BA} & K_{BB} \end{array}\right] \left[\begin{array}{c} q_A \\ \hline q_B \end{array}\right] = \left[\begin{array}{c} Q_A \\ \hline Q_B \end{array}\right]$$

$Q_B \rightarrow$ known force quantities;
in this case, $Q_2 = 500$ or
$Q_4 = -500$



in general, you also need to account for fixed end forces (though they = 0 for this particular problem)

(1) $K_{AA} \cdot q_A + K_{AB} \cdot q_B = Q_A$

sign can change

(2) $K_{BA} \cdot q_A + K_{BB} \cdot q_B = Q_B = \pm M_p$

$$q_B = K_{BB}^{-1} \left[M_p - K_{BA} \cdot q_A \right]$$

$$Q_A = K_{AA} q_A + K_{AB} \cdot K_{BB}^{-1} \left[M_p - K_{BA} \cdot q_A \right]$$

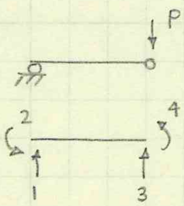
$$= K_{AA} q_A + K_{AB} K_{BB}^{-1} (M_p) - K_{AB} K_{BB}^{-1} K_{BA} q_A$$

$$\underbrace{Q_A - K_{AB} K_{BB}^{-1} (M_p)}_{\text{load vector}} = \underbrace{(K_{AA} - K_{AB} K_{BB}^{-1} K_{BA})}_{\text{stiffness matrix}} q_A$$

HOMEWORK #5

consider each element:

(1)



$$\begin{array}{cc} & K_{AA} \qquad\qquad\qquad\qquad K_{AB} \\ K = EI & \left[\begin{array}{ccc|c} 12/L^3 & 6/L^2 & -12/L^3 & 6/L^2 \\ 6/L^2 & 4/L & -6/L^2 & 2/L \\ -12/L^3 & -6/L^2 & 12/L^3 & -6/L^2 \\ \hline 6/L^2 & 2/L & -6/L^2 & 4/L \end{array}\right] \\ & K_{BA} \qquad\qquad\qquad\qquad K_{BB} \end{array}$$

$$K_{mod1} = K_{AA} - K_{AB} \cdot K_{BB}^{-1} \cdot K_{BA}$$

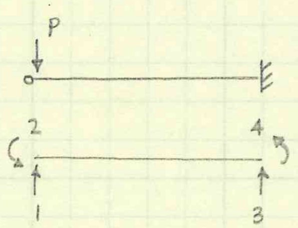
$$= 3EI \begin{bmatrix} 1/L^3 & 1/L^2 & -1/L^3 \\ 1/L^2 & 1/L & -1/L^2 \\ -1/L^3 & -1/L^2 & 1/L^3 \end{bmatrix}$$

L refers to element length, L/4

$$Q_{A1} = \begin{bmatrix} 0 \\ 0 \\ -P/2 \end{bmatrix} \qquad Q_{mod1} = Q_A - M_p \cdot K_{AB} \cdot K_{BB}^{-1}$$

$$Q_{mod1} = \begin{bmatrix} -6M_p/L \\ -M_p/2 \\ -\frac{P}{2} + 6M_p/L \end{bmatrix}$$

(2)



We must permute stiffness matrix (same as above) to gather terms (Q_2 needs to be separate)

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cc} & K_{AA} \qquad\qquad\qquad\qquad K_{AB} \\ K' = P^T K P = EI & \left[\begin{array}{ccc|c} 12/L^3 & 6/L^2 & -12/L^3 & 6/L^2 \\ 6/L^2 & 4/L & -6/L^2 & 2/L \\ -12/L^3 & -6/L^2 & 12/L^3 & -6/L^2 \\ \hline 6/L^2 & 2/L & -6/L^2 & 4/L \end{array}\right] \\ & K_{BA} \qquad\qquad\qquad\qquad K_{BB} \end{array}$$

L again refers to element length; now $L_e = \frac{3}{4} L$

$$Q_{A2} = \begin{bmatrix} -P/2 \\ 0 \\ 0 \end{bmatrix} \qquad Q_{mod2} = Q_A + M_p K_{AB} K_{BB}^{-1}$$

$$Q_{mod2} = \begin{bmatrix} -P/2 + 2M_p/L \\ M_p/2 \\ -2M_p/L \end{bmatrix}$$

L = total length

mathcad sheets follow →

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$$k(EI,L) := EI\cdot\begin{pmatrix} \frac{12}{L^3} & \frac{6}{L^2} & \frac{-12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & \frac{-6}{L^2} & \frac{2}{L} \\ \frac{-12}{L^3} & \frac{-6}{L^2} & \frac{12}{L^3} & \frac{-6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & \frac{-6}{L^2} & \frac{4}{L} \end{pmatrix}$$

$$L_1 := \frac{L}{4} \qquad L_2 := \frac{3}{4}\cdot L$$

$$Per := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$k_1(EI,L) := k(EI,L_1)$$

$$k_2(EI,L) := Per^T\cdot k(EI,L_2)\cdot Per$$

Considering partitioning:

$$k_{aa1}(EI,L) := \text{submatrix}(k_1(EI,L),0,2,0,2) \qquad k_{aa2}(EI,L) := \text{submatrix}(k_2(EI,L),0,2,0,2)$$

$$k_{ba1}(EI,L) := \text{submatrix}(k_1(EI,L),3,3,0,2) \qquad k_{ba2}(EI,L) := \text{submatrix}(k_2(EI,L),3,3,0,2)$$

$$k_{ab1}(EI,L) := \text{submatrix}(k_1(EI,L),0,2,3,3) \qquad k_{ab2}(EI,L) := \text{submatrix}(k_2(EI,L),0,2,3,3)$$

$$k_{bb1}(EI,L) := 16\frac{EI}{L} \qquad k_{bb2}(EI,L) := \frac{16EI}{3L}$$

$$k_{bb1inv}(EI,L) := \frac{L}{16EI} \qquad k_{bb2inv}(EI,L) := \frac{3\cdot L}{16\cdot EI}$$

Solving for the new stiffness matrix:

$$k_{mod1}(EI,L) := k_{aa1}(EI,L) - k_{ab1}(EI,L)\cdot k_{bb1inv}(EI,L)\cdot k_{ba1}(EI,L)$$

$$k_{mod2}(EI,L) := k_{aa2}(EI,L) - k_{ab2}(EI,L)\cdot k_{bb2inv}(EI,L)\cdot k_{ba2}(EI,L)$$

Stiffness matrices now use L as total length, not element length

$$k_{mod1}(EI,L) \rightarrow \begin{pmatrix} \frac{192}{L^3}\cdot EI & \frac{48}{L^2}\cdot EI & \frac{-192}{L^3}\cdot EI \\ \frac{48}{L^2}\cdot EI & \frac{12}{L}\cdot EI & \frac{-48}{L^2}\cdot EI \\ \frac{-192}{L^3}\cdot EI & \frac{-48}{L^2}\cdot EI & \frac{192}{L^3}\cdot EI \end{pmatrix}$$

$$k_{mod2}(EI,L) \rightarrow \begin{pmatrix} \frac{64}{9\cdot L^3}\cdot EI & \frac{16}{3\cdot L^2}\cdot EI & \frac{-64}{9\cdot L^3}\cdot EI \\ \frac{16}{3\cdot L^2}\cdot EI & \frac{4}{L}\cdot EI & \frac{-16}{3\cdot L^2}\cdot EI \\ \frac{-64}{9\cdot L^3}\cdot EI & \frac{-16}{3\cdot L^2}\cdot EI & \frac{64}{9\cdot L^3}\cdot EI \end{pmatrix}$$

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$$Q_{a1}(P) := \begin{pmatrix} 0 \\ 0 \\ \frac{-P}{2} \end{pmatrix} \qquad Q_{a2}(P) := \begin{pmatrix} \frac{-P}{2} \\ 0 \\ 0 \end{pmatrix}$$

$$Q_{mod1}(EI,L,P,M_p) := Q_{a1}(P) - M_p \cdot k_{ab1}(EI,L) \cdot k_{bb1inv}(EI,L)$$

$$Q_{mod2}(EI,L,P,M_p) := Q_{a2}(P) + M_p \cdot k_{ab2}(EI,L) \cdot k_{bb2inv}(EI,L)$$

Load matrices now use L as total length, not element length

$$Q_{mod1}(EI,L,P,M_p) \rightarrow \begin{pmatrix} \frac{-6}{L} \cdot M_p \\ \frac{-1}{2} \cdot M_p \\ \frac{-1}{2} \cdot P + \frac{6}{L} \cdot M_p \end{pmatrix} \qquad Q_{mod2}(EI,L,P,M_p) \rightarrow \begin{pmatrix} \frac{-1}{2} \cdot P + \frac{2}{L} \cdot M_p \\ \frac{1}{2} \cdot M_p \\ \frac{-2}{L} \cdot M_p \end{pmatrix}$$

Now, add back the rows that were removed, and for beam 2, repermute

$$k_{modfull1}(EI,L) := EI \begin{pmatrix} \frac{192}{L^3} & \frac{48}{L^2} & \frac{-192}{L^3} & 0 \\ \frac{48}{L^2} & \frac{12}{L} & \frac{-48}{L^2} & 0 \\ \frac{-192}{L^3} & \frac{-48}{L^2} & \frac{192}{L^3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad k_{temp2}(EI,L) := EI \begin{pmatrix} \frac{64}{9L^3} & \frac{16}{3L^2} & \frac{-64}{9L^3} & 0 \\ \frac{16}{3L^2} & \frac{4}{L} & \frac{-16}{3L^2} & 0 \\ \frac{-64}{9L^3} & \frac{-16}{3L^2} & \frac{64}{9L^3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$k_{modfull2}(EI,L) := Per^T k_{temp2}(EI,L) \cdot Per$$

$$k_{modfull2}(EI,L) \rightarrow \begin{pmatrix} \frac{64}{9 \cdot L^3} \cdot EI & 0 & \frac{-64}{9 \cdot L^3} \cdot EI & \frac{16}{3 \cdot L^2} \cdot EI \\ 0 & 0 & 0 & 0 \\ \frac{-64}{9 \cdot L^3} \cdot EI & 0 & \frac{64}{9 \cdot L^3} \cdot EI & \frac{-16}{3 \cdot L^2} \cdot EI \\ \frac{16}{3 \cdot L^2} \cdot EI & 0 & \frac{-16}{3 \cdot L^2} \cdot EI & \frac{4}{L} \cdot EI \end{pmatrix}$$

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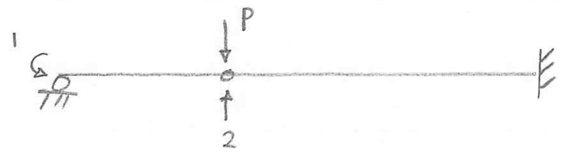
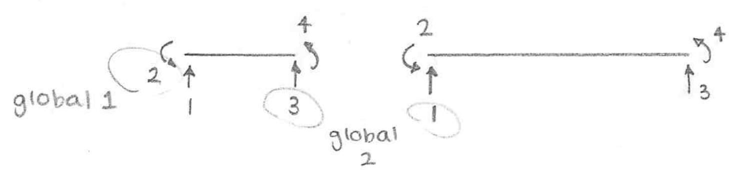
Modify load matrices to reflect additional line, permutations

$$Q_{modfull1}(EI,L,P,M_p) := \begin{pmatrix} Q_{mod1}(EI,L,P,M_p)_0 \\ Q_{mod1}(EI,L,P,M_p)_1 \\ Q_{mod1}(EI,L,P,M_p)_2 \\ 0 \end{pmatrix} \qquad Q_{modfull1}(EI,L,P,M_p) \rightarrow \begin{pmatrix} \frac{-6}{L}\cdot M_p \\ \frac{-1}{2}\cdot M_p \\ \frac{-1}{2}\cdot P + \frac{6}{L}\cdot M_p \\ 0 \end{pmatrix}$$

$$Q_{modfull2}(EI,L,P,M_p) := \begin{pmatrix} Q_{mod2}(EI,L,P,M_p)_0 \\ 0 \\ Q_{mod2}(EI,L,P,M_p)_2 \\ Q_{mod2}(EI,L,P,M_p)_1 \end{pmatrix} \qquad Q_{modfull2}(EI,L,P,M_p) \rightarrow \begin{pmatrix} \frac{-1}{2}\cdot P + \frac{2}{L}\cdot M_p \\ 0 \\ \frac{-2}{L}\cdot M_p \\ \frac{1}{2}\cdot M_p \end{pmatrix}$$

Combine the stiffness and load vectors to consider global DOFs





$$k_{hinge}(EI,L) := \begin{bmatrix} \left(k_{modfull1}(EI,L)^{\langle 1\rangle}\right)_1 & \left(k_{modfull1}(EI,L)^{\langle 2\rangle}\right)_1 \\ \left(k_{modfull1}(EI,L)^{\langle 1\rangle}\right)_2 & \left(k_{modfull1}(EI,L)^{\langle 2\rangle}\right)_2 + \left(k_{modfull2}(EI,L)^{\langle 2\rangle}\right)_2 \end{bmatrix}$$

$$Q_{hinge}(EI,P,L,M_p) := \begin{pmatrix} Q_{modfull1}(EI,L,P,M_p)_1 \\ Q_{modfull1}(EI,L,P,M_p)_2 + Q_{modfull2}(EI,L,P,M_p)_0 \end{pmatrix}$$

Global stiffness and load matrices:

$$k_{hinge}(EI,L) \rightarrow \begin{pmatrix} \frac{12}{L}\cdot EI & \frac{-48}{L^2}\cdot EI \\ \frac{-48}{L^2}\cdot EI & \frac{1792}{9\cdot L^3}\cdot EI \end{pmatrix} \qquad Q_{hinge}(EI,P,L,M_p) \rightarrow \begin{bmatrix} \frac{-1}{2}\cdot M_p \\ (-P) + \frac{8}{L}\cdot M_p \end{bmatrix}$$

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Solve for rotation, deflection

$$q_{hinge}(EI,L,P,M_p) := k_{hinge}(EI,L)^{-1} \cdot Q_{hinge}(EI,P,L,M_p)$$

$$q_{hinge}(EI,L,P,M_p) \rightarrow \begin{bmatrix} \frac{-7}{6}\cdot\frac{L}{EI}\cdot M_p + \frac{9}{16}\cdot\frac{L^2}{EI}\cdot\left[(-P) + \frac{8}{L}\cdot M_p\right] \\ \frac{-9}{32}\cdot\frac{L^2}{EI}\cdot M_p + \frac{9}{64}\cdot\frac{L^3}{EI}\cdot\left[(-P) + \frac{8}{L}\cdot M_p\right] \end{bmatrix}$$

Rotation at left end

Vertical displacement at hinge

Insert values for EI, L, Mp

$L := 160$

$EI := 1000000$

$M_p := 500$

Solve for the rotation at the hinge, to use in shape function / midspan deflection calculations

$$q_{a2}(EI,L,P,M_p) := \begin{pmatrix} q_{hinge}(EI,L,P,M_p)_1 \\ 0 \\ 0 \end{pmatrix}$$

$$q_{b2}(M_p,P,EI,L) := k_{bb2inv}(EI,L)\cdot\left(-M_p - k_{ba2}(EI,L)\cdot q_{a2}(EI,L,P,M_p)\right)$$

$$q_{b2}(M_p,P,EI,L) \rightarrow \frac{-3}{20} + \frac{9}{1250}\cdot P$$

Use Hermitian shape functions to calculate midspan deflections

$$H_1(x,L) := 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 \qquad H_1\left(\frac{L}{3},L\right) \rightarrow \frac{20}{27}$$

$$H_2(x,L) := x - 2\frac{x^2}{L} + \frac{x^3}{L^2} \qquad H_2\left(\frac{L_e}{3},L_e\right) \rightarrow \frac{4}{27}\cdot L_e \qquad L_e := L_2(L)$$

$$u_{mid}(P) := \frac{20}{27}\cdot q_{hinge}(EI,L,P,M_p)_1 + \frac{L}{9}\cdot q_{b2}(M_p,P,EI,L)$$

$$\boxed{u_{mid}(P) \rightarrow \frac{16}{3} - \frac{112}{375}\cdot P}$$

$u_{mid}(19.75309) = -0.56626$ Value matches the calculated deflection at point of hinging.

CE 381P: ANALYSIS

C. HOVELL
21 March 07

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HOMEWORK #5

Find when hinging occurs at wall; this is when the structure becomes unstable

$M_{wall}(P,L) := 0.117 \cdot P \cdot L$ Moment at the wall when the first hinge forms

$$M_{wall}(19.75309, L) \rightarrow 369.77784480$$

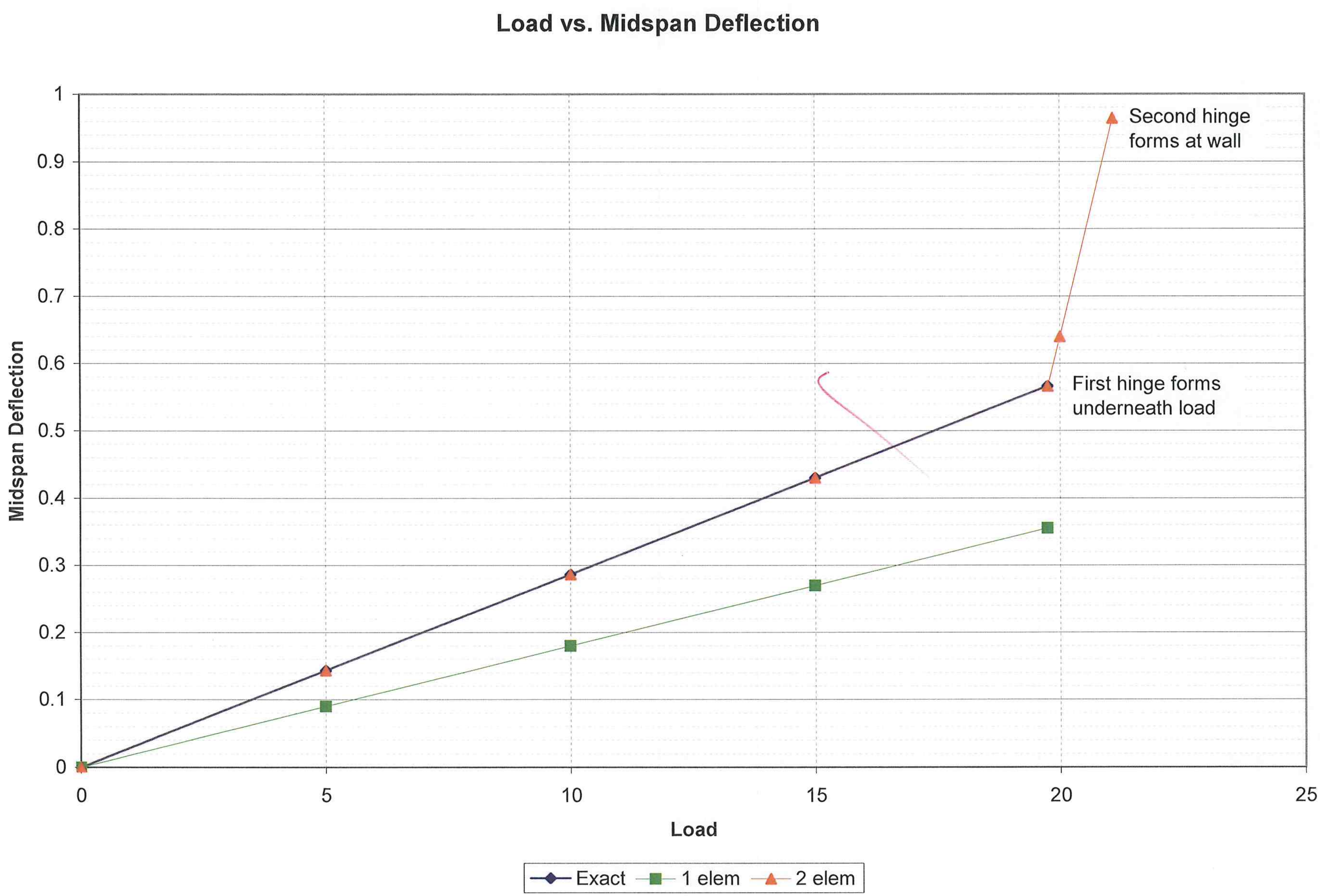
$M_{remain} := M_p - M_{wall}(19.75309, L)$ Moment capacity that remains at the wall

$$M_{remain} \rightarrow 130.22215520$$

Consider a cantilevered beam with applied load ΔP (all of the load goes to the lefthand beam). Calculate ΔP.

$$\Delta P := \frac{M_{remain}}{L_2(L)}$$

ΔP float, 5 $\rightarrow 1.0852$ Additional load that can be applied until second hinge occurs



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21 March 07

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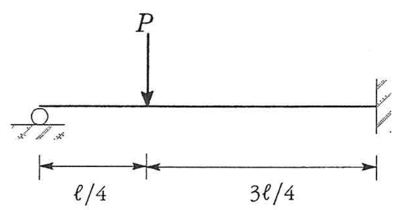
University of Texas
Dept. of Civil, Arch. & Env. Eng.

Instructor: E. B. Williamson
CE 381P

Homework 5 **Due: 21 MAR 2007**

The propped cantilever shown in the figure below is loaded with a concentrated force of P. For this assignment, you must develop a graph of load P versus the deflection at mid-span. In developing the graph, you will begin by assuming the load P is equal to zero. The corresponding deflection, of course, is also zero. To calculate the next point on the graph, increment the load P and determine the corresponding value of mid-span deflection. Eventually, as the load P is increased, a plastic hinge will form when the moment reaches the plastic moment capacity of the section M_p. As such, you will need to develop a new force-deformation relationship corresponding to values of load P that are larger than the one required to cause plastic hinging in the member.



(a). Model the structure shown in the figure using just ***a single beam element***. Utilizing the stiffness method of analysis and the concept of equivalent nodal loads to compute the response of the member, is it possible to determine the onset of plastic hinging? If so, explain where plastic hinging will occur. Otherwise, suggest a strategy for accounting for the onset of inelastic material behavior.

(b) Model the structure above using two beam elements – one of length $\ell/4$ and the other of length $3\ell/4$. Develop two sets of equations to capture the response of the structure. One set of equations will correspond to the elastic response of the element, and the other set will correspond to the case in which the element has formed a plastic hinge. For both sets of equations, you must form the relationship for the stiffness matrix and the computation of the ***mid-span deflection*** as a function of the nodal values. When forming the stiffness matrix for the case of inelastic material response, you must use the concept of a modified element stiffness matrix in which you know the value of moment (i.e., the plastic moment) acting at one end of the member.

Plot the response of the structure (i.e., load versus mid-span deflection) using the following parameters: $\ell = 160$, $EI = 1 \times 10^6$, $M_p = 500$, and $\Delta P = 5$. All quantities have been given in a consistent set of units.

For discussion: How does the solution you graph compare with the "exact" solution? Is the displacement computed at mid-span using the stiffness method the same one that you would compute using, for example, the moment-area theorems?

HOMEWORK #6

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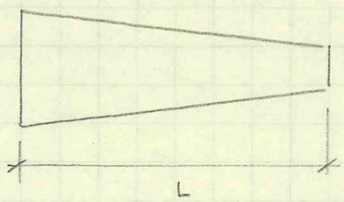


$h(x) = h_o(1 - x/3L)$

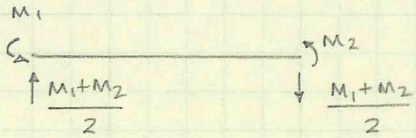
$h_o = 36\,in$

$b = 4\,in$

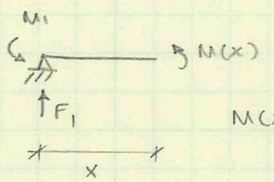
$E = 10{,}000\,ksi$

$L = 15\,ft$ or $180\,in$

1. choose independent, dependent variables



2. Develop equilibrium equations



$$M(x) = (x/L - 1)M_1 + x/L\, M_2$$

$$M(x) = \underbrace{\begin{bmatrix} x/L - 1 & x/L \end{bmatrix}}_{\underset{\sim}{D}} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

3. Develop virtual system
   - use same geometries as real

4. From complimentary virtual work,

$$\underset{\sim}{f} = \int_0^L \underset{\sim}{D}^T \frac{1}{E} \frac{1}{I(x)} \underset{\sim}{D}\, dx$$

$$I(x) = 1/12\, b h_o^3 (1 - x/3L)^3$$

$$\underset{\sim}{f} = \frac{12}{b h_o^3 E} \int_0^L \begin{bmatrix} \frac{x}{L} - 1 \\ \frac{x}{L} \end{bmatrix} \begin{bmatrix} \frac{x}{L} - 1 & \frac{x}{L} \end{bmatrix} (1 - x/3L)^{-3}\, dx$$

HOMEWORK #6

Using mathcad to solve for $\underset{\sim}{f}$:

$$f(b,h_o,E,L) := \frac{12}{b\cdot h_o^{\,3}\cdot E}\cdot\begin{bmatrix} \displaystyle\int_0^L \left(\frac{x}{L}-1\right)^2\cdot\left(1-\frac{x}{3L}\right)^{-3} dx & \displaystyle\int_0^L \left(\frac{x}{L}-1\right)\frac{x}{L}\cdot\left(1-\frac{x}{3L}\right)^{-3} dx \\ \displaystyle\int_0^L \left(\frac{x}{L}-1\right)\frac{x}{L}\cdot\left(1-\frac{x}{3L}\right)^{-3} dx & \displaystyle\int_0^L \left(\frac{x}{L}\right)^2\cdot\left(1-\frac{x}{3L}\right)^{-3} dx \end{bmatrix}$$

Sub in variable values:

$b := 4$ $\quad L := 180$

$h_o := 36$ $\quad E := 10000$

$$f(b,h_o,E,L) = \begin{pmatrix} 5.18\times10^{-7} & -3.5\times10^{-7} \\ -3.5\times10^{-7} & 9.52\times10^{-7} \end{pmatrix}$$

Transform $\underset{\sim}{f}$ to $\underset{\sim}{k}$

Relate independent, dependent forces

$$\underset{\underset{\sim}{Q_1}}{\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}} = \underset{\underset{\sim}{\phi}}{\begin{bmatrix} 1/L & 1/L \\ -1/L & -1/L \end{bmatrix}} \underset{\underset{\sim}{Q_0}}{\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}}$$

as $F_1 = \frac{M_1}{L} + \frac{M_2}{L}$

$F_2 = -\frac{M_1}{L} - \frac{M_2}{L}$

Rules for assembling:

$$\begin{bmatrix} k_{00} & k_{01} \\ k_{10} & k_{11} \end{bmatrix}, \quad k_{00} = f^{-1} \quad k_{01} = f^{-1}\phi^T$$

$$k_{10} = \phi f^{-1} \quad k_{11} = \phi f^{-1}\phi^T$$

HOMEWORK #6

$$\Phi_o(L) := \frac{1}{L}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$ Phi matrix relates independent and dependent forces

$$k_{00}(b,h_o,E,L) := f(b,h_o,E,L)^{-1}$$

$$k_{00}(b,h_o,E,L) = \begin{pmatrix} 2.569\times10^6 & 9.445\times10^5 \\ 9.445\times10^5 & 1.398\times10^6 \end{pmatrix}$$

$$k_{10}(b,h_o,E,L) := \Phi_o(L)\cdot f(b,h_o,E,L)^{-1}$$

$$k_{10}(b,h_o,E,L) = \begin{pmatrix} 1.952\times10^4 & 1.301\times10^4 \\ -1.952\times10^4 & -1.301\times10^4 \end{pmatrix}$$

$$k_{01}(b,h_o,E,L) := f(b,h_o,E,L)^{-1}\cdot\Phi_o(L)^T$$

$$k_{01}(b,h_o,E,L) = \begin{pmatrix} 1.952\times10^4 & -1.952\times10^4 \\ 1.301\times10^4 & -1.301\times10^4 \end{pmatrix}$$

$$k_{11}(b,h_o,E,L) := \Phi_o(L)\cdot f(b,h_o,E,L)^{-1}\cdot\Phi_o(L)^T$$

$$k_{11}(b,h_o,E,L) = \begin{pmatrix} 180.72 & -180.72 \\ -180.72 & 180.72 \end{pmatrix}$$

Combine each part of the matrix together into one big k matrix

$$k(b,h_o,E,L) = \begin{pmatrix} 2.569\times10^6 & 9.445\times10^5 & 1.952\times10^4 & -1.952\times10^4 \\ 9.445\times10^5 & 1.398\times10^6 & 1.301\times10^4 & -1.301\times10^4 \\ 1.952\times10^4 & 1.301\times10^4 & 180.72 & -180.72 \\ -1.952\times10^4 & -1.301\times10^4 & -180.72 & 180.72 \end{pmatrix}$$ ✓

Rearrange k matrix to reflect F1,M1,F2,M2 ordering

$$P := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$k_{perm}(b,h_o,E,L) := P^T\cdot k(b,h_o,E,L)\cdot P$$ ✓

b)

$$k_{perm}(b,h_o,E,L) = \begin{pmatrix} 180.72 & 1.952\times10^4 & -180.72 & 1.301\times10^4 \\ 1.952\times10^4 & 2.569\times10^6 & -1.952\times10^4 & 9.445\times10^5 \\ -180.72 & -1.952\times10^4 & 180.72 & -1.301\times10^4 \\ 1.301\times10^4 & 9.445\times10^5 & -1.301\times10^4 & 1.398\times10^6 \end{pmatrix}$$ ✓

HOMEWORK #6

Now, use stiffness-based approach to develop $\underset{\sim}{K}$

2nd derivatives of cubic shape functions:

$$H_1'' = \frac{-6}{L^2} + \frac{12x}{L^3}$$

$$H_2'' = \frac{-4}{L} + \frac{6x}{L^2}$$

$$H_3'' = \frac{6}{L^2} - \frac{12x}{L^3}$$

$$H_4'' = \frac{-2}{L} + \frac{6x}{L^2}$$

$$\underset{\sim}{K} = \int_0^L (H'')^T E \frac{bh_o^3}{12}\left(1 - x/3L\right)^3 H''\,dx$$

using mathcad,

$$k_{stiff}(b, h_o, E, L) = \begin{pmatrix} 198.519 & 2.123\times10^4 & -198.519 & 1.451\times10^4 \\ 2.123\times10^4 & 2.733\times10^6 & -2.123\times10^4 & 1.088\times10^6 \\ -198.519 & -2.123\times10^4 & 198.519 & -1.451\times10^4 \\ 1.451\times10^4 & 1.088\times10^6 & -1.451\times10^4 & 1.523\times10^6 \end{pmatrix}$$

compare values to $\underset{\sim}{K}$ from flexibility method. While not the same, they're in the same range.

Calculate fixed end forces in mathCAD ⟶

CE 381P: ANALYSIS

HOMEWORK #6

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18 April 07

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Hermitian shape functions and second derivatives for the points of interest (H3, H4)

$$\mathrm{Herm}(x,L) := \begin{bmatrix} 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 \\ -\frac{x^2}{L} + \frac{x^3}{L^2} \end{bmatrix} \qquad H_{doubleprime}(x,L) := \begin{pmatrix} \frac{6}{L^2} - \frac{12x}{L^3} \\ \frac{-2}{L} + \frac{6x}{L^2} \end{pmatrix}$$

State constants, equations for thermal curvature and moment distribution

$$\Delta T := -100 \qquad w_o := \frac{1}{3} \qquad w_1 := \frac{1}{6}$$

$$\alpha := 6.5\cdot 10^{-6}$$

$$\chi(\alpha,\Delta T,h_o,x,L) := \frac{\alpha\cdot\Delta T}{h_o\left(1-\frac{x}{3L}\right)} \qquad w(x) := -\left(w_o + w_1\cdot\frac{x}{L}\right)$$

Calculate fixed end loads from thermal and applied loads

$$F_{thermstiff} := \begin{bmatrix} \int_0^L E\cdot\left(\frac{6}{L^2} - \frac{12x}{L^3}\right)\cdot\frac{b}{12}\cdot h_o^{\,3}\cdot\left(1-\frac{x}{3L}\right)^3\cdot\frac{\alpha\cdot\Delta T}{h_o\cdot\left(1-\frac{x}{3L}\right)}\,dx \\ \int_0^L E\cdot\left(\frac{-2}{L} + \frac{6x}{L^2}\right)\cdot\frac{b}{12}\cdot h_o^{\,3}\cdot\left(1-\frac{x}{3L}\right)^3\cdot\frac{\alpha\cdot\Delta T}{h_o\cdot\left(1-\frac{x}{3L}\right)}\,dx \end{bmatrix} \qquad F_{thermstiff} = \begin{pmatrix} -8.667 \\ -1.196\times 10^3 \end{pmatrix}$$

$$F_{loadstiff} := \begin{bmatrix} \int_0^L \left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right]\cdot -\left(w_o + w_1\cdot\frac{x}{L}\right)dx \\ \int_0^L \left(-\frac{x^2}{L} + \frac{x^3}{L^2}\right)\cdot -\left(w_o + w_1\cdot\frac{x}{L}\right)dx \end{bmatrix} \qquad F_{loadstiff} = \begin{pmatrix} -40.5 \\ 1.17\times 10^3 \end{pmatrix}$$

$$F_{stifftot} := F_{loadstiff} + F_{thermstiff} \qquad F_{stifftot} = \begin{pmatrix} -49.167 \\ -26 \end{pmatrix}$$

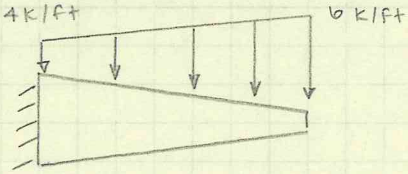
Solve for deflection, rotation of right end

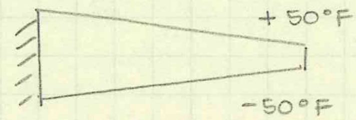
$$k_{stifftot}(b,h_o,E,L) := \mathrm{submatrix}(k_{stiff}(b,h_o,E,L),2,3,2,3)$$

$$q_{stiff} := k_{stifftot}(b,h_o,E,L)^{-1}\cdot F_{stifftot} \qquad q_{stiff} = \begin{pmatrix} -0.819\ \text{in} \\ -7.814\times 10^{-3} \end{pmatrix} \begin{matrix} \Delta \\ \theta \end{matrix}$$

HOMEWORK #6

consider applied loads:





$\Delta T = 100°F$

$\alpha = 6.5\times10^{-6}/°F$

$$Q_{FE} = -f^{-1} q_{bf_0}$$

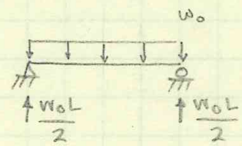
$$q_{bf_0} = \int_0^L \underset{\sim}{D}^T \left[\varepsilon_T(x) + \frac{Q_P(x)}{E\cdot\beta(x)} \right] dx$$

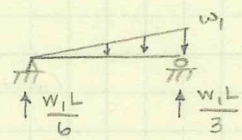
ε_T = thermal curvature

$$k(x) = \frac{\alpha}{I(x)} \int_{-h(x)/2}^{h(x)/2} \Delta T_0 \frac{y}{h} y \cdot b\,dy \;, \quad \Delta T_0 = -100\,°F$$

$Q_P(x)$ = distribution of moment over the length

split load:





$$M(x) = \frac{w_0 L}{2} x - \frac{w_0 x^2}{2} \qquad M(x) = \frac{w_1 L}{6} x - \frac{w_1 x^2}{6}\cdot\frac{x}{L}$$

$$M(x) = \left(\frac{w_0}{2} + \frac{w_1}{6}\right) L x - \frac{w_0 x^2}{2} - \frac{w_1 x^3}{6L}$$

$$w_0 = 1/3\ k/in \quad w_1 = 1/6\ k/in$$

$$M(x) = \frac{7}{36} L x - 1/6\, x^2 - \frac{x^3}{36L}$$

For flexure, $\beta(x) = I(x) = 1/12\, b h_0^3 (1 - x/3L)^3$

continued in mathcad

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HOMEWORK #6

Find applied load vector using the flexibility method:

Calculate discrete member deformations due to thermal and applied loads individually

$$\kappa(x,\alpha,b,h_o,L,\Delta T) := \frac{\alpha \cdot b}{\frac{1}{12} b \cdot h_o^{3}\left(1-\frac{x}{3L}\right)^{3}} \cdot \int_{-\frac{h_o}{2}\left(1-\frac{x}{3L}\right)}^{\frac{h_o}{2}\left(1-\frac{x}{3L}\right)} \frac{\Delta T \cdot y^2}{h_o \cdot \left(1-\frac{x}{3L}\right)}\,dy$$

✓

$$q_{thermflex}(L,b,h_o) := \begin{bmatrix} \int_0^L \left(\frac{x}{L}-1\right)\cdot\kappa(x,\alpha,b,h_o,L,\Delta T)\,dx \\ \int_0^L \left(\frac{x}{L}\right)\cdot\kappa(x,\alpha,b,h_o,L,\Delta T)\,dx \end{bmatrix}$$

✓

$$q_{thermflex}(L,b,h_o) = \begin{pmatrix} 1.843\times 10^{-3} \\ -2.11\times 10^{-3} \end{pmatrix}$$

✓

$$M(x) := \left(\frac{w_o}{2}+\frac{w_1}{6}\right)\cdot L\cdot x - \frac{w_o\cdot x^2}{2} - \frac{w_1\cdot x^3}{6L}$$

$$q_{loadflex}(E,L,b,h_o) := \frac{12}{b\cdot h_o^{3}\cdot E}\cdot\begin{bmatrix} \int_0^L \left(\frac{x}{L}-1\right)\cdot\frac{M(x)}{\left(1-\frac{x}{3L}\right)^3}\,dx \\ \int_0^L \left(\frac{x}{L}\right)\cdot\frac{M(x)}{\left(1-\frac{x}{3\cdot L}\right)^3}\,dx \end{bmatrix}$$

$$q_{loadflex}(E,L,b,h_o) = \begin{pmatrix} -1.031\times 10^{-3} \\ 1.351\times 10^{-3} \end{pmatrix}$$

✓

$$q_{flextot}(E,L,b,h_o) := q_{thermflex}(L,b,h_o) + q_{loadflex}(E,L,b,h_o)$$

✓

$$q_{flextot}(E,L,b,h_o) = \begin{pmatrix} 8.122\times 10^{-4} \\ -7.592\times 10^{-4} \end{pmatrix}$$

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HOMEWORK #6

Use discrete member deformations to calculate fixed end forces (total deformation = 0)

$$Q_{FEflex}(E,L,b,h_o) := \left(-f(b,h_o,E,L)\right)^{-1} \cdot q_{flextot}(E,L,b,h_o)$$

$$Q_{FEflex}(E,L,b,h_o) = \begin{pmatrix} -1.369 \times 10^3 \\ 293.96 \end{pmatrix}$$

Calculate dependent forces using equilibrium

$$F_1(w_o,w_1,L,M_1,M_2) := \frac{w_o \cdot L}{2} + \frac{w_1 \cdot L}{6} + \frac{(M_1 + M_2)}{L}$$

$$M_1 := Q_{FEflex}(E,L,b,h_o)_0$$

$$F_2(w_o,w_1,L,M_1,M_2) := \frac{w_o \cdot L}{2} + \frac{w_1 \cdot L}{3} - \frac{(M_2 + M_1)}{L}$$

$$M_2 := Q_{FEflex}(E,L,b,h_o)_1$$

$$Q_{flexapp} := \begin{pmatrix} F_1(w_o,w_1,L,M_1,M_2) \\ M_1 \\ F_2(w_o,w_1,L,M_1,M_2) \\ M_2 \end{pmatrix} \qquad Q_{flexapp} = \begin{pmatrix} 29.026 \\ -1.369 \times 10^3 \\ 45.974 \\ 293.96 \end{pmatrix}$$

To apply boundary conditions, first two rows of k, Q matrices should be removed

$$k_{flex}(b,h_o,E,L) := \text{submatrix}\left(k_{perm}(b,h_o,E,L),2,3,2,3\right)$$

$$Q_{flexfinal} := \text{submatrix}\left(Q_{flexapp},2,3,0,0\right)$$

$$q_{flex} := k_{flex}(b,h_o,E,L)^{-1} \cdot -Q_{flexfinal}$$

$$q_{flex} = \begin{pmatrix} -0.818 \text{ in} \\ -7.821 \times 10^{-3} \end{pmatrix} \begin{matrix} \Delta \\ \theta \end{matrix}$$

Discussion: Because the flexibility method is based on equilibrium (which is exact), while the stiffness method requires an appropriately assumed deflected shape, the flexibility method should be more accurate. However, the difference in solution in both rotation and deflection is around 0.1%, indicating the cubic shape functions approximate the shape with excellent accuracy.

very different $\underset{\sim}{K}$, $\underset{\sim}{F}$ matrices, but similar answers

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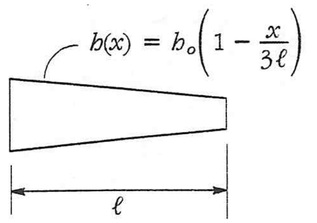
Homework 6 **Due: 16 APR 2007**

Shown in Fig. 1 below is a flexural element with a depth that varies linearly over the member length. The width of the member, b, is constant. Numerical values for the properties of the member are given in Fig. 1.



$h_o = 36\ in$
$b = 4\ in$
$E = 10,000\ ksi$
$\ell = 15\ ft$

Fig. 1 Tapered Flexural Member

(a). Develop the member flexibility matrix $\boldsymbol{f}$ for the element shown in Fig. 1. Assume that the member is elastic with a constant modulus of E and that shear and axial deformations can be ignored.

(b). Compute the element stiffness matrix $\boldsymbol{k}$ using the element flexibility matrix $\boldsymbol{f}$ and the matrix $\boldsymbol{\Phi}$ that relates the dependent nodal force quantities to the independent quantities.

For the next part of the homework assignment, you are to compare the response of the member using the stiffness matrix developed in (b) with the stiffness matrix that would result for the traditional displacement-based approach in which you assume the displacement to be described by the cubic Hermitian shape functions for a two-noded element. The loading on the member is due to a distributed load that varies linearly over the member length (Fig. 2) as well as a temperature change that varies linearly through the depth of the member. For this problem, assume that the top of the member increases in temperature by 50 degrees Fahrenheit over the entire length of the beam, and the bottom of the member is cooled by 50 degrees Fahrenheit over the entire length of the beam. Assume that the coefficient of thermal expansion is $\alpha = 6.5 \times 10^{-6}/°F$.

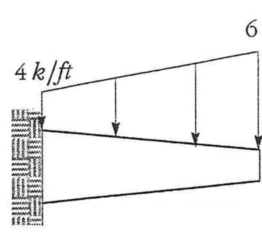


Fig. 2 Tapered Flexural Member with Transverse Loading

(c). Compute the fixed end forces for the member using the flexibility-based approach.

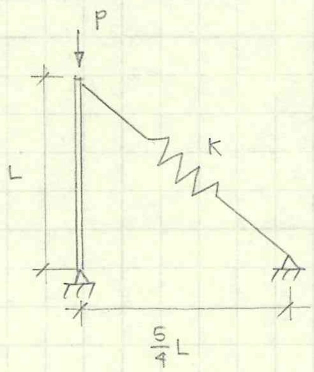
(d). Compare the tip deflection calculated using $\boldsymbol{k}$ derived from the flexibility-based approach with that computed using a traditional displacement-based approach.

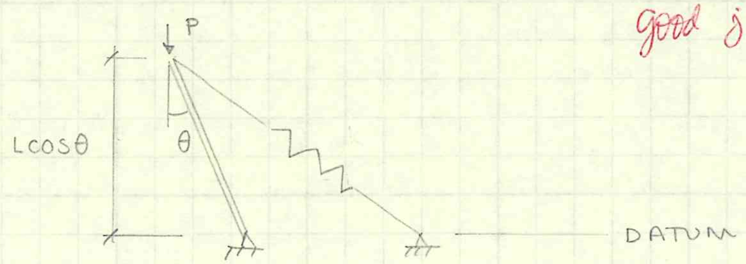
For Discussion: Which method gives a better prediction of the tip deflection? Justify your answer. Also, comment on the appropriateness of assuming a cubic variation in displacement for the current problem.

HOMEWORK #7

9.5/10

good job.





a)

$V = PL\cos\theta$

$U = \frac{1}{2} k x^2$

↳ x = elongation of spring
function of θ

$L_{si} = 1.601L$

$L_{sf} = \left[(L\cos\theta)^2 + (5/4 L + L\sin\theta)^2 \right]^{\frac{1}{2}}$

$= L\left[\cos^2\theta + \frac{25}{16} + \frac{5}{2}\sin\theta + \sin^2\theta\right]^{\frac{1}{2}}$

$= L\left[41/16 + 5/2 \sin\theta\right]^{\frac{1}{2}}$

$x = L\left[1.601 - (41/16 + 5/2\sin\theta)^{\frac{1}{2}}\right]$

Δ SHOULD BE DEFINED AS $L_{sf} - L_{si}$, which is THE OPPOSITE OF WHAT you HAVE

$\Pi = U + V$

$\Pi = PL\cos\theta + \frac{1}{2}kL^2\left[\overset{1.601}{L_{so}} - (41/16 + 5/2\sin\theta)^{1/2}\right]^2$

For equilibrium,

$\frac{d\Pi}{d\theta} = 0$

$\frac{d\Pi}{d\theta} = -PL\sin\theta + kL^2\left[L_{so} - (41/16 + 5/2\sin\theta)^{\frac{1}{2}}\right]\frac{-1/2 \cdot 5/2\cos\theta}{(41/16 + 5/2\sin\theta)^{\frac{1}{2}}} = 0$

$P = \frac{-5/4\cos\theta\, kL}{\sin\theta} \frac{\left[L_{so} - (41/16 + 5/2\sin\theta)^{\frac{1}{2}}\right]}{(41/16 + 5/2\sin\theta)^{1/2}}$

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Correct here, but not consistent w/ previous page

$$\Pi(\theta,P,L,k) := P\cdot L\cdot\cos(\theta) + \frac{1}{2}\cdot k\cdot L^2\cdot\left[\sqrt{\frac{41}{16}+\frac{5}{2}\cdot\sin(\theta)} - \sqrt{1^2+\left(\frac{5}{4}\right)^2}\right]^2$$

$$\frac{d}{d\theta}\Pi(\theta,P,L,k) \rightarrow (-P)\cdot L\cdot\sin(\theta) + 5\cdot k\cdot L^2\cdot\frac{\frac{1}{4}\cdot(41+40\cdot\sin(\theta))^{\frac{1}{2}} - \frac{1}{4}\cdot 41^{\frac{1}{2}}}{(41+40\cdot\sin(\theta))^{\frac{1}{2}}}\cdot\cos(\theta)$$

✓

$$P(\theta,L,k) := 5\cdot k\cdot L\cdot\frac{\frac{1}{4}\cdot(41+40\cdot\sin(\theta))^{\frac{1}{2}} - \frac{1}{4}\cdot 41^{\frac{1}{2}}}{(41+40\cdot\sin(\theta))^{\frac{1}{2}}}\cdot\frac{\cos(\theta)}{\sin(\theta)}$$

provided $\sin\theta \neq 0$

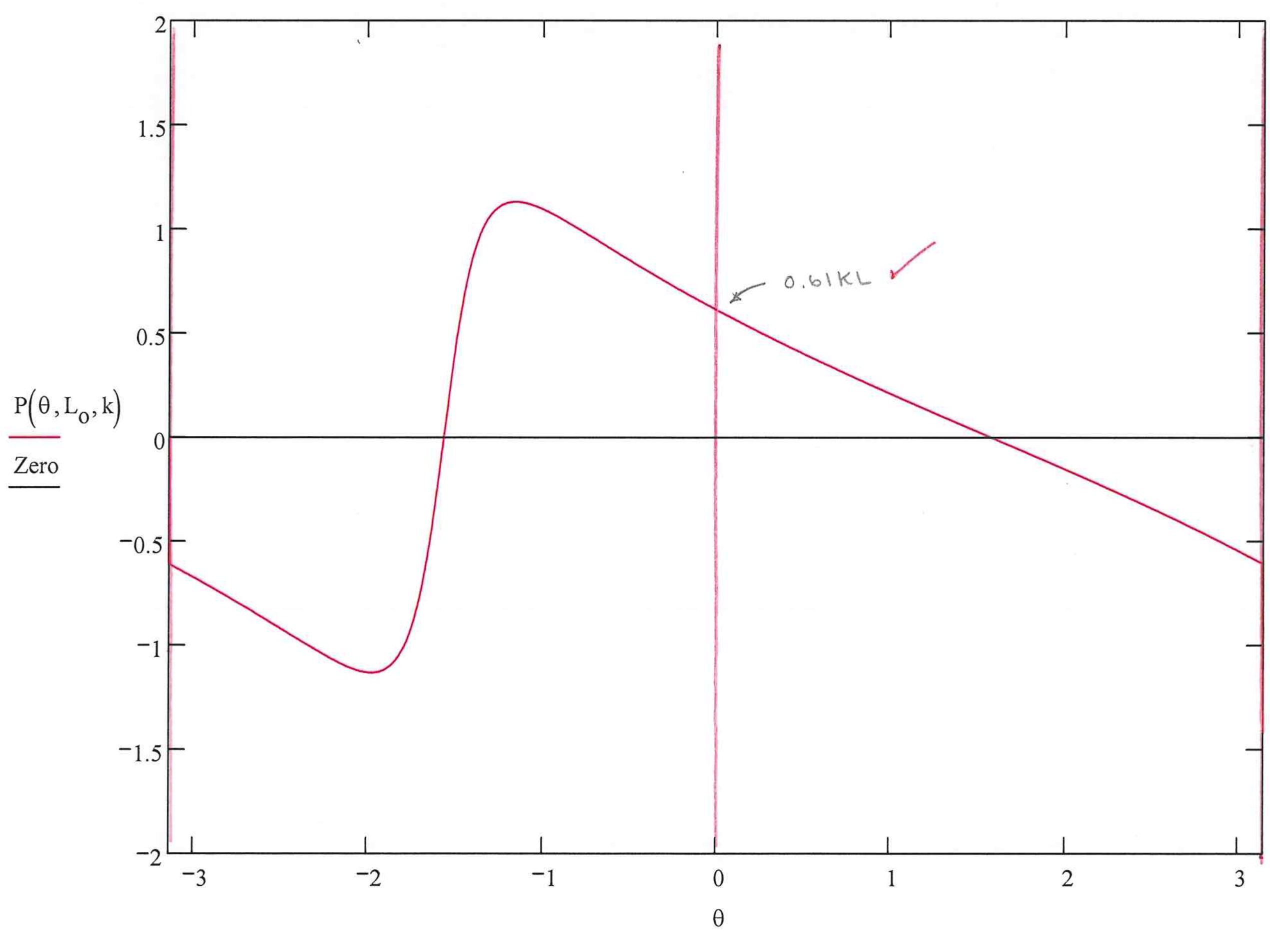
$k := 1$

$L_o := 1$

$Zero := 0$

Then, assuming k, L = 1, graph P(θ)

Equilibrium equations:



To check stability:

$$\Pi_P(\theta,P,L,k) := P\cdot L\cdot\cos(\theta) + \frac{1}{2}\cdot k\cdot L^2\cdot\left[\sqrt{\frac{41}{16}+\frac{5}{2}\cdot\sin(\theta)} - \sqrt{1^2+\left(\frac{5}{4}\right)^2}\right]^2$$

$$\frac{d^2}{d\theta^2}\Pi_P(\theta,P,L,k) \rightarrow (-P)\cdot L\cdot\cos(\theta) + 25\cdot k\cdot\frac{L^2}{41+40\cdot\sin(\theta)}\cdot\cos(\theta)^2 - 100\cdot k\cdot L^2\cdot\frac{\frac{1}{4}\cdot(41+40\cdot\sin(\theta))^{\frac{1}{2}} - \frac{1}{4}\cdot 41^{\frac{1}{2}}}{(41+40\cdot\sin(\theta))^{\frac{3}{2}}}\cdot\cos(\theta)^2 - 5\cdot k\cdot L^2\cdot\frac{\frac{1}{4}\cdot(41+40\cdot\sin(\theta))^{\frac{1}{2}} - \frac{1}{4}\cdot 41^{\frac{1}{2}}}{(41+40\cdot\sin(\theta))^{\frac{1}{2}}}\cdot\sin(\theta)$$

$$P\cdot L\cdot\cos(\theta) < 25\cdot k\cdot\frac{L^2}{41+40\cdot\sin(\theta)}\cdot\cos(\theta)^2 - 100\cdot k\cdot L^2\cdot\frac{\frac{1}{4}\cdot(41+40\cdot\sin(\theta))^{\frac{1}{2}} - \frac{1}{4}\cdot 41^{\frac{1}{2}}}{(41+40\cdot\sin(\theta))^{\frac{3}{2}}}\cdot\cos(\theta)^2 - 5\cdot k\cdot L^2\cdot\frac{\frac{1}{4}\cdot(41+40\cdot\sin(\theta))^{\frac{1}{2}} - \frac{1}{4}\cdot 41^{\frac{1}{2}}}{(41+40\cdot\sin(\theta))^{\frac{1}{2}}}\cdot\sin(\theta)$$

evaluate at equilibrium positions

Define potential and internal strain energies from inequality

after 2 derivatives, these terms no longer correspond to energy

$$Pot(\theta,P,L) := P\cdot L\cdot\cos(\theta)$$

$$Int(\theta,L,k) := 25\cdot k\cdot\frac{L^2}{41+40\cdot\sin(\theta)}\cdot\cos(\theta)^2 - 100\cdot k\cdot L^2\cdot\frac{\frac{1}{4}\cdot(41+40\cdot\sin(\theta))^{\frac{1}{2}} - \frac{1}{4}\cdot 41^{\frac{1}{2}}}{(41+40\cdot\sin(\theta))^{\frac{3}{2}}}\cdot\cos(\theta)^2 - 5\cdot k\cdot L^2\cdot\frac{\frac{1}{4}\cdot(41+40\cdot\sin(\theta))^{\frac{1}{2}} - \frac{1}{4}\cdot 41^{\frac{1}{2}}}{(41+40\cdot\sin(\theta))^{\frac{1}{2}}}\cdot\sin(\theta)$$

$$Pot(0,P,L) \rightarrow P\cdot L$$

$$Int(0,L,k) \rightarrow \frac{25}{41}\cdot k\cdot L^2$$

Potential energy is less than internal strain energy at θ=0 when P is less than 0.61kL.

$$P_{stable}(k,L) := 0.61\cdot k\cdot L$$

$P < 0.61kL \longrightarrow$ stable

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When considering stability along the curve, use P(θ) equation derived earlier ✓ correct!

$$Pot_{curve}(\theta, L, k) := 5 \cdot k \cdot L^2 \cdot \frac{\frac{1}{4} \cdot (41 + 40 \cdot \sin(\theta))^{\frac{1}{2}} - \frac{1}{4} \cdot 41^{\frac{1}{2}}}{(41 + 40 \cdot \sin(\theta))^{\frac{1}{2}}} \cdot \frac{\cos(\theta)^2}{\sin(\theta)}$$ ✓

$$Int_{curve}(\theta, L, k) := Int(\theta, L, k)$$

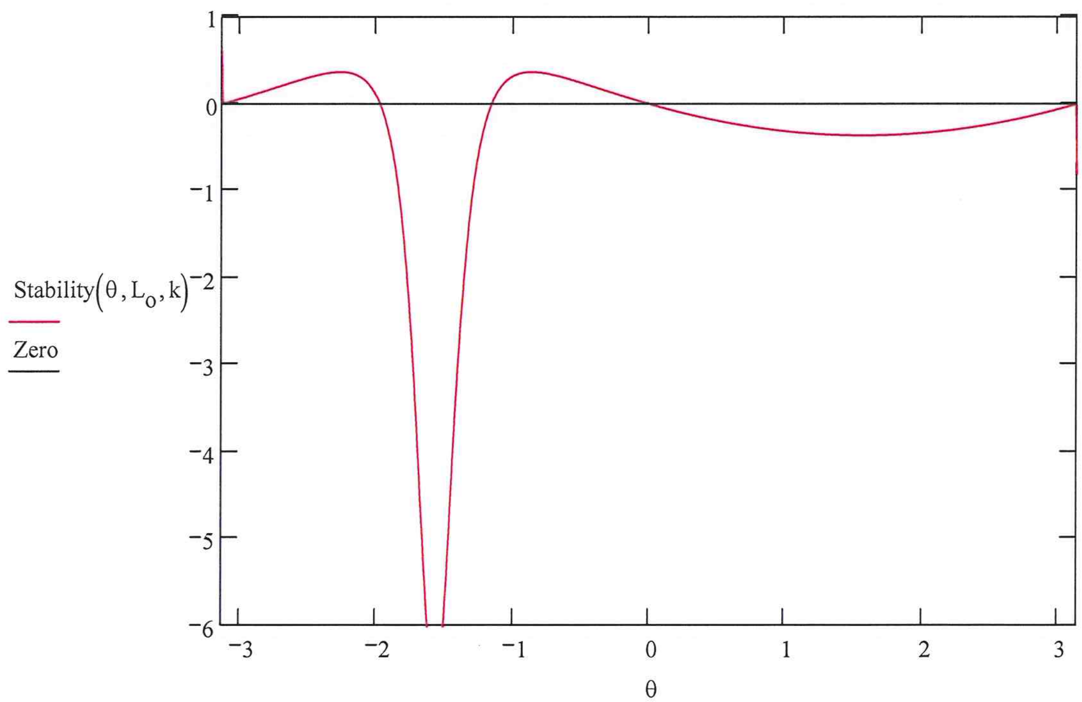
Stability occurs when the internal strain energy is larger than the potential energy from the load, or when the following graph is positive ✓

$$Stability(\theta, L, k) := Int_{curve}(\theta, L, k) - Pot_{curve}(\theta, L, k)$$

$k := 1$

$L_o := 1$

$Zero := 0$ ✓



The graph changes from positive to negative at the local maximum and minimum values for P

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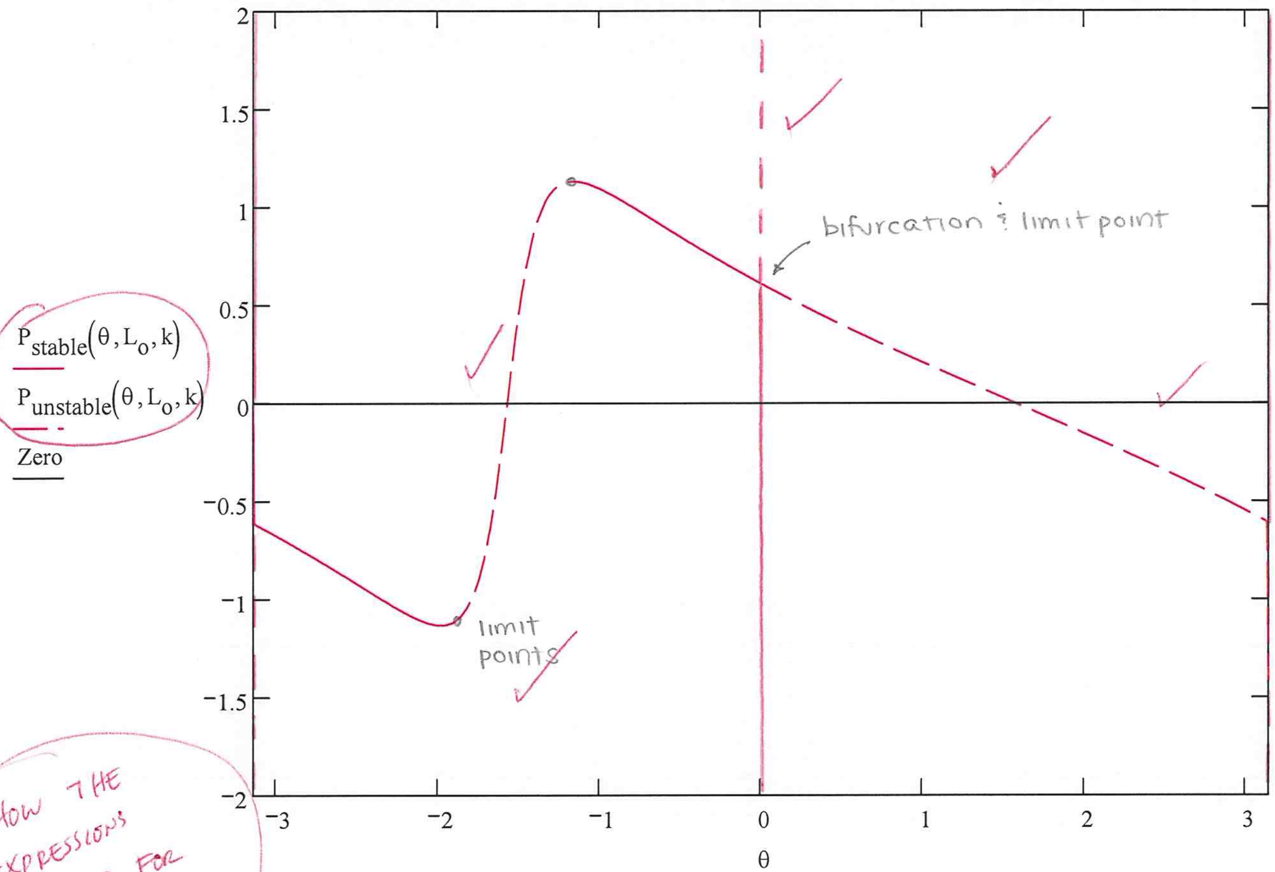
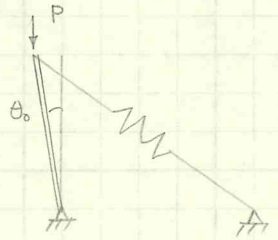
Considering the stability requirements found above, draw the bifurcation diagram

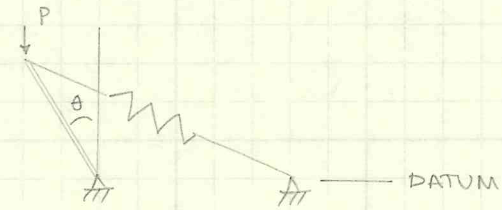


SHOW THE EXPRESSIONS YOU USED FOR THESE FUNCTIONS

HOMEWORK # 7

c) consider an imperfection θ_0





$$V = PL\cos\theta$$

$$U = \frac{1}{2} k x^2$$

$$\quad \llcorner \; L_f - L_i$$

$$L_i = \left[(L\cos\theta_0)^2 + (5/4L + L\sin\theta_0)^2 \right]^{\frac{1}{2}}$$

$$L_f = \left[(L\cos\theta)^2 + (5/4L + L\sin\theta)^2 \right]^{1/2}$$

$$\Pi = PL\cos\theta + \frac{1}{2}kL^2 \left[\underbrace{\left[\cos^2\theta_0 + (5/4 + \sin\theta_0)^2\right]^{\frac{1}{2}}}_{L_i} - \underbrace{\left[\cos^2\theta + (5/4 + \sin\theta)^2\right]^{\frac{1}{2}}}_{L_f} \right]^2$$

$\Delta = L_f - L_i$

using L_i, L_f,

$$\frac{d\Pi}{d\theta} = -PL\sin\theta - \frac{1}{2}kL^2 \; \frac{L_i - L_f}{L_f} \left[2(5/4 + \sin\theta)\cos\theta - 2\cos\theta\sin\theta \right] = 0$$

$$P(\theta) = \frac{-kL}{2\sin\theta} \; \frac{L_i - L_f}{L_f} \left[2(5/4 + \sin\theta)\cos\theta - 2\cos\theta\sin\theta \right]$$

(written w/ incorrect signs,
$L_i - L_f$. mathcad
sheet shows correct version)

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$$\Delta x_k(\theta,\theta_o,L) := \sqrt{(L\cdot\cos(\theta))^2 + \left(\frac{5}{4}\cdot L + L\cdot\sin(\theta)\right)^2} - \sqrt{(L\cdot\cos(\theta_o))^2 + \left(\frac{5}{4}\cdot L + L\cdot\sin(\theta_o)\right)^2}$$

$$\Pi_{ii}(\theta,\theta_o,P,L,k) := P\cdot L\cdot\cos(\theta) + \frac{1}{2}\cdot k\cdot L^2\cdot\left(\sqrt{\frac{41}{16}+\frac{5}{2}\cdot\sin(\theta)} - \sqrt{\frac{41}{16}+\frac{5}{2}\cdot\sin(\theta_o)}\right)^2$$

$$\frac{d}{d\theta}\Pi_{ii}(\theta,\theta_o,P,L,k) \rightarrow (-P)\cdot L\cdot\sin(\theta) + 5\cdot k\cdot L^2\cdot\frac{\frac{1}{4}\cdot(41+40\cdot\sin(\theta))^{\frac{1}{2}} - \frac{1}{4}\cdot(41+40\cdot\sin(\theta_o))^{\frac{1}{2}}}{(41+40\cdot\sin(\theta))^{\frac{1}{2}}}\cdot\cos(\theta)$$

$$P_{ii}(\theta,\theta_o,L,k) := 5\cdot k\cdot L\cdot\frac{\frac{1}{4}\cdot(41+40\cdot\sin(\theta))^{\frac{1}{2}} - \frac{1}{4}\cdot(41+40\cdot\sin(\theta_o))^{\frac{1}{2}}}{(41+40\cdot\sin(\theta))^{\frac{1}{2}}}\cdot\frac{\cos(\theta)}{\sin(\theta)}$$

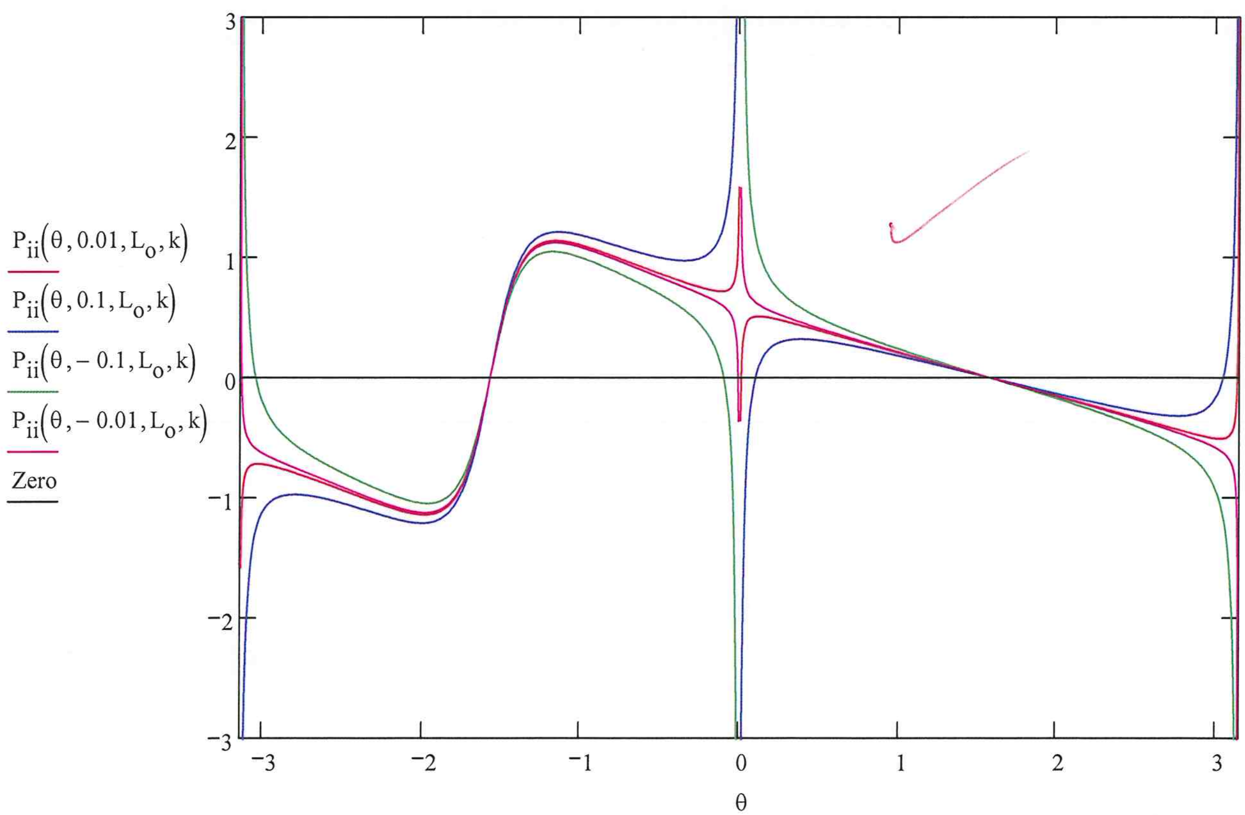
$L_o := 1$

$k := 1$

$\text{Zero} := 0$

Assume and/or assign values to L, k, and the initial imperfection, then graph P(θ)

Equilibrium plot



To find stability,

$$\Pi_{ii}(\theta,\theta_o,P,L,k) := P\cdot L\cdot\cos(\theta) + \frac{1}{2}\cdot k\cdot L^2\cdot\left(\sqrt{\frac{41}{16}+\frac{5}{2}\cdot\sin(\theta)} - \sqrt{\frac{41}{16}+\frac{5}{2}\cdot\sin(\theta_o)}\right)^2$$

$$\Pi stab_{ii}(\theta,\theta_o,P,L,k) := -P\cdot L\cdot\cos(\theta) + \frac{k\cdot L^2\cdot\left[25\cdot\cos(\theta)^2 - \left(\sqrt{41+40\cdot\sin(\theta)} - \sqrt{41+40\cdot\sin(\theta_o)}\right)\cdot\left(\frac{25\cdot\cos(\theta)^2}{\sqrt{41+40\cdot\sin(\theta)}} + \frac{5}{4}\cdot\sin(\theta)\cdot\sqrt{41+40\cdot\sin(\theta)}\right)\right]}{41+40\cdot\sin(\theta)}$$

$$P\cdot L\cdot\cos(\theta) < \frac{k\cdot L^2\cdot\left[25\cdot\cos(\theta)^2 - \left(\sqrt{41+40\cdot\sin(\theta)} - \sqrt{41+40\cdot\sin(\theta_o)}\right)\cdot\left(\frac{25\cdot\cos(\theta)^2}{\sqrt{41+40\cdot\sin(\theta)}} + \frac{5}{4}\cdot\sin(\theta)\cdot\sqrt{41+40\cdot\sin(\theta)}\right)\right]}{41+40\cdot\sin(\theta)}$$

Define potential and internal strain ~~energies~~ from inequality

$$Pot(\theta,P,L) := P\cdot L\cdot\cos(\theta)$$

$$Int(\theta,\theta_o,L,k) := \frac{k\cdot L^2\cdot\left[25\cdot\cos(\theta)^2 - \left(\sqrt{41+40\cdot\sin(\theta)} - \sqrt{41+40\cdot\sin(\theta_o)}\right)\cdot\left(\frac{25\cdot\cos(\theta)^2}{\sqrt{41+40\cdot\sin(\theta)}} + \frac{5}{4}\cdot\sin(\theta)\cdot\sqrt{41+40\cdot\sin(\theta)}\right)\right]}{41+40\cdot\sin(\theta)}$$

The previous equilibrium equation, θ=0, is no longer valid, so it is not checked

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When considering stability along the curve, use P(θ) equation derived earlier ✓

$$Pot_{curve}(\theta,\theta_o,L,k) := 5\cdot k\cdot L^2\cdot\frac{\frac{1}{4}\cdot(41+40\cdot\sin(\theta))^{\frac{1}{2}}-\frac{1}{4}\cdot\left(41+40\cdot\sin(\theta_o)\right)^{\frac{1}{2}}}{(41+40\cdot\sin(\theta))^{\frac{1}{2}}}\cdot\frac{\cos(\theta)^2}{\sin(\theta)}$$

$$Int_{curve}(\theta,\theta_o,L,k) := Int(\theta,\theta_o,L,k)$$

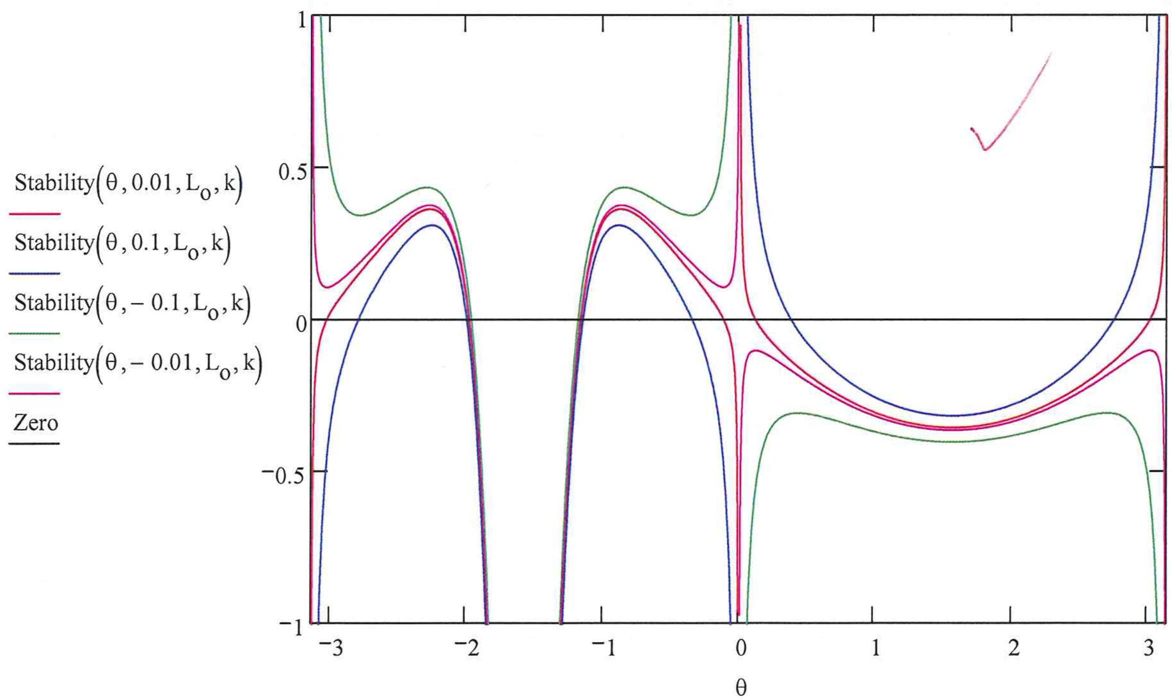
Stability occurs when the internal strain energy is larger than the potential energy from the load, or when the following graph is positive (above blue dashed line)

$$Stability(\theta,\theta_o,L,k) := Int_{curve}(\theta,\theta_o,L,k) - Pot_{curve}(\theta,\theta_o,L,k)$$ ✓

$k := 1$

$L_o := 1$

$Zero := 0$



Again, compare plots of Pii and Stability and determine where the positive values correspond.

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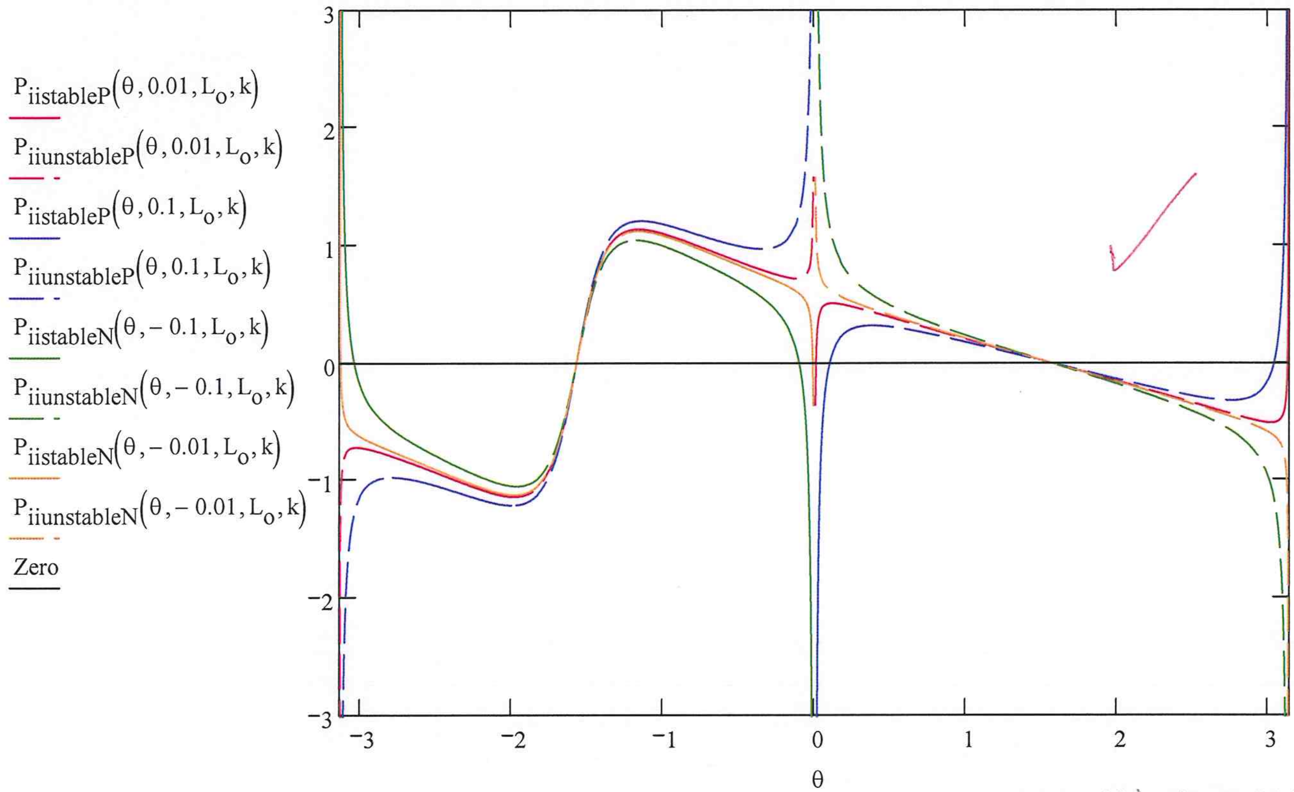
Considiering the stability requirements, draw the bifurcation diagram:



limit points exist where one curve changes from stable to unstable.

Discussion:

In the perfect case, when the critical load is reached (P = 0.61kL), three things can happen. As more load is added, the column can stay vertical, but it will do so unstably. If disturbed away from the spring, the geometry will snap through to the stable equilibrium that occurs at π. If disturbed towards the spring, more load can be added without failure up to the point θ=1.131kL, where it will snap through to the $\theta=\pi$ equilibrium curve.

When an initial imperfection exists, the response is highly dependent on the sign of that imperfection. When the initial imperfection is towards the spring (a negative θ_o), the response mirrors the perfect case for negative rotations. If the initial imperfection is away from the spring, the load reaches 0.51kL (for θ_o=0.01) before becoming unstable, snapping through to an asymptote along $\theta=\pi$. For both cases, it is physically impossible for the column to deflect away from the direction of initial imperfection. For positive θ_o, the negative rotation is similar to the perfect case, except near θ=0 and $\theta=-\pi$. For negative θ_o, all positive rotations result in an unstable system.

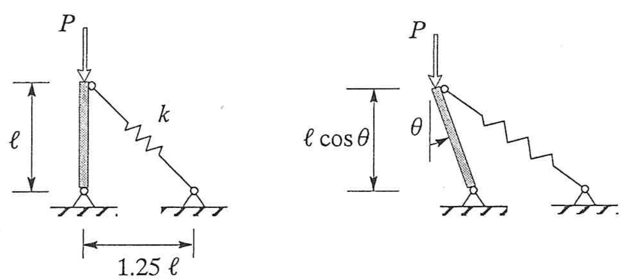
University of Texas
Dept. of Civil, Arch. & Env. Eng.

Instructor: E. B. Williamson
CE 381P

CE 381P Computer Methods in Structural Analysis

Homework 7 **Due: 30 APR 2007**

Shown in the figure below is a rigid column that is restrained by an inclined spring. The elastic spring is undeformed when $\theta = 0$.



(a). Compute the critical load for the structure.

(b). Construct the bifurcation diagram for the structure. Clearly indicate all equilibrium paths and indicate their stability properties.

(c). Consider the effects of an initial imperfection of an amount θ_o and draw the bifurcation diagram for the imperfect case.

For Discussion: Describe the response of the system at load values near the critical load. Specifically, will the system experience "snap-through" or can it continue to carry load?

BONUS HOMEWORK

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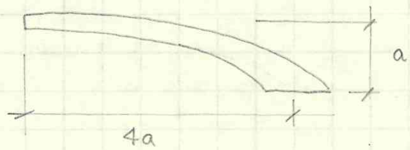
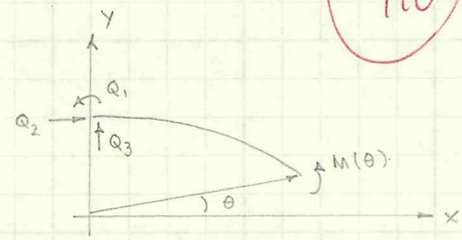


E, b

$d = d_0 \left(1 + (x/4a)^2\right)$

or, $d = d_0 (1 + \cos^2\theta)$



$x = 4a\cos\theta$
$y = a\sin\theta$

$M(x,y) = -Q_1 + Q_2(a-y) + Q_3(x)$

$M(\theta) = -Q_1 + Q_2 a(1-\sin\theta) + Q_3(4a\cos\theta)$

Thus,

a) $\underset{\sim}{D}^T = \begin{bmatrix} -1 & a(1-\sin\theta) & 4a\cos\theta \end{bmatrix}$

such that

$M(\theta) = \underset{\sim}{D}^T \underset{\sim}{F}_F$,

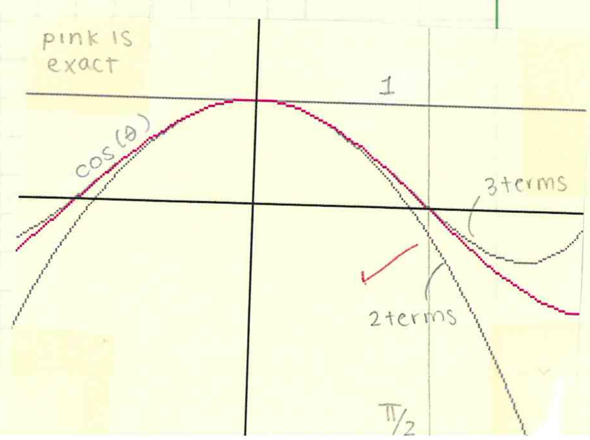
with $\underset{\sim}{F}_F = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}$

Determine the flexibility matrix, f:

$$f = \int_0^{\pi/2} \underset{\sim}{D}^T \frac{\underset{\sim}{D}}{E(x)\, I(x)}\, d\theta$$

$$f = \int_0^{\pi/2} \frac{\begin{bmatrix} -1 & a(1-\sin\theta) & 4a\cos\theta \end{bmatrix}}{E \cdot \frac{1}{12} b d_0^3 (1+\cos^2\theta)^3} \begin{bmatrix} -1 \\ a(1-\sin\theta) \\ 4a\cos\theta \end{bmatrix} R(\theta)\, d\theta$$

- six unique terms
- sine, cosine should not be approximated with just x, 1 — does not match well through $\pi/2$ →



BONUS HOMEWORK

Method 1 for finding *f*: using Gaussian Quadrature and linear sine and cosine approximations

Define general equations for the stations and weightings of 3rd order Gauss functions

$$x(a,b,\xi) := \frac{1}{2}\cdot\left[a\cdot(1-\xi) + b\cdot(1+\xi)\right]$$

$$x_1(a,b) := x\left(a,b,-\sqrt{\frac{3}{5}}\right) \qquad w_1 := \frac{5}{9}$$

$$x_2(a,b) := x(a,b,0) \qquad w_2 := \frac{8}{9}$$

$$x_3(a,b) := x\left(a,b,\sqrt{\frac{3}{5}}\right) \qquad w_3 := \frac{5}{9}$$

Define D matrix, and assign a value to a for computations

$$D(a,\theta) := \begin{bmatrix} -1 \\ a\cdot(1-\sin(\theta)) \\ 4\cdot a\cdot\cos(\theta) \end{bmatrix} \qquad a := 1$$

RATHER THAN USING A LINEAR APPROXIMATION FOR FUNCTION OVER CERTAIN RANGE, YOU COULD USE MORE TERMS FOR SERIES EXPANSION OR SUBDIVIDE INTEGRAL

Using Excel, I found linear best-fit lines for sine and cosine, breaking interval [0, π/2] into two sections that were each approximately linear (see attached Excel graph).

$$\cos1(\theta) := -0.3784\cdot\theta + 1.0548 \qquad \text{For } 0 < \theta < 7/32\pi$$

$$\cos2(\theta) := -0.8829\cdot\theta + 1.4075 \qquad \text{For } 7/32\pi < \theta < \pi/2$$

$$\sin1(\theta) := 0.8377\cdot\theta + 0.0416 \qquad \text{For } 0 < \theta < 5/16\pi$$

$$\sin2(\theta) := 0.287\cdot\theta + 0.5723 \qquad \text{For } 5/16\pi < \theta < \pi/2$$

graphed on following page

To make the equations smaller, the depth and radius (which vary with location) were written for each segment

$$d1(\theta) := 1 + \cos1(\theta)^2 \qquad d2(\theta) := 1 + \cos2(\theta)^2$$

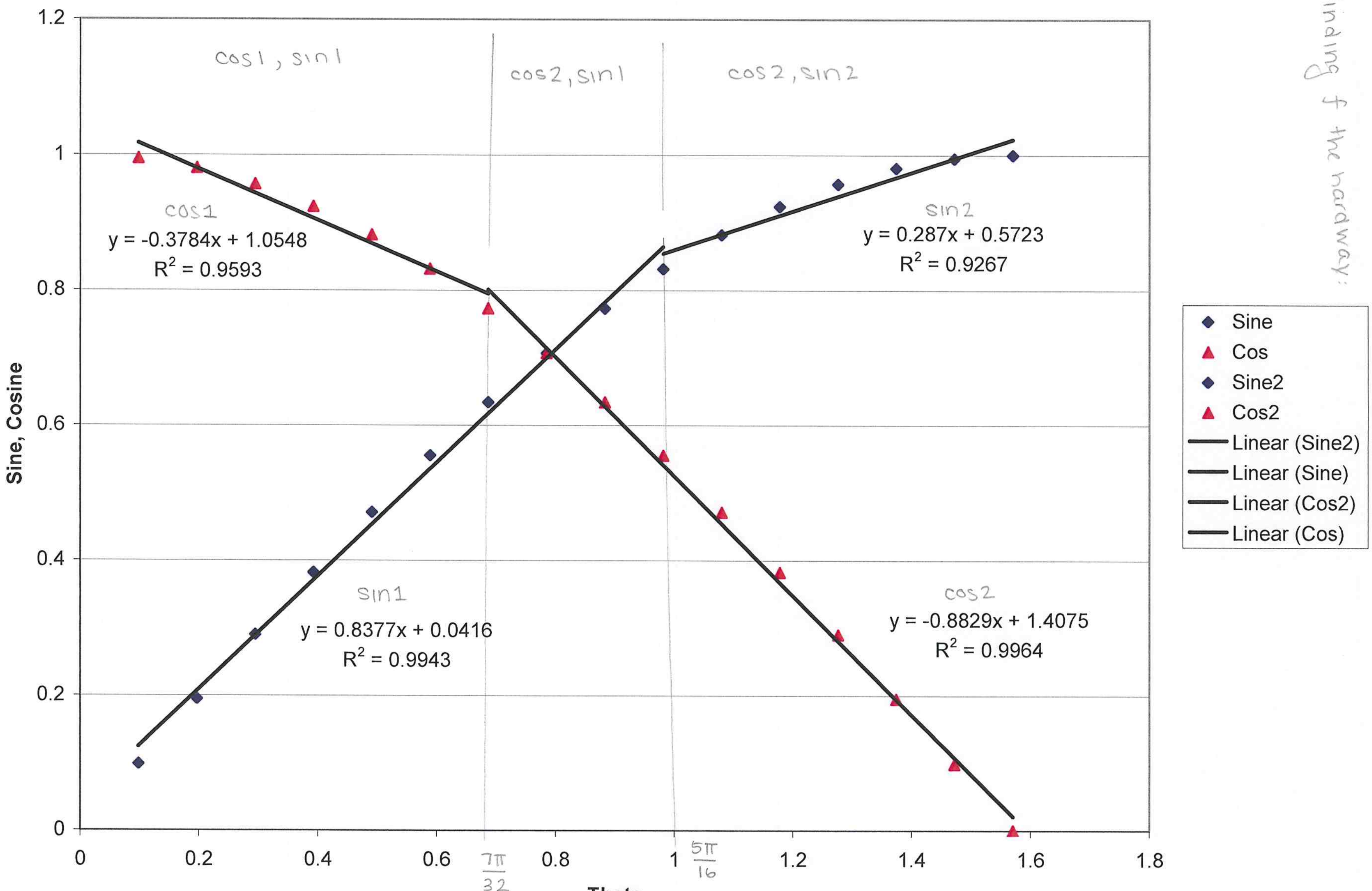
$$r1(\theta) := \sqrt{(a\cdot\cos1(\theta))^2 + (-4\cdot a\cdot\sin1(\theta))^2} \qquad r2(\theta) := \sqrt{(a\cdot\cos2(\theta))^2 + (-4\cdot a\cdot\sin1(\theta))^2}$$

$$r3(\theta) := \sqrt{(a\cdot\cos2(\theta))^2 + (-4\cdot a\cdot\sin2(\theta))^2}$$

CE 381P: ANALYSIS

BONUS HOMEWORK

Finding f the hardway:



Fit two linear curves to parts of sine, cosine graphs on $[0, \pi/2]$. ✓

Use in future calculations

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BONUS HOMEWORK

3rd Order Gaussian Quadrature approximates the function f(x) on the interval [a, b] using:

$$\int_a^b f(x)\,dx := \frac{b-a}{2}\cdot\left(w_1\cdot f(x_1) + w_2\cdot f(x_2) + w_3\cdot f(x_3)\right)$$

The weights and statitions (w and x values) are given above. Because the sine and cosine functions are approximated with two equations (and three different combinations), the following f(x) functions are needed.

$$F11_1(\theta) := \frac{(-1)^2}{d1(\theta)^3}\cdot r1(\theta) \qquad F12_1(\theta) := \frac{(-1)\cdot a\cdot(1-\sin1(\theta))}{d1(\theta)^3}\cdot r1(\theta) \qquad F13_1(\theta) := \frac{(-1)\cdot 4\cdot a\cdot\cos1(\theta)}{d1(\theta)^3}\cdot r1(\theta)$$

$$F11_3(\theta) := \frac{(-1)^2}{d2(\theta)^3}\cdot r3(\theta) \qquad F12_2(\theta) := \frac{(-1)\cdot[a\cdot(1-\sin1(\theta))]}{d2(\theta)^3}\cdot r2(\theta) \qquad F13_3(\theta) := \frac{(-1)\cdot 4\cdot a\cdot\cos2(\theta)}{d2(\theta)^3}\cdot r3(\theta)$$

$$F12_3(\theta) := \frac{(-1)\cdot[a\cdot(1-\sin2(\theta))]}{d2(\theta)^3}\cdot r3(\theta)$$

$$F22_1(\theta) := \frac{[a\cdot(1-\sin1(\theta))]^2}{d1(\theta)^3}\cdot r1(\theta) \qquad F23_1(\theta) := \frac{a\cdot(1-\sin1(\theta))\cdot 4\cdot a\cdot\cos1(\theta)}{d1(\theta)^3}\cdot r1(\theta)$$

$$F22_2(\theta) := \frac{[a\cdot(1-\sin1(\theta))]^2}{d2(\theta)^3}\cdot r2(\theta) \qquad F23_2(\theta) := \frac{a\cdot(1-\sin1(\theta))\cdot 4\cdot a\cdot\cos2(\theta)}{d2(\theta)^3}\cdot r2(\theta)$$

$$F22_3(\theta) := \frac{[a\cdot(1-\sin2(\theta))]^2}{d2(\theta)^3}\cdot r3(\theta) \qquad F23_3(\theta) := \frac{a\cdot(1-\sin2(\theta))\cdot 4\cdot a\cdot\cos2(\theta)}{d2(\theta)^3}\cdot r3(\theta)$$

$$F33_1(\theta) := \frac{(4\cdot a\cdot\cos1(\theta))^2}{d1(\theta)^3}\cdot r1(\theta)$$

$$F33_3(\theta) := \frac{(4\cdot a\cdot\cos2(\theta))^2}{d2(\theta)^3}\cdot r3(\theta)$$

The numbering scheme for each function refers to the location in the f matrix and the linear approximation number.

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BONUS HOMEWORK

Because the flexibility matrix is symmetric, only six unique values must be calculated

$$f11_1 := \frac{7\pi}{64}\cdot\left(w_1\cdot F11_1\left(x_1\left(0,\frac{7\pi}{32}\right)\right) + w_2\cdot F11_1\left(x_2\left(0,\frac{7\pi}{32}\right)\right) + w_3\cdot F11_1\left(x_3\left(0,\frac{7\pi}{32}\right)\right)\right)$$

$$f11_2 := \frac{9\pi}{64}\cdot\left(w_1\cdot F11_3\left(x_1\left(\frac{7\pi}{32},\frac{\pi}{2}\right)\right) + w_2\cdot F11_3\left(x_2\left(\frac{7\pi}{32},\frac{\pi}{2}\right)\right) + w_3\cdot F11_3\left(x_3\left(\frac{7\pi}{32},\frac{\pi}{2}\right)\right)\right)$$

$f_{11} := f11_1 + f11_2$ $\qquad f_{11} = 2.262$

$$f12_1 := \frac{7\pi}{64}\cdot\left(w_1\cdot F12_1\left(x_1\left(0,\frac{7\pi}{32}\right)\right) + w_2\cdot F12_1\left(x_2\left(0,\frac{7\pi}{32}\right)\right) + w_3\cdot F12_1\left(x_3\left(0,\frac{7\pi}{32}\right)\right)\right)$$

$$f12_2 := \frac{3\pi}{64}\cdot\left(w_1\cdot F12_2\left(x_1\left(\frac{7\pi}{32},\frac{5\pi}{16}\right)\right) + w_2\cdot F12_2\left(x_2\left(\frac{7\pi}{32},\frac{5\pi}{16}\right)\right) + w_3\cdot F12_2\left(x_3\left(\frac{7\pi}{32},\frac{5\pi}{16}\right)\right)\right)$$

$$f12_3 := \frac{3\pi}{32}\cdot\left(w_1\cdot F12_3\left(x_1\left(\frac{5\pi}{16},\frac{\pi}{2}\right)\right) + w_2\cdot F12_3\left(x_2\left(\frac{5\pi}{16},\frac{\pi}{2}\right)\right) + w_3\cdot F12_3\left(x_3\left(\frac{5\pi}{16},\frac{\pi}{2}\right)\right)\right)$$

$f_{12} := f12_1 + f12_2 + f12_3$ $\qquad f_{12} = -0.273$

$$f13_1 := \frac{7\pi}{64}\cdot\left(w_1\cdot F13_1\left(x_1\left(0,\frac{7\pi}{32}\right)\right) + w_2\cdot F13_1\left(x_2\left(0,\frac{7\pi}{32}\right)\right) + w_3\cdot F13_1\left(x_3\left(0,\frac{7\pi}{32}\right)\right)\right)$$

$$f13_2 := \frac{9\pi}{64}\cdot\left(w_1\cdot F13_3\left(x_1\left(\frac{7\pi}{32},\frac{\pi}{2}\right)\right) + w_2\cdot F13_3\left(x_2\left(\frac{7\pi}{32},\frac{\pi}{2}\right)\right) + w_3\cdot F13_3\left(x_3\left(\frac{7\pi}{32},\frac{\pi}{2}\right)\right)\right)$$

$f_{13} := f13_1 + f13_2$ $\qquad f_{13} = -3.234$

$$f22_1 := \frac{7\pi}{64}\cdot\left(w_1\cdot F22_1\left(x_1\left(0,\frac{7\pi}{32}\right)\right) + w_2\cdot F22_1\left(x_2\left(0,\frac{7\pi}{32}\right)\right) + w_3\cdot F22_1\left(x_3\left(0,\frac{7\pi}{32}\right)\right)\right)$$

$$f22_2 := \frac{3\pi}{64}\cdot\left(w_1\cdot F22_2\left(x_1\left(\frac{7\pi}{32},\frac{5\pi}{16}\right)\right) + w_2\cdot F22_2\left(x_2\left(\frac{7\pi}{32},\frac{5\pi}{16}\right)\right) + w_3\cdot F22_2\left(x_3\left(\frac{7\pi}{32},\frac{5\pi}{16}\right)\right)\right)$$

$$f22_3 := \frac{3\pi}{32}\cdot\left(w_1\cdot F22_3\left(x_1\left(\frac{5\pi}{16},\frac{\pi}{2}\right)\right) + w_2\cdot F22_3\left(x_2\left(\frac{5\pi}{16},\frac{\pi}{2}\right)\right) + w_3\cdot F22_3\left(x_3\left(\frac{5\pi}{16},\frac{\pi}{2}\right)\right)\right)$$

$f_{22} := f22_1 + f22_2 + f22_3$ $\qquad f_{22} = 0.1$

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BONUS HOMEWORK

$$f23_1 := \frac{7\pi}{64}\cdot\left(w_1\cdot F23_1\left(x_1\left(0,\frac{7\pi}{32}\right)\right) + w_2\cdot F23_1\left(x_2\left(0,\frac{7\pi}{32}\right)\right) + w_3\cdot F23_1\left(x_3\left(0,\frac{7\pi}{32}\right)\right)\right)$$

$$f23_2 := \frac{3\pi}{64}\cdot\left(w_1\cdot F23_2\left(x_1\left(\frac{7\pi}{32},\frac{5\pi}{16}\right)\right) + w_2\cdot F23_2\left(x_2\left(\frac{7\pi}{32},\frac{5\pi}{16}\right)\right) + w_3\cdot F23_2\left(x_3\left(\frac{7\pi}{32},\frac{5\pi}{16}\right)\right)\right)$$

$$f23_3 := \frac{3\pi}{32}\cdot\left(w_1\cdot F23_3\left(x_1\left(\frac{5\pi}{16},\frac{\pi}{2}\right)\right) + w_2\cdot F23_3\left(x_2\left(\frac{5\pi}{16},\frac{\pi}{2}\right)\right) + w_3\cdot F23_3\left(x_3\left(\frac{5\pi}{16},\frac{\pi}{2}\right)\right)\right)$$

$$f_{23} := f23_1 + f23_2 + f23_3 \qquad f_{23} = 0.745$$

$$f33_1 := \frac{7\pi}{64}\cdot\left(w_1\cdot F33_1\left(x_1\left(0,\frac{7\pi}{32}\right)\right) + w_2\cdot F33_1\left(x_2\left(0,\frac{7\pi}{32}\right)\right) + w_3\cdot F33_1\left(x_3\left(0,\frac{7\pi}{32}\right)\right)\right)$$

$$f33_2 := \frac{9\pi}{64}\cdot\left(w_1\cdot F33_3\left(x_1\left(\frac{7\pi}{32},\frac{\pi}{2}\right)\right) + w_2\cdot F33_3\left(x_2\left(\frac{7\pi}{32},\frac{\pi}{2}\right)\right) + w_3\cdot F33_3\left(x_3\left(\frac{7\pi}{32},\frac{\pi}{2}\right)\right)\right)$$

$$f_{33} := f33_1 + f33_2 \qquad f_{33} = 6.977$$

$$f_{M1} := \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix} \qquad f_{M1} = \begin{pmatrix} 2.262 & -0.273 & -3.234 \\ -0.273 & 0.1 & 0.745 \\ -3.234 & 0.745 & 6.977 \end{pmatrix}$$

Close - some numerical error, but solution is close to correct answer

Consider the influence of a, as well as other constants (L, b, etc):

b)

$$f_{final}(E,a,I_o) := \frac{a}{E\cdot I_o}\cdot\begin{pmatrix} 2.262 & -0.273\cdot a & -3.234\cdot a \\ -0.273\cdot a & 0.1\cdot a^2 & 0.745\cdot a^2 \\ -3.234\cdot a & 0.745\cdot a^2 & 6.977\cdot a^2 \end{pmatrix}$$

BONUS HOMEWORK

Method 2 for finding *f*: using Mathcad functions

Using:

$$D(a,\theta) \to \begin{pmatrix} -1 \\ 1-\sin(\theta) \\ 4\cdot\cos(\theta) \end{pmatrix} \qquad a = 1 \qquad r(\theta,a) := \sqrt{(a\cdot\cos(\theta))^2 + (-4\cdot a\cdot\sin(\theta))^2}$$

$$f_{M2} := \begin{bmatrix} \int_0^{\frac{\pi}{2}} \frac{(D(a,\theta)_0)^2}{(1+\cos(\theta)^2)^3}\cdot r(\theta,a)\,d\theta & \int_0^{\frac{\pi}{2}} \frac{D(a,\theta)_0\cdot D(a,\theta)_1}{(1+\cos(\theta)^2)^3}\cdot r(\theta,a)\,d\theta & \int_0^{\frac{\pi}{2}} \frac{D(a,\theta)_0\cdot D(a,\theta)_2}{(1+\cos(\theta)^2)^3}\cdot r(\theta,a)\,d\theta \\ \int_0^{\frac{\pi}{2}} \frac{D(a,\theta)_1\cdot D(a,\theta)_0}{(1+\cos(\theta)^2)^3}\cdot r(\theta,a)\,d\theta & \int_0^{\frac{\pi}{2}} \frac{(D(a,\theta)_1)^2}{(1+\cos(\theta)^2)^3}\cdot r(\theta,a)\,d\theta & \int_0^{\frac{\pi}{2}} \frac{D(a,\theta)_1\cdot D(a,\theta)_2}{(1+\cos(\theta)^2)^3}\cdot r(\theta,a)\,d\theta \\ \int_0^{\frac{\pi}{2}} \frac{D(a,\theta)_2\cdot D(a,\theta)_0}{(1+\cos(\theta)^2)^3}\cdot r(\theta,a)\,d\theta & \int_0^{\frac{\pi}{2}} \frac{D(a,\theta)_2\cdot D(a,\theta)_1}{(1+\cos(\theta)^2)^3}\cdot r(\theta,a)\,d\theta & \int_0^{\frac{\pi}{2}} \frac{(D(a,\theta)_2)^2}{(1+\cos(\theta)^2)^3}\cdot r(\theta,a)\,d\theta \end{bmatrix}$$

$$f_{M2} = \begin{pmatrix} 2.228 & -0.265 & -3.168 \\ -0.265 & 0.099 & 0.736 \\ -3.168 & 0.736 & 6.886 \end{pmatrix}$$

b2)

$$f_{final2}(E,a,I_o) := \frac{a}{E\cdot I_o}\cdot\begin{pmatrix} 2.228 & -0.265\cdot a & -3.168\cdot a \\ -0.265\cdot a & 0.099\cdot a^2 & 0.736\cdot a^2 \\ -3.168\cdot a & 0.736\cdot a^2 & 6.886\cdot a^2 \end{pmatrix}$$

For comparison, calculate the error between Methods 1 and 2:

To demonstrate,

$$\int_0^{\frac{\pi}{2}} \frac{[a\cdot(1-\sin(\theta))]^2}{(1+\cos(\theta)^2)^3}\cdot r(\theta,a)\,d\theta = 0.099 \quad \text{vs.} \quad f_{22} = 0.1$$

$$Error_{22} := \left| \frac{\left(f_{M2}{}^{\langle 1\rangle}\right)_1 - f_{22}}{\left(f_{M2}{}^{\langle 1\rangle}\right)_1}\cdot 100 \right|$$

$$Error_{22} = 1.403$$

$$Error = \begin{pmatrix} 1.523 & 3.392 & 2.083 \\ 3.392 & 1.403 & 1.15 \\ 2.083 & 1.15 & 1.318 \end{pmatrix}$$

Not too bad, considering linear approximations were not perfect, with R^2 values as low as 0.9267

CE 381P: ANALYSIS

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BONUS HOMEWORK

Linearly varying thermal forces in flexibility analyses have the form:

$$q_{thermal} := D^T \cdot \frac{\alpha \cdot \Delta T}{d}$$

$$q_{fotherm}(a) := \begin{bmatrix} \displaystyle\int_0^{\frac{\pi}{2}} (-1) \cdot \frac{1}{1 + \cos(\theta)^2} \cdot r(\theta, a)\, d\theta \\ \displaystyle\int_0^{\frac{\pi}{2}} [a \cdot (1 - \sin(\theta))] \cdot \frac{1}{1 + \cos(\theta)^2} \cdot r(\theta, a)\, d\theta \\ \displaystyle\int_0^{\frac{\pi}{2}} (4 \cdot a \cdot \cos(\theta)) \cdot \frac{1}{1 + \cos(\theta)^2} \cdot r(\theta, a)\, d\theta \end{bmatrix}$$

$$Q_{ind}(a) := f_{M2}^{-1} \cdot q_{fotherm}(a)$$

$$Q_{ind}(a) = \begin{pmatrix} -0.94 \\ 3.199 \\ 0.113 \end{pmatrix} \quad \frac{\alpha \cdot \Delta T \cdot E \cdot I_o}{a \cdot d_o}$$

From independent forces, calculate dependent forces using the ϕ matrix

$$\text{phi}(a) := \begin{pmatrix} -1 & a & 4a \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$Q_{dep}(a) := \text{phi}(a) \cdot Q_{ind}(a)$$

$$Q_{dep}(a) = \begin{pmatrix} 4.592 \\ -3.199 \\ -0.113 \end{pmatrix}$$

Combining forces:

c)

$$Q(a) := \begin{pmatrix} -0.94 \cdot a \\ 3.199 \\ 0.113 \\ 4.592 \cdot a \\ -3.199 \\ -0.113 \end{pmatrix} \frac{\alpha \cdot \Delta T \cdot E \cdot I_o}{a \cdot d_o}$$

University of Texas
Dept. of Civil, Arch. & Env. Eng.

Instructor: E. B. Williamson
CE 381P

CE 381P Computer Methods in Structural Analysis

Homework (BONUS)

The Principle of Virtual Forces can be used to develop the flexibility matrix for a general line element. For this exercise, you will be considering the response of a non-prismatic, elliptical ring beam. The dimensions of the ellipse are such that the major axis is ~~twice~~ 4x the length of the minor axis (Fig. 1).

(*a*). Using the force quantities Q_1, Q_2, and Q_3 shown in Fig. 1, establish the equilibrium relationship for the elliptical beam. The parametric equation that describes the ellipse is given as

$$x = 4a\cos\theta$$
$$y = a\sin\theta$$

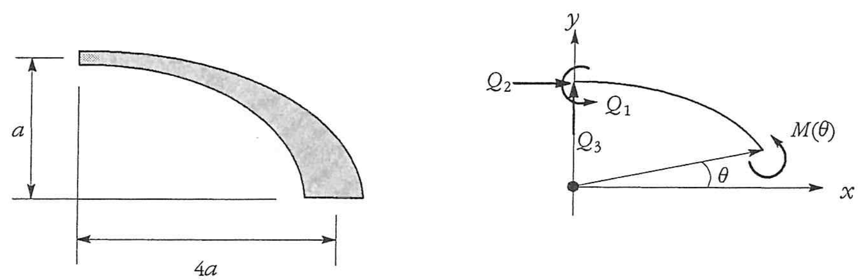


Fig. 1 Description of elliptical ring beam

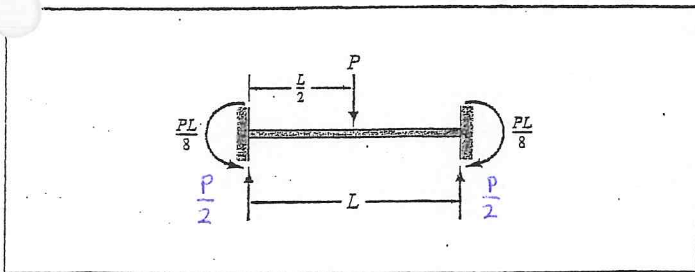
Assume that the material modulus E is uniform over the length of the beam, axial and shear deformations can be ignored, and curved beam theory need not be considered. The width of the elliptical beam, b, is constant over the length, but the depth of the beam varies according to the expression

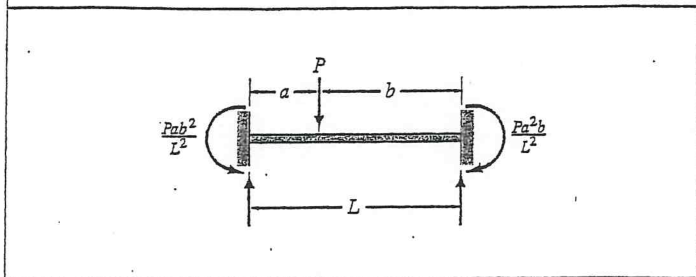
$$d(x) = d_o\left[1 + \left(\frac{x}{4a}\right)^2\right]$$

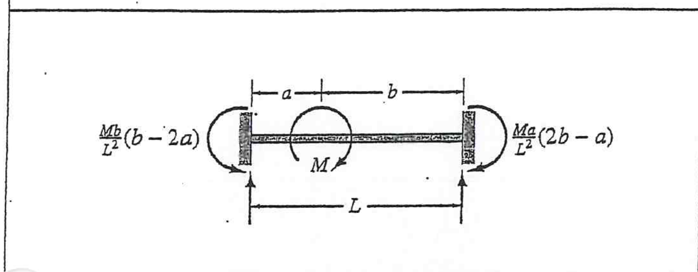
(*b*). Use numerical integration to compute values for the terms in the flexibility matrix.

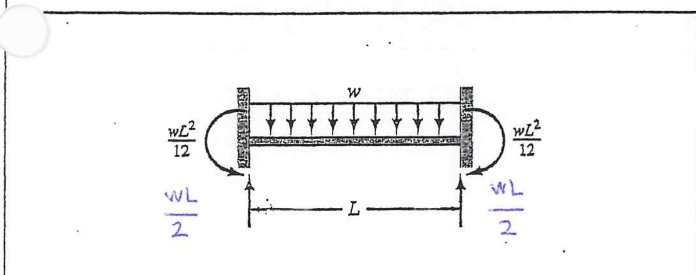
(*c*). Assuming that the elliptical beam is subjected to a thermal gradient of ΔT that acts uniformly over the length of the element, varies linearly through the depth, and gives rise to positive curvature, compute the equivalent nodal forces to use for analysis. Assume that the coefficient of thermal expansion is α.

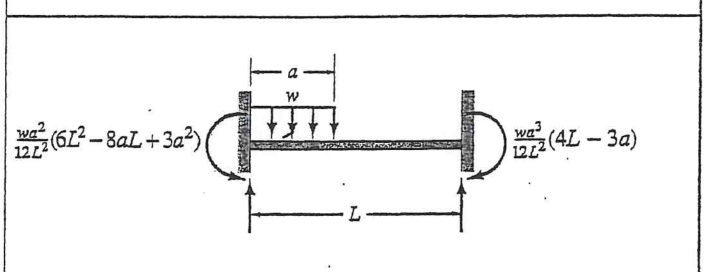
FIXED-END MOMENTS

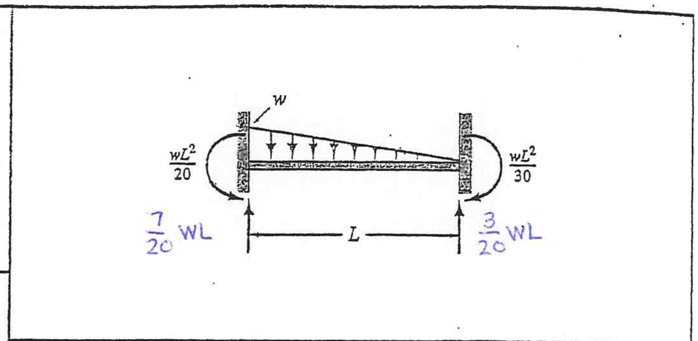


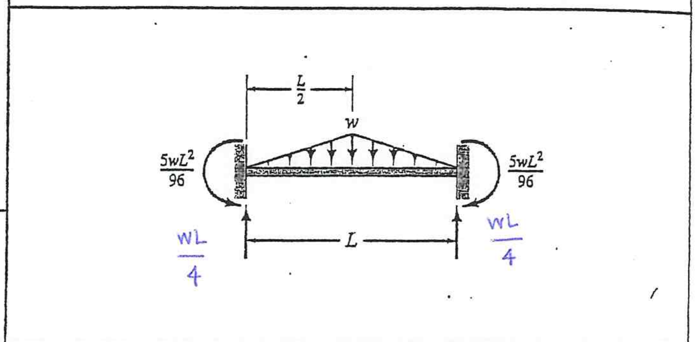


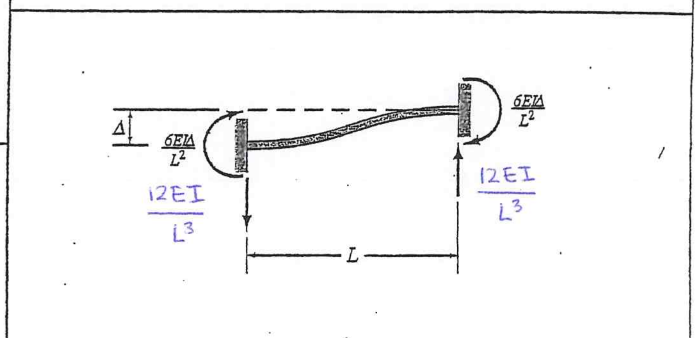


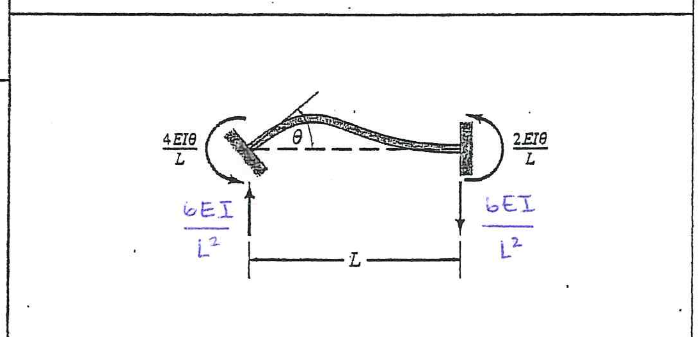












BASIC MECHANICS / REVIEW

Three keys to mechanics problems

1. Equilibrium
2. compatibility
3. constitutive properties (material behavior)

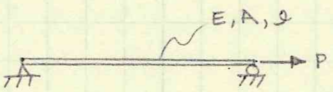
Relate displacements to deformation (strain)

Relate forces and stresses

stress to strain, or strain to stress using E, ν...

simple member: truss

Ex.



$$\Delta = \frac{PL}{EA} \text{ , but how?}$$

$\sigma = \frac{F}{A}$ – equilibrium

$\varepsilon = \Delta/L$ – compatibility

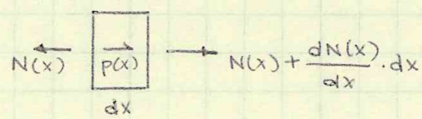
$\sigma = E\varepsilon$ – constitution

So, $E\varepsilon = F/A$

$$\frac{\Delta}{L} \cdot E = \frac{F}{A}$$

$$\Delta = \frac{FL}{EA} \quad \text{or, P not F}$$

Elaborating:



N(x) is the internal axial force

$$N(x) \equiv \sigma A(x)$$
$\quad\quad \hookrightarrow \sigma(x)$

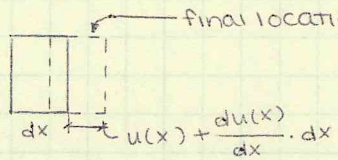
→ Equilibrium says $\Sigma F_x = 0$

$$-N(x) + N(x) + \frac{dN(x)}{dx} \cdot dx + p(x) \cdot dx = 0$$

$$\frac{dN(x)}{dx} + p(x) = 0$$

$$\underline{N'(x) + p(x) = 0}$$

→ Kinematics:



$$L_f = \frac{du(x)}{dx} \cdot dx + dx$$

$$\varepsilon = \frac{L_f - L_i}{L_i} = \frac{du(x)}{dx} = \underline{u'(x)}$$

→ constitution:

$\sigma = E\varepsilon$, if elastic and linear

NEXT: boundaries

$$\boxed{\varepsilon(x) = \frac{du(x)}{dx} = u'} \longrightarrow \text{compatibility (kinematics)}$$

Axial Force Member

① Equilibrium: $\frac{dN(x)}{dx} + p(x) = 0$

② Constitution: $\sigma = E\varepsilon$

③ Kinematics: $\varepsilon = \frac{du}{dx}$

} Equations of the "interior"

Boundary Conditions — essential BC → displ. BC

$u(0) = 0$ left end has zero displacement

$N(\ell) = P$ axial force at $x = \ell$ equals P

natural BC ⇒ force BC

The equations of the "interior" + essential BC + natural BC = Boundary Value Problem.

Solve BVP for axial force member

Simplify system of equations

③ into ② : ④ $\sigma = E \cdot \frac{du}{dx} = Eu'$

⑤ $\sigma(x) = N(x)/A(x) \rightarrow \sigma(x)A(x) = N(x)$

④ into ⑤ : ⑥ $E(x)A(x)u'(x) = N(x)$

⑦ $\boxed{\frac{d}{dx}\left(E(x)A(x)u'(x)\right) + p(x) = 0}$ ← ⑥ into ①

For this particular axial force member:

$A(x) = A$; $E(x) = E$

$\Longrightarrow EA \frac{d}{dx}(u') + p(x) = 0$

$\boxed{EA\, u''(x) + p(x) = 0}$ OR $EA\, u''(x) = -p(x)$

Homogeneous Solution:

$$p(x) = 0 \longrightarrow EA\,u''(x) = 0$$

$$u(x) = c_0 + c_1 x$$

c_0 and c_1 are constants that depend on BCs

$u(0) = 0$ + $N(\ell) = P$

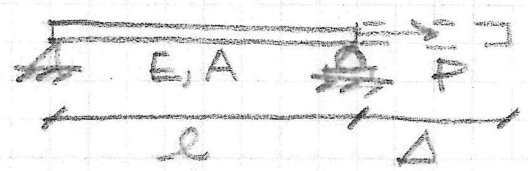
$$u(0) = 0 = c_0 + c_1 \cdot 0 \Rightarrow \boxed{c_0 = 0}$$

$$N(\ell) = P = EA\,u'(\ell) = EA \cdot c_1(\ell) = EA \cdot c_1$$

$$\Rightarrow \boxed{c_1 = P/EA}$$

Then, $\boxed{u(x) = \dfrac{P}{EA} \cdot x}$

Physical Interpretation



flexibility coeff

$$u(\ell) = \Delta = P\ell/EA = [\ell/EA] \cdot P$$

OR

$$P = [EA/\ell] \cdot \Delta$$

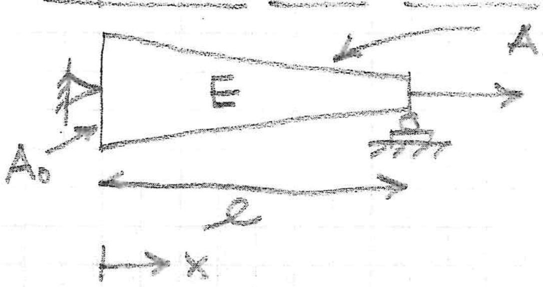
stiffness coeff

For a solution

- Need a "well posed" BVP
  1. at any location, only one type of BC can be known
  2. not all combinations of BCs are acceptable

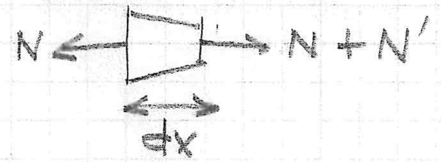
Tapered Axial Force Member

$$A(x) = A_0 [1 - \frac{x}{2\ell}]$$



Equations on "interior" | Boundary Conditions

Equilibrium : $N'(x) = 0$ | $u(0) = 0$, $N(\ell) = P$



Constitution : $\sigma = E\varepsilon$
Kinematics : $u' = \varepsilon$

since $A(x)$ is not constant,

$$\frac{d}{dx}\left(E(x)A(x)u'(x)\right) = E \cdot \frac{d}{dx}\left[A_0\left(1-\frac{x}{2\ell}\right)\cdot u'(x)\right] = 0$$

$E \neq 0$ so, $\frac{d}{dx}\left[A_0\left(1-\frac{x}{2\ell}\right)\cdot u'(x)\right] = 0$

$$\underbrace{\int d\left[A_0\left(1-\frac{x}{2\ell}\right)\cdot u'(x)\right]}_{A_0\left(1-\frac{x}{2\ell}\right)\cdot u'(x)} = \underbrace{\int 0 \cdot dx}_{0}$$

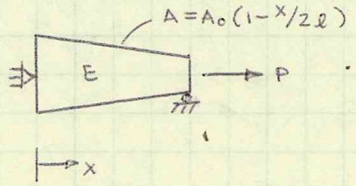
$$A_0\left(1-\frac{x}{2\ell}\right)\cdot\frac{du}{dx} = 0 + c_1 = c_1$$

$$du = c_1\, dx \big/ A_0\left(1-\frac{x}{2\ell}\right)$$

$$\int du = \int c_2 \cdot \frac{dx}{1-\frac{x}{2\ell}} \Rightarrow u = c_2 \cdot \int \frac{dx}{1-\frac{x}{2\ell}}$$

MORE COMPLEX PROBLEMS

Example



$$\frac{d}{dx}\left(E(x) \times A(x) \times \frac{du(x)}{dx}\right) = -p(x)$$

$$\frac{d}{dx}\left[E \cdot A_0 (1 - x/2L)\frac{du}{dx}\right] \quad = 0 \text{ in this problem}$$

$$E \cdot \frac{d}{dx}\left[A_0 \frac{du}{dx}(1 - x/2L)\right] = 0$$

$$\int d\left[\left(A_0 (1 - x/2L)\frac{du}{dx}\right]\right. = \int 0 \, dx$$

$$A_0 (1 - x/2L)\frac{du}{dx} = c_0$$

$$du = \frac{c_0 \, dx}{A_0 (1 - x/2L)}$$

Now, let

$$b_0 = \frac{c_0}{A_0} \quad \text{(constants)}$$

$$du = b_0 \frac{dx}{1 - x/2L} \quad \text{deflection varies logrithmically}$$

integrating both sides,

$$u = \underbrace{b_0(-2L)}_{a_0} \ln(1 - x/2L) + b_1$$

$$\underline{u = a_0 \ln(1 - x/2L) + a_1}$$

need to then solve for constants

Now consider boundaries:

$u(0) = 0 \quad N(L) = P$

$$u(0) = a_0 \underbrace{\ln(1)}_{=0} + a_1 = 0 \quad \text{thus, } \underline{a_1 = 0}$$

$$N(x) = E(x) A(x) u'(x)$$

$$N(x) = E \cdot A_0 (1 - x/2L) \cdot \frac{a_0}{1 - x/2L} \cdot \frac{-1}{2L}$$

$$N(x) = E \cdot A_0 \cdot a_0 \cdot {}^{-1}/2L = P$$

$$\underline{a_0 = \frac{-2L \cdot P}{EA_0}}$$

All together,

$$\boxed{u(x) = \frac{-2L \cdot P}{EA_0} \ln(1 - x/2L)} \quad \text{general solution}$$

$$u(L) = 1.39 \frac{PL}{EA_0}$$

GENERALIZATIONS

consider simplifications

Assume $A = 3/4 A_o$ (average)

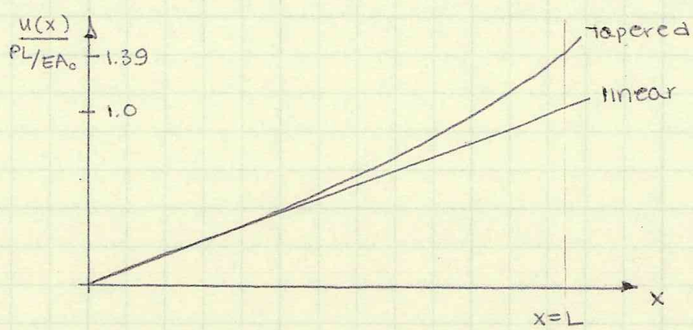
if not tapered, use

$$u = \frac{PL}{EA} = \frac{PL}{\frac{3}{4}EA_o} = \frac{4}{3}\frac{PL}{EA} \text{ or } 1.33 \frac{PL}{EA_o}$$

Not very far off exact solution (1.39)

or, consider multiple elements along beam

solution summary



observations

- solution is valid for all x (everywhere)
- math is somewhat difficult, even for simple example
- approximate solution may be acceptable

PRINCIPLE OF VIRTUAL WORK

An approach to obtain a practical approximate method

Equilibrium

p

axial force member

$$N'(x) + p(x) = 0$$

$$N'(x) = -p(x)$$

Multiply by an arbitrary function:

$$\bar{u}(x) N'(x) = -\bar{u}(x) p(x)$$

Integrate equation from 0 to L

$$\underbrace{\int_0^L N'(x)\bar{u}(x)\,dx}_{\text{integrate by parts}} = \int_0^L p(x)\bar{u}(x)\,dx$$

$$\int u\,dv = uv - \int v\,du$$

$\cancel{N'(x)\cdot x - \int_0^L \bar{u}(x)\,dN'(x)}$ ugly, messy way no one likes

$$u = \bar{u} \qquad du = d\bar{u}$$

$$dv = \frac{dN}{dx}\cdot dx \qquad v = N$$

$$\int_0^L N'(x)\bar{u}(x)\,dx = N(x)\cdot\bar{u}(x)\Big|_0^L - \int_0^L N(x)\bar{u}'(x)\,dx$$

All together again:

$$N(L)\cdot\bar{u}(L) - N(0)\cdot\bar{u}(0) - \int_0^L N(x)\cdot d\bar{u}(x)\,dx = -\int_0^L p(x)\bar{u}(x)\,dx$$

Re-arranging,

$$\boxed{\int_0^L N(x)\,d\bar{u}(x)\,dx = \int_0^L p(x)\bar{u}(x)\,dx + N(L)\cdot\bar{u}(L) - N(0)\bar{u}(0)}$$

Principle of virtual work

Internal work = External work

assuming that $\bar{u}$ is a displacement value

Principle of Virtual Displacements

Observation: *For a deformable structure in equilibrium under the action of a system of applied forces, the external virtual work due to an* ***admissible*** *virtual displaced is equal to the internal virtual work due to the same virtual displacements.*

The above observation by itself is not all that useful. If we happened to be lucky enough to know the stress distribution required to satisfy equilibrium for a problem we wanted to solve, then we could select virtual displacements at random and demonstrate that internal virtual work always balanced external virtual work. Usually, we are not given the stress distribution; rather, we are trying to find it. What the principle of virtual work does is to reverse the observation to say if the external virtual work is equal to the internal virtual work for *all* admissible virtual displacements, then the system is in equilibrium. The subtle swapping of the word "any" for the word "all" is not a trivial operation.

According to the principle of virtual work, if we satisfy the virtual work equation

$$\delta W_{int} = \delta W_{ext} \tag{1}$$

for all admissible virtual displacement functions, then the equilibrium equations are automatically satisfied. In other words, we have managed to swap a differential equation for an integral equation. We call the integral equation the *weak form* of the differential equation. Here is the catch. The strong form and the weak form are only identically equivalent if the weak form is really satisfied for *all* choices of the virtual displacement function $\bar{u}(x)$. Because $\bar{u}(x)$ is a continuous function, there are an infinite number of possible variations of this function. Making sure that Eq. (1) is satisfied for all possible choices of $\bar{u}(x)$ would seem an impossible task, and indeed there are not very many problems for which we can accomplish this task.

There is a distinct advantage, however, to the weak form. Integration acts to smooth rough things out, while differentiation has the opposite effect. An approximation can be viewed as a rough thing. Thus, if we approximate the solution to our problem, then the weak form will forgive us but the strong form will not. The advantage of the weak form of the differential equation is in its power of approximation. In fact, this approach forms the basis of the finite element method.

In order to make use of the principle of virtual displacements, both the real and the virtual displacements must satisfy certain requirements. These requirements result from the way in which the principle was developed. As noted in the statement above, the principle of virtual displacements simply states that when a structure satisfies equilibrium, the external virtual work is equal to internal virtual work. The principle, therefore, does not implicitly place any restrictions on material properties or compatibility. For the solution to a structural mechanics problem to be exact, however, all relevant conditions of constitution, equilibrium, and compatibility (kinematics) must be met. As a result, in order for a displacement function to be *admissible*, it is necessary that it satisfy the condition of displacement compatibility implicitly. Thus, both the real and virtual displacements must be of such a form that the displacement continuity conditions are met from the outset.

For cases in which the real displacements are approximated and do not satisfy the exact solution (the one that would be obtained from solving the Boundary Value Problem), enforcing the condition that the external virtual work be equal to internal virtual work leads to an approximate answer. Although the approximate solution obtained from the virtual work approach cannot give exact satisfaction of equilibrium throughout the structure, it can be shown that equilibrium is satisfied in a weighted-average sense. If more terms are employed in the approximation of the displacement, the resulting approximation comes closer to satisfying the correct solution.

ANALYSIS METHODS

Recap: virtual work

If axial force is in equilibrium, then $\delta W_{ext} = \delta W_{int}$ for all admissible virtual displacements $(\bar{u}(x))$

Principle of virtual displacement

If $\delta W_{int} = \delta W_{ext}$ for all admissible virtual displacements $u(x)$, then the structure is in equilibrium

what does this mean?

- restricts possible functions for $\bar{u}(x)$
- typically includes considering essential boundary conditions (not required)
- function must have a derivative
  - function must be continuous, but 1st derivative does not need to be

compare boundary value problem to virtual work

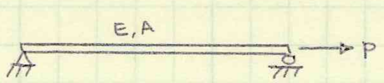
$$\int_0^L N(x)\bar{u}'(x)\,dx = \int_0^L p(x)\bar{u}(x)\,dx + N(L)\bar{u}(L) - N(0)\bar{u}(0)$$

virtual work — weak form of sol'n

$$\frac{d}{dx}\left[E(x)A(x)\frac{du}{dx}\right] = -p(x)$$

boundary value — strong form of solution

Example for demonstration



using the principle of virtual displacements

1. Assume a solution for $u(x)$ → $u(x) = a_0 + a_1 x$
2. Select a function for $\bar{u}(x)$
3. Enforce $\delta W_{int} = \delta W_{ext}$
4. solve for unknown parameters

enforce essential boundaries:

$u(x) = a_1 x$

good idea to match $u(x)$ in form: $\bar{u}(x) = \bar{a}_1 x$

(leads to a symmetric stiffness matrix

$\delta W_{int} = \int_0^L N(x)\bar{u}'(x)\,dx$

real axial force in member; not known

CALCULATION METHODS

Example, cont'd

Modify δW equations to only include one variable

$$\delta W_{int} = \int_0^L N(x)\,\bar{u}'(x)\,dx$$

compatibility: $\varepsilon(x) = u'(x)$

constitution: $\sigma(x) = E(x)\varepsilon(x)$

$$N(x) = \cancel{E/(A}\ \sigma(x)\cdot A(x) = E(x)\cdot u'(x)\cdot A(x)$$

$$\delta W_{int} = \int_0^L E(x)\cdot A(x)\cdot u'(x)\cdot \bar{u}'(x)\,dx$$

for this problem,

$$= \int_0^L E\cdot A\cdot a_1\cdot \bar{a}_1\,dx = EA\cdot a_1\cdot \bar{a}_1\cdot L$$

$$\delta W_{ext} = p\cdot \bar{u}(L) = p\cdot \bar{a}_1\cdot L$$

Returning to steps,

4. $\delta W_{int} = \delta W_{ext}$

$$EAa_1\bar{a}_1\cdot L = p\cdot \bar{a}_1\cdot L \longrightarrow (EAa_1 L - pL)\bar{a}_1 = 0$$

$\bar{a}_1 = 0$ is trivial solution

$$EAa_1 = p$$

$$a_1 = \frac{p}{EA}$$

compare to boundary value method

$$\frac{d}{dx}\left[EA\,u'(x)\right] = -p(x)$$

$$u(x) = \frac{px}{EA}$$

correct because guess for $u(x)$ was right

consider if $\bar{u}(x) = \bar{a}x^2$

$u(x) = a_1 x$

$$\delta W_{int} = EAa_1\int_0^L 2\bar{a}x\,dx = EAa_1\cdot \bar{a}x^2\Big|_0^L$$
$$= EAa_1\cdot \bar{a}L^2$$

$$\delta W_{ext} = p\cdot \bar{u}(L) = p\cdot \bar{a}\cdot L^2$$

$$EA\cdot a_1\cdot \bar{a}\cdot L^2 = p\cdot \bar{a}\cdot L^2$$

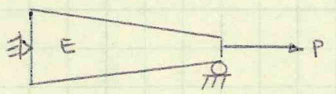
only real difference; term exists on both sides, so it's no real problem to change it

$$(EAa_1 - p)\,\bar{a} = 0$$

$$a_1 = \frac{p}{EA}$$

CALCULATION METHODS

Another example



$A = A_o(1 - x/2L)$

1. Guess poorly on $u(x)$:

$$u(x) = a_1 x$$

2. Match $\bar{u}(x)$ to $u(x)$

$$\bar{u}(x) = \bar{a}_1 x$$

3.

$$\delta W_{int} = \int_0^L \bar{u}'(x) E \cdot A(x) u'(x)\, dx$$

$$= E \cdot A_o \cdot \bar{a}_1 \cdot a_1 \int_0^L (1 - x/2L)\, dx = \frac{3}{4} E A_o \bar{a}_1 \cdot a_1 \cdot L$$

$$\delta W_{ext} = P \cdot \bar{a}_1 \cdot L$$

4.

$$\delta W_{ext} = \delta W_{int}$$

$$\frac{3}{4} E A_o \bar{a}_1 \cdot a_1 \cdot L = P \cdot \bar{a}_1 \cdot L$$

$$a_1 = \frac{4}{3} \frac{P}{EA_o}$$ – same solution we got when assuming $A = 3/4 A_o$

consider accuracy:

$$u(L) = \frac{4}{3}\frac{PL}{EA_o} \quad \text{vs.} \quad 1.39 \frac{PL}{EA_o}$$ → 4% error from exact solution – at that given point

Error can/does vary along length

ASK: is system in equilibrium?

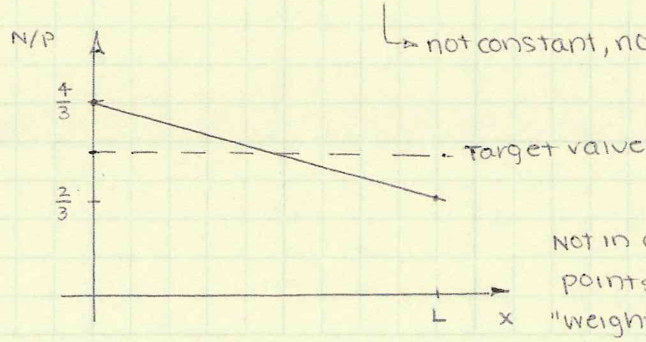
need axial force

$$N(x) = E A(x) u'(x)$$

$$N(x) = E \cdot A_o (1 - x/2L) \frac{4}{3} \frac{P}{EA_o}$$ – must equal P at all locations in the rod

$$= \frac{4}{3} P (1 - x/2L)$$

↳ not constant, not always equal to P



Not in equilibrium at all points along rod ∴ "weighted average equilibrium"

CALCULATION METHODS

Tapered bar example

Attempt #2: $\bar{u}(x) = \bar{a}_1 x^2$

$u(x) = a_1 x$

$$\delta W_{int} = E \cdot \int_0^L \bar{u}'(x) \cdot A(x) \cdot u'(x)\, dx$$

$$= EA_0 2\bar{a}_1 \cdot a_1 \int_0^L x \cdot (1 - x/2L)\, dx$$

$$x - x^2/2L$$

$$\frac{x^2}{2} - \frac{x^3}{6L} \Big|_0^L = \frac{L^2}{2} - \frac{L^2}{6} = \frac{L^2}{3}$$

$$= \frac{2}{3} E \cdot A_0 \cdot \bar{a}_1 \cdot a_1 \cdot L^2$$

$$\delta W_{ext} = P \cdot \bar{u}(L) = P \cdot \bar{a}_1 L^2$$

$$\frac{2}{3} EA_0 \bar{a}_1 \cdot a_1 \cdot L^2 = P \cdot \bar{a}_1 \cdot L^2$$

$a_1 = \frac{3}{2} \frac{P}{EA_0}$ — not as awesome; 8% error in endpoint displacement

In comparing to exact, need to consider order of $\bar{u} \cdot u$

x^3 is not as good as x^2 when comparing to logrithmic

Improving approximation

- include additional terms in $u(x), \bar{u}(x)$

  don't skip terms: include all, make it a "complete polynomial"

  $a_0 + a_1 x + a_2 x^2 \ldots$

$$\delta W_{int} = \int_0^L EA_0 (1 - x/2L)(\bar{a}_1 + 2\bar{a}_2 x)(a_1 + 2a_2 x)\, dx$$

lots of solving and ugliness later,

$$= EA_0 \left[\left(3/4\, a_1 L + \tfrac{2}{3} a_2 L^2\right)\bar{a}_1 + \left(2/3\, a_1 L^2 + 5/6\, a_2 L^3\right)\bar{a}_2 \right]$$

$$\delta W_{ext} = P(\bar{a}_1 L + \bar{a}_2 L^2)$$

Equilibrium:

$$\left[EA_0\left(3/4\, a_1 L + 2/3\, a_2 L^2\right) - PL\right]\bar{a}_1 + \left[EA_0\left(2/3\, a_1 L^2 + 5/6\, a_2 L^3\right) - PL^2\right]\bar{a}_2 = 0$$

$\bar{a}_1 = \bar{a}_2 = 0$ is the trivial solution

if only one is zero, answer is the same as previous cases

$$\left.\begin{aligned} EA_0\left(3/4\, a_1 L + 2/3\, a_2 L^2\right) &= PL \\ EA_0\left(2/3\, a_1 L^2 + 5/6\, a_2 L^3\right) &= PL^2 \end{aligned}\right\} \rightarrow EA_0 \underbrace{\begin{bmatrix} 3/4\, L & 2/3\, L^2 \\ \frac{2}{3} L^2 & \frac{5}{6} L^3 \end{bmatrix}}_{\text{stiffness-esque matrix}} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} PL \\ PL^2 \end{bmatrix}$$

CALCULATION METHODS

improving Accuracy o' approximation

- using additional terms (cont'd)

$$\underbrace{EA_0\begin{bmatrix} \frac{3}{4}L & \frac{2}{3}L^2 \\ \frac{2}{3}L^2 & \frac{5}{6}L^3 \end{bmatrix}}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} PL \\ PL^2 \end{bmatrix}$$

→ like a stiffness matrix

symmetric because $\bar{u}(x)$, $u(x)$ are of the same form

we like this fact, cause K should be symmetric

solving,

$$a_1 = \frac{12}{13}\,P/EA_0$$

$$a_2 = \frac{6}{13}\,P/EA_0 L$$

$$u(x) = \frac{PL}{EA_0}\left[\frac{12}{13}\left(\frac{x}{L}\right) + \frac{6}{13}\left(\frac{x}{L}\right)^2\right] \longrightarrow u(L) = \frac{18}{13}\,\frac{PL}{EA_0},$$

less than 1% error, woot.

more terms = more awesomer

rate of return per term decreases

(law of diminishing returns)

WORKING WITH EQUATIONS

Generalizations and Modifications

Do $u(x)$ and $\bar{u}(x)$ have to satisfy essential boundary conditions?

NO.

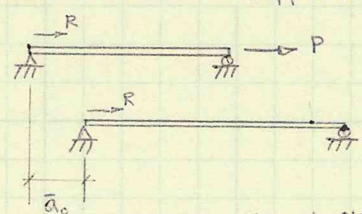
$u(x) = a_1 x$

$\bar{u}(x) = \bar{a}_0 + \bar{a}_1 x$

$\delta W_{int} = \int_0^L E \cdot A \cdot u' \cdot \bar{u}' dx = \int_0^L E \cdot A \cdot a_1 \cdot \bar{a}_1 dx$

$\delta W_{ext} = P(\bar{a}_0 + \bar{a}_1 L) + R \cdot \bar{a}_0$

reaction force at support



virtual displacements

distance over which reaction force was moved ($W = F \cdot d$)

$EA a_1 \bar{a}_1 L = P\bar{a}_0 + P\bar{a}_1 L + R \cdot \bar{a}_0$

$(EAL a_1 - PL)\bar{a}_1 - (P+R)\bar{a}_0 = 0$

solving,

$R = -P$ — easy problem, so we know this

$a_1 = P/EA$ — solution from before

No, but don't forget about the additional external work term.

Can it be specified that $\bar{u}(x) = 0$ at locations that do not correspond with EBCs?

NO.

$\bar{u}(x) = \bar{a}_0 (1 - x/L)$

$\delta W_{ext} = P \cdot \bar{a}_0 (1 - L/L) = P \cdot 0 = 0$

bad news. no external work?
Not taking advantage of /or recognizing the load at that point

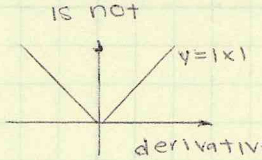
To what degree of continuity must $u(x)$ and $\bar{u}(x)$ satisfy?

$C^0, C^1 \ldots C^n$ – classes of functions

ex. polynomial – C^∞

C^1: function, 1st derivative are continuous, 2nd isn't necessarily

C^0: function is continuous, 1st derivative is not



derivative is discontinuous at zero

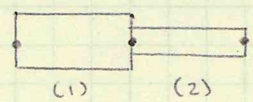
WORKING WITH EQUATIONS

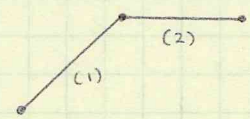
Gens., Mods., and Questions (cont'd)

Degree of continuity:

C^0 – two truss elements, only need displacements to match at joint

C^1 – beam elements, need displacements and slopes to match





etc.

deals with inter-element connectivity

Do real and virtual displacements have to match? (be of the same form)

NO.

$u(x) = a_1 x$

$\bar{u}(x) = \bar{a}_1 x^2$

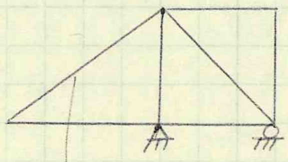
> example from the other day (pg 11)

However, if they are, the stiffness matrix will be symmetric (not required mathematically)

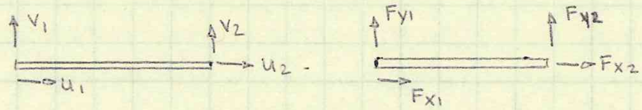
→ Galerkin Approximation

STIFFNESS MATRIX

Virtual work approach for a truss



→ random unrestrained bar in the middle of a truss somewhere



$u(x) = a_0 + a_1 x$

Boundaries: $u(0) = u_1$, so $a_0 = u_1$

$u(L) = u_2$, $a_1 = \dfrac{u_2 - u_1}{L}$

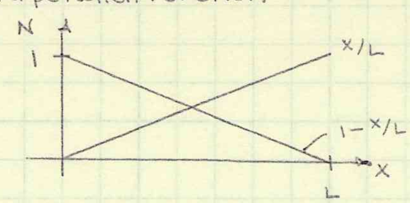
$$u(x) = u_1 + \frac{u_2 - u_1}{L} \cdot x$$

consider equation for $u(x)$, rearrange:

$$u(x) = (1 - x/L)\, u_1 + (x/L)\, u_2$$

nodal displacement

shape function, or interpolation function



Interpolation functions: define contribution of nodal displacements to overall displacement; sum to 1.0

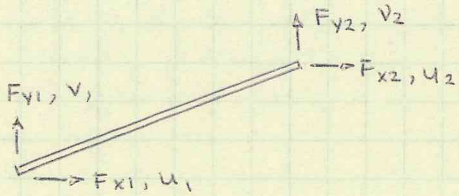
$u(x) = \underset{\sim}{L} \cdot \underset{\sim}{u}$

$\underset{\sim}{L} = \begin{bmatrix} 1 - x/L & 0 & x/L & 0 \end{bmatrix}$

$\underset{\sim}{u} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$ same as equation above when multiplied

ELEMENTAL ANALYSIS

Truss member formulations



$$u(x) = L_x \underset{\sim}{u} = \begin{bmatrix} 1 - x/L & 0 & x/L & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

$$v(x) = L_y \underset{\sim}{u} = \begin{bmatrix} 0 & 1 - x/L & 0 & x/L \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

$$\varepsilon(x) = u'(x) = \frac{du}{dx}$$

$$= -1/L \, u_1 + 1/L \, u_2$$

Define this as $\underset{\sim}{B_x} \cdot \underset{\sim}{u}$

$$B_x = \begin{bmatrix} -1/L & 0 & 1/L & 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 \end{bmatrix}$$

$B_x \equiv$ strain/displacement matrix

for small displacements, u and v strains are uncoupled

$\varepsilon_y = 0$ – only from Poisson's, and we ignore that in truss analysis
– take derivitives with respect to y, $B_y = [0]$

virtual displacements

$\bar{u}(x) = (1 - x/L)\bar{u}_1 + (x/L)\bar{u}_2$ – same form due to Gallerkin Approx.

Principle of virtual work

$$\delta W_{int} = \int_0^L E(x) A(x) u'(x) \bar{u}'(x) \, dx$$

rearranged, $\int_0^L \bar{u}'(x) E(x) A(x) u'(x) \, dx$

ELEMENTAL ANALYSIS

Truss members – PVW

$$\delta W_{int} = \int_0^L E(x)A(x)u'(x)\bar{u}'(x)\,dx$$

$$= \int_0^L \underset{\sim}{B}_x \underset{\sim}{\bar{u}}\; E(x)A(x)\,\underset{\sim}{B}_x \underset{\sim}{u}\,dx$$

$\underset{\sim}{B}_x \underset{\sim}{\bar{u}}$ = scalar value – 1×1 matrix
thus, it's symmetric
$=(B_x\bar{u})^T = \bar{u}^T B_x^T$

$$\delta W_{int} = \int_0^L \underset{\sim}{\bar{u}}^T \underset{\sim}{B}_x^T E(x)A(x)\underset{\sim}{B}_x \underset{\sim}{u}\,dx$$

$$= \underset{\sim}{\bar{u}}^T \int_0^L \underset{\sim}{B}_x^T E(x)A(x)\underset{\sim}{B}_x\,dx \cdot \underset{\sim}{u}$$

scalar value that does not vary with x

$$\delta W_{ext} = \underset{\sim}{\bar{u}}^T \underset{\sim}{F}$$

$$\underset{\sim}{F} = \begin{bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{bmatrix}$$

Applying Equilibrium:

$$\underset{\sim}{\bar{u}}^T \int_0^L \underset{\sim}{B}_x^T E(x)A(x)\underset{\sim}{B}_x\,dx \cdot \underset{\sim}{u} = \underset{\sim}{\bar{u}}^T \cdot \underset{\sim}{F}$$

$\underset{\sim}{\bar{u}} = 0$ – trivial

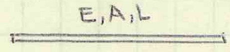
$$\underbrace{\left[\int_0^L \underset{\sim}{B}_x^T E(x)A(x)\underset{\sim}{B}_x\,dx\right]}_{\underset{\sim}{K}}\underset{\sim}{u} = \underset{\sim}{F}$$

$\underset{\sim}{K}$ – stiffness matrix in local coordinates

ELEMENTAL ANALYSIS

Example using K formulation

E, A, L



$$K = \int_0^L \begin{bmatrix} -1/L \\ 0 \\ 1/L \\ 0 \end{bmatrix} E \cdot A \begin{bmatrix} -1/L & 0 & 1/L & 0 \end{bmatrix} dx = EA \begin{bmatrix} 1/L & 0 & -1/L & 0 \\ 0 & 0 & 0 & 0 \\ -1/L & 0 & 1/L & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

look familiar? ✓

Methods of improving solution accuracy

1. Improve guess of displaced shape – $u(x), \bar{u}(x)$ → P. type refinement (p for polynomial)
2. Physically subdivide the element → h. type refinement OR n. type (number) (h has to do with characteristic length)
   go from 1 to 2 to 4 to 100...

Example with p. type refinement

$$u(x) = N_1 u_1 + N_2 u_2 + N_3 u_3$$

Need 3rd node for quadratic definition

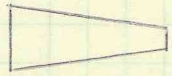
$$\left.\begin{aligned} N_1 &= 1 - 3x/L + 2(x/L)^2 \\ N_2 &= 4(x/L - (x/L)^2) \\ N_3 &= -x/L + 2(x/L)^2 \end{aligned}\right\} \text{ick.}$$

$$B_{x1} = -3/L + 4x/L^2$$

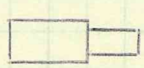
$$B_{x2} = 4(1/L - 2x/L^2)$$

$$B_{x3} = -1/L + 4x/L^2$$

does not result in the same K as if two linear elements are used] well, depends on situation



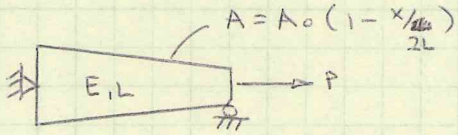
– known it is logrithmic, quad. was okay – use 3-node



– 2 2-noded bars

ELEMENTAL ANALYSIS

Example



3-node $\quad k_{11} = \int_0^L \left(-3/L + 4x/L^2\right)^2 EA_0 (1 - x/2L)\, dx = \frac{25}{12} \frac{EA_0}{L}$

K matrix − rows, columns sum to zero
means equilibrium has been
established ⟶ no boundaries, can displace without strain
singular, cannot be inversed

(not necessarily true for multiple elements with summed K matrix)

k_{11} = mess o' stuff
left elem.

Note: $L = L/2$, $A(x)$ = different functions for either element; $A_{02} = \frac{3}{4} A_0$

To enforce boundaries, remove rows/columns
where you know the displacement to be 0

p-type has limitations
generally only used up to quadratic (or, fourier series)
n/h-type is more effective, can have more and more elements

ELEMENT FORMULATIONS

General case

virtual strain

$$\bar{\varepsilon} = \begin{bmatrix} \bar{\varepsilon}_x \\ \bar{\varepsilon}_y \\ \bar{\varepsilon}_z \\ \bar{\gamma}_{xy} \\ \bar{\gamma}_{xz} \\ \bar{\gamma}_{yz} \end{bmatrix} \qquad \sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}$$

virtual

$\frac{\delta W_{int}}{Vol.} \equiv$ oh, nevermind

or $\delta W_{int} = \int_v \bar{\varepsilon}^T \sigma \, dvol$ but, we don't know these things

$\sigma = E \cdot \varepsilon$, $E \equiv$ constitutive matrix $f(E, G, \nu)$

$\delta W_{int} = \int_v \bar{\varepsilon}^T \cdot E \cdot \varepsilon \, dvol$ size is problem-dependent

symmetric if real, virtual strain are of the same form; E is symmetric (isotropic) — ~~Galerkin~~ Galerkin

Assume:

$\varepsilon = B u$, $\bar{\varepsilon} = B \bar{u}$

if forms match (Galerkin), B is the same for $\bar{u}, u$

$$\delta W_{int} = \int_{vol} (B\bar{u})^T E B u \, dvol$$

$$= \int_{vol} \bar{u}^T B^T E B u \, dvol$$

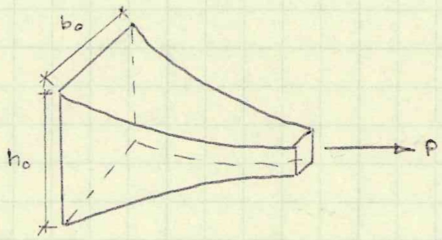
$$= \bar{u}^T \underbrace{\int_{vol} B^T E B \, dvol}_{K} \cdot u$$

K – stiffness matrix for a given member

basis for finite element analysis

(oh, to have known this before...)

EXAMPLE PROBLEM



$$A(x) = b_o h_o (1 - x/2L)^2 (1 - \sin \pi x/4L)$$

ugly to solve exactly
(e, imaginary #s...)

Assumptions:
- quadratic variation for $u, \bar{u}$
- linear elastic material

$$k_{11} = \int_0^L B_1 E(x) A(x) B_1 \, dx$$

using:

$$N_1 = 1 - 3x/L + 2(x/L)^2$$

$$B_1 = \partial N_1/\partial x = -3/L + 4\,x/L^2$$

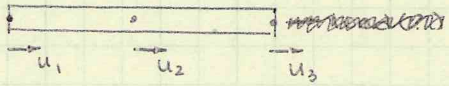
$$k_{11} = \int_0^L (-3/L + 4x/L^2)^2 E b_o h_o (1 - x/2L)^2 (1 - \sin \pi x/4L) \, dx$$

much more attractive, but
still kind of a pain

Up next: numerical integration

NUMERICAL INTEGRATION

Lagrange Shape Functions



Assume n nodes for an axial member

Let x_i be the location of node i

Shape functions:

$$L_1 = 1 - 3x/L + 2(x/L)^2$$

$$= \frac{(x_2 - x)(x_3 - x)\dots(x_n - x)}{(x_2 - x_1)(x_3 - x_1)\dots(x_n - x_1)}$$

thus making $L_1 = 1$ at node 1 and $L_1 = 0$ at other nodes

$$L_j = \frac{\prod_{\substack{i=1 \\ i\neq j}}^{N} (x_i - x)}{\prod_{\substack{i=1 \\ i\neq j}}^{N} (x_i - x_j)}$$

Big Π is similar to big Σ except multiply, not add

Properties of shape functions (c^0 – continuous, derivative isn't necessarily)

1. $L_i = 1$ at x_i, $L_i = 0$ at all other nodes
2. At any location, x, $\sum L_i = 1.0$

Numerical Integration

Approximate a solution that is difficult to solve in closed form

Newton Cotes
Gaussian
> quadrature methods → "numerical integration"

Features of Newton Cotes (ex. trapezoidal rule)

- all points of evaluation are equally spaced
- will integrate polynomial exactly of order n+1, with an n order rule ← order = number of terms in polynomial; 3rd order is quadratic (degree is 2) $a + bx + cx^2$

$$\int_a^b f(x)\,dx = \left[\tfrac{1}{2}(b-a)\right] \sum_{i=1}^{N} w_i \cdot f(x_i)$$

N ← numerical rule order

$f(x_i)$ ← function evaluated at x_i

w_i ← weighting factor

NUMERICAL INTEGRATION

Main concepts

- pass a polynomial function through a set of data points
- integrate polynomial exactly

--> thus, accuracy depends upon how well the polynomial represents the function

Basic mathematical form:

don't forget this!

$$\int_a^b f(x)\,dx \longrightarrow \text{evaluate numerically} = \boxed{\left(\frac{b-a}{2}\right)} \sum_{i=1}^{N} w_i \cdot f(x_i)$$

closed-form sol. is too difficult to compute

specific location for evaluation

weighting factor

Newton-Cotes Quadrature

- all points of evaluation are equally spaced in [a,b]
- will integrate polynomial exactly of order $n+1$

  └ # of terms

$$f(x) = a + bx + cx^2 + dx^3$$

4th order, needs 3rd-order Newton-Cotes rule

- because of equal spacing limitation, N-C is usually used in a composite scheme — subdivide integral

$$\int_a^b f(x)\,dx = \underbrace{\int_a^{a_1} f(x)\,dx + \int_{a_1}^{a_2} f(x)\,dx \ldots + \int_{a_n}^{b} f(x)\,dx}_{\text{sub-divide problem domain into sub-intervals}}$$

NUMERICAL INTEGRATION

Gaussian Quadrature

Both ~~locations~~ locations of sampling points (x_i) and the weighting factors (w_i) can vary

- optimized to give most accurate solution
- significantly more accurate than Newton-Cotes
- Gaussian rule of order n will evaluate exactly a polynomial of order $\underline{2n}$ ← so much better!

Standard values tabulated in handout

Example

$$\int_0^L x \cdot \sin(\pi x/2L)\,dx$$

Exact solution: $\cancel{4\pi^2/L^2}$ $4L^2/\pi^2 = 0.4053L^2$

(damn you, EBW)

Using N.C.:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

thus $f(x) \propto x^2$ as first term — third-order already!

sampling locations:

$0, L/2, L$ — $x_i = (b-a)\,i/N + a$, $i = 0, 1 \ldots N$ (order)

N = order of the rule

weighting factors:

$w_1 = 1/3$
$w_2 = 4/3$
$w_3 = 1/3$
} from handout (or, Simpson's rule)

$$\int_0^L x\sin(\pi x/2L)\,dx = \frac{1/3(0)\sin(0) + 4/3(L/2)\sin(\pi/4) + 1/3(L)\sin(\pi/2)}{\text{all} \times (L-0)/2}$$

$= 0.4024L^2$ (0.7% error)

For improved accuracy, subdivide integral

$x_i = 0, L/4, L/2;\ x_i = L/2, 3L/4, L$

$$\frac{(L-0)}{2\cdot 2}\left[1/3(0) + 4/3(L/4)\sin\pi/8 + 1/3(L/2)\sin\pi/4 + \cdots\right] = 0.40512L^2$$

(0.4% off)

NUMERICAL INTEGRATION

Example, again

using Gaussian quadrature

$$\int_0^L x \cdot \sin(\pi x / 2L)\, dx$$

3rd order equation ⟶ 2nd order rule

weighting, locations in terms of mapped coordinates –

$\xi = -1/\sqrt{3}, +1/\sqrt{3}$ – does not fall on $[0, L]$

must shift to ξ scale

$$x(\xi) = \tfrac{1}{2}\left[a(1-\xi) + b(1+\xi)\right]$$

on domain $[a, b]$

$a \rightarrow \xi = -1$

$b \rightarrow \xi = +1$

locations

$x(-1/\sqrt{3}) = \frac{1}{2}\left[0 + L(1 - 1/\sqrt{3})\right] = 0.2113L$

$x_2 = 0.7887L$

weighting = 1.0

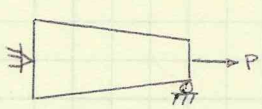
$$\int_0^L f(x) = \frac{L}{2}\left[0.2113L \cdot \sin\_ + (1.0)(0.7887L)\sin\frac{0.7887\pi}{2}\right]$$

$= 0.40724L^2$ (overestimates 0.5%)

N.C underestimates, but this is problem-dependent

3-point Gauss is essentially exact

Example, again (or, a new)



$E, L, A = A_0(1 - x/2L)$

Assumptions:

– use 2-noded element linear variation

$\underset{\sim}{L} = [1 - x/L \quad x/L]$, $\underset{\sim}{B} = [-1/L \quad 1/L]$

stiffness matrix:

$$k = \int_0^L \underset{\sim}{B}^T E A(x) \underset{\sim}{B}\, dx$$ ↙ strain-displacement [ ]

$$k_{11} = \int_0^L (-1/L)^2 E A_0 (1 - x/2L)\, dx$$

$$= \frac{EA_0}{L^2}\int_0^L 1 - x/2L \, dx$$

Now: omg, I can't solve this! let's approximate! :)

$$\int_0^L f(x)\,dx = 2f(0) \cdot \frac{L}{2}$$

or, $L/2$ – center of domain

$$= \frac{EA_0}{L^2} \cdot \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4}\frac{EA_0}{L^2}$$

same as exact answer we got before (but not exact to actual solution)

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Numerical Integration

Numerical integration consists of evaluating a definite integral through a combination of weighted function evaluations. We write this approximation in the form

$$\int_a^b f(x)dx = \tfrac{1}{2}(b-a)\sum_{i=1}^{N} w_i f(x_i) \tag{1}$$

where w_i is the weighting factor associated with integration station i. Although not required, we will find it convenient to make a change of variables so that the integral is defined on the interval [-1,1]. The change of variables can be accomplished using the mapping

$$x(\xi) = \tfrac{1}{2}[a(1-\xi) + b(1+\xi)] \tag{2}$$

Note that at $\xi=-1$ we have $x=a$ and at $\xi=1$ we have $x=b$. With this change of variable, the original approximation problem can be restated as

$$\int_{-1}^{1} f(\xi)d\xi = \sum_{i=1}^{N} w_i f(\xi_i) \tag{3}$$

where w_i are the weights associated with the function evaluated at location ξ_i and N is the number of points at which the function is evaluated (*i.e.*, the number in integration stations). In essence the problem we are trying to solve can be stated as follows: find the area under the curve $f(\xi)$ between the limits of $\xi = -1$ and $\xi = +1$, as shown in Fig. 1. There are many numerical

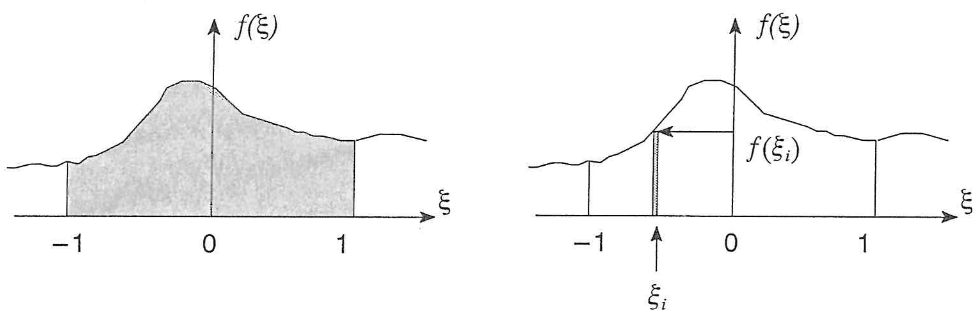

Fig. 1. Numerical Integration

methods of the form described by Eq. (3), each one having its own properties and inherent accuracy. In this handout we discuss two families of numerical quadrature methods: *Newton-Cotes Quadrature* and *Gaussian Quadrature*. Both methods are based upon the same criterion: The method should integrate exactly the highest order polynomial possible. It is this criterion that defines the method. One would rarely use numerical integration to evaluate polynomial integrals because they are simple to integrate directly. However, the largest exact polynomial degree is an indication of the accuracy of the method for integrating other functions.

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Newton-Cotes Family - The Newton-Cotes family is defined by the requirement that the sampling locations be equally spaced within the interval of integration. Among the more important Newton-Cotes formulas are the Trapezoidal Rule and Simpson's Rule. The approximation of an integral using Newton-Cotes is accomplished as follows

$$\int_{-1}^{1} f(\xi)d\xi = \sum_{i=0}^{N} c_i f(\xi_i) \qquad (4)$$

where the quadrature locations are equally spaced within the interval according to the relationship

$$\xi_i = \frac{2i}{N} - 1 \qquad i = 0, \ldots, N \qquad (5)$$

and the weighting coefficients are given in Table 1 below. Note that, for this case, the summation starts with $i = 0$ rather than $i = 1$. In practice, the Trapezoidal Rule and Simpson's Rule are most often used in a composite scheme (*i.e.*, by breaking the integral into sub-interval integrals). A Newton-Cotes scheme of order N will exactly integrate a polynomial of order N+1, except for Simpson's rule, which will actually integrate cubic functions exactly (N=2, but a cubic is a 4th order polynomial). Note that the *order* of a polynomial refers to the number of terms in the expression of the function and *degree* refers to the highest exponent. Thus, a quadratic function containing constant, linear, and quadratic terms is said to be a third-order polynomial. Such a function could also be described as a second degree polynomial.

Table 1. Newton-Cotes Coefficients

N	c0	c1	c2	c3	c4	c5	c6
1	1	1			(Trapezoidal Rule)		
2	1/3	4/3	1/3		(Simpson's Rule)		
3	1/4	3/4	3/4	1/4			
4	7/45	32/45	12/45	32/45	7/45		
5	19/144	75/144	50/144	50/144	75/144	19/144	
6	41/420	216/420	27/420	272/420	27/420	216/420	41/420

Gaussian Quadrature - The basic idea behind Gaussian Quadrature is to improve the accuracy of numerical integration by determining optimal sampling locations in addition to determining the optimal weighting coefficients, w_i. Thus, for a two-point integration rule, one has four parameters to adjust to get the greatest accuracy: w_1, w_2, ξ_1, and ξ_2. Gaussian quadrature is the best one can do without additional information about the integrand because all possible information about the integration scheme is optimized. Gaussian quadrature integrates polynomials most accurately with the fewest function evaluations. In fact, Gaussian quadrature will integrate a polynomial of degree 2N−1 (order 2N) with only N integration stations and weights. For example, a cubic polynomial $f(x)=a + bx + cx^2 + dx^3$ can be integrated between any limits *exactly* with two point Gaussian integration. The first four integration formulas for the integral of a function on the interval $\xi = [-1,1]$ are summarized in Table 2.

Table 2. Gaussian Quadrature Stations and Weights

Integration Order	Station ξ_i	Weight w_i
1	0	2
2	$\sqrt{1/3}$ $-\sqrt{1/3}$	1 1
3	$-\sqrt{3/5}$ 0 $\sqrt{3/5}$	5/9 8/9 5/9
4	$\sqrt{(3+2\sqrt{6/5})/7}$ $\sqrt{(3-2\sqrt{6/5})/7}$ $-\sqrt{(3-2\sqrt{6/5})/7}$ $-\sqrt{(3+2\sqrt{6/5})/7}$	$(3-\sqrt{5/6})/6$ $(3+\sqrt{5/6})/6$ $(3+\sqrt{5/6})/6$ $(3-\sqrt{5/6})/6$

$\times \frac{a-b}{2}$

Integrals on Domain x=[a,b] - The integration formula (3) can be transformed to any general interval $x=[a,b]$ by noting that this interval can be mapped onto $\xi = [-1,1]$ by the equation

$$x(\xi) = \tfrac{1}{2}[a(1-\xi) + b(1+\xi)] \tag{6}$$

Now the integral in question can be written as

$$\int_a^b f(x)dx = \tfrac{1}{2}(b-a)\sum_{i=1}^{N} w_i f(x_i) \tag{7}$$

where $x_i = \frac{1}{2}[a(1-\xi_i) + b(1+\xi_i)]$ is the location of the integration station in terms of the variable defined on the interval $x=[a,b]$; the value of ξ_i can be obtained from Eq. (5) for Newton-Cotes quadrature or from Table 2 for Gaussian quadrature. Alternatively, the original integral can be analytically transformed to an integral on the interval [-1,1] through the change of variables given by Eq. (2). Then the original numerical quadrature formulas can be used.

Composite Rules - Because of the linearity of the integration operation, any integral can be decomposed into a sum of sub-integrals as

$$\int_a^b f(x)dx = \int_a^{a_1} f(x)dx + \int_{a_1}^{a_2} f(x)dx + \ldots + \int_{a_n}^{b} f(x)dx \tag{8}$$

if $a < a_1 < a_2 < \ldots < a_n < b$. Numerical quadrature can be applied to any of the subinterval integrations. If the function to be integrated has a jump discontinuity or cusp as shown in Fig. 2, then it

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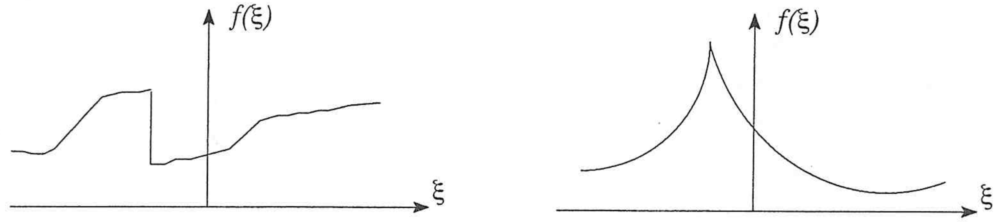


Fig. 2. Functions which require composite integration

is extremely important to subdivide the integral before applying a numerical scheme. Remember that the integration schemes are equivalent to passing the highest order polynomial possible through the data at the points of function evaluation (*e.g.*, a cubic for two point Gaussian quadrature) and then exactly integrating the resulting polynomial. The accuracy of the method then depends upon how well the polynomial approximates the actual function. Clearly, a low order polynomial will be a poor approximation of any function with jump discontinuities or singularities.

References

Bathe, K.-J. and E.L. Wilson, **Numerical Methods in Finite Element Analysis**, Prentice Hall, Inc., Englewood Cliffs, N.J., 1976.

Dahlquist, G. and A. Bjork, **Numerical Methods**, Prentice Hall, Inc., Englewood Cliffs, N.J., 1974.

FLEXURAL MEMBERS

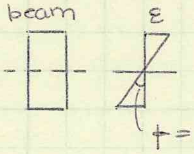
main objective: develop K using virtual work principles

$K = \int_V \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B} \, dvol$ from $\int_V \bar{\varepsilon}^T \sigma \, dvol$

mixture of terms, so, we switched things

already assumes that $u, \bar{u}$ are of same form

beam ε



$\varepsilon = y \cdot K$ – assumes small angles

ϕ = curvature, or K (kappa)

List o' Assumptions:

- plane sections remain plane
- small displacements or rotations
- $u(x), \bar{u}(x)$ are of same form
- curvature is second derivitive o' displace.

make problem easier:

- material properties: linear elastic
- planar beam (no twist)
- ignore shear deformations

beam BVP:

$$\frac{d^2}{dx^2}\left[EI v''\right] = -w(x)$$

vertical disp.

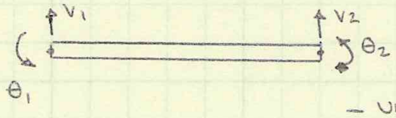
homogeneous: $EIv'''' = 0$

cubic function

For a prismatic bar,

exact solution is $v(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

not a bad place to start for non-constant cross-sections, modulus, etc.



– unlike truss, each node has two values, not one

– must match Δ and Δ' (rotation) (or slope) → need C' fxns – continuous in 1st derivitive

boundaries to consider:

$v(0) = v_1$, $v(L) = v_2$

$v'(0) = \theta_1$, $v'(L) = \theta_2$

use to replace a_0 through a_3

$a_0 = v_1$, etc.

FLEXURAL MEMBERS

creating governing equations

$$v(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

calculate coefficients from boundary values

Hermitian Interpolation Functions

$$\boxed{\begin{aligned} H_1 &= 1 - 3(x/L)^2 + 2(x/L)^3 \longrightarrow v_1 \\ H_2 &= x - 2x^2/L + x^3/L^2 \longrightarrow \theta_1 \\ H_3 &= 3(x/L)^2 - 2(x/L)^3 \longrightarrow v_2 \\ H_4 &= \frac{-x^2}{L} + \frac{x^3}{L^2} \longrightarrow \theta_2 \end{aligned}}$$

can write expression for displacement in terms of interpolation functions and nodal values

$$v(x) = \underset{\sim}{H}\,\underset{\sim}{v}$$

$$= \begin{bmatrix} H_1 & H_2 & H_3 & H_4 \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

So, $\underset{\sim}{B} = \frac{d^2}{dx^2} \underset{\sim}{H}$

H for Hermitian; L previously was for Lagrangian

because $\varepsilon(x) = y \cdot v''(x)$

y: dist. from N.A.

$\underset{\sim}{B} \equiv$ strain-displacement matrix

needs relationship between strain... and... displacement

$$\varepsilon(x) = y \cdot \frac{d^2}{dx^2}\left[\underset{\sim}{H} \cdot \underset{\sim}{v}\right] = y \cdot \underset{\sim}{H}'' \cdot v$$

doesn't vary in x

$$H'' = \begin{bmatrix} H_1'' & H_2'' & H_3'' & H_4'' \end{bmatrix} = \underset{\sim}{B}$$

$$H_1'' = -6/L^2 + 12\,x/L^3 \quad \text{etc.}$$

$$\underset{\sim}{K} = \int_{vol} \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B}\, d\text{vol}$$

$\underset{\sim}{E}$ must be 1x1 to match in size, so $\underset{\sim}{K} = \int_{vol} \underset{\sim}{B}^T E \underset{\sim}{B}\, d\text{vol} = \int_V y \cdot [\underset{\sim}{H}'']^T E \underset{\sim}{H}'' \cdot y\, d\text{vol}$

$\underset{\sim}{B}$ needs to include multiplying by y

$$\underset{\sim}{K} = \int (\underset{\sim}{H}'')^T E (H'')(y^2 dA)\, dx$$

definition for $I \equiv \int_A y^2 dA$

$$\boxed{\underset{\sim}{K} = \int_0^L (H'')^T E(x) I(x) H''\, dx}$$ for beams

FLEXURAL MEMBERS

Develop K using virtual work principles

$$\delta W_{int} = \int_{vol} \bar{\varepsilon} \cdot \sigma \, dvol$$

$\bar{\varepsilon} \equiv$ virtual strain

$\sigma \equiv$ real stress

scalar value because beams only consider one strain value

Δx along neutral axis, as other locations can be calculated (plane sections remain...)

using constitution,

$$\sigma = E \varepsilon$$

$$\delta W_{int} = \int_{vol} \bar{\varepsilon} \cdot E \cdot \varepsilon \, dvol$$

does not have to be a constant : $E(x,y,z)$

review:

$\varepsilon = y K$ (y × curvature)

$\varepsilon = y v'' = y \underset{\sim}{H}'' \underset{\sim}{v}$ as $v(x) = \underset{\sim}{H}\underset{\sim}{v}$

does not vary over x

$$\delta W_{int} = \int_{vol} y \cdot \underset{\sim}{H}'' \underset{\sim}{\bar{v}} \cdot E(x,y,z) \cdot y \cdot \underset{\sim}{H}'' \underset{\sim}{v} \, dvol$$

y = scalar, can come out (or, can move around)

$\underset{\sim}{H}\underset{\sim}{v}$ = scalar, $(\underset{\sim}{H}\underset{\sim}{v})^T$ = same scalar = $\underset{\sim}{v}^T \underset{\sim}{H}^T$

$$\int_{vol} \bar{\underset{\sim}{v}}^T \underset{\sim}{H}''^T E \underset{\sim}{H}'' y^2 \underset{\sim}{v} \, dvol$$

$\bar{\underset{\sim}{v}}, \underset{\sim}{v}$ don't vary over volume

$$\bar{\underset{\sim}{v}}^T \int_0^L \int_A (\underset{\sim}{H}'')^T E H'' \underbrace{y^2 \, dA}_{I} \, dx \cdot \underset{\sim}{v}$$

$$\delta W_{int} = \bar{\underset{\sim}{v}}^T \int_0^L (\underset{\sim}{H}'')^T E \underset{\sim}{H}'' I \, dx \cdot \underset{\sim}{v}$$

$$\delta W_{int} = \bar{\underset{\sim}{v}}^T \underline{\int_0^L (\underset{\sim}{H}'')^T E(x) I(x) H'' \, dx} \cdot \underset{\sim}{v}$$

$$\boxed{K = \int_0^L \underset{\sim}{B}^T E(x) I(x) \underset{\sim}{B} \, dx}$$

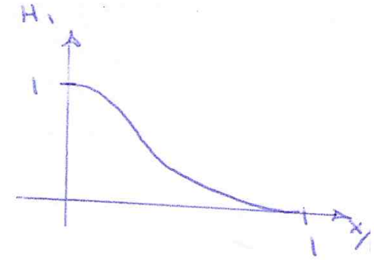
$$\underset{\sim}{B} = \underset{\sim}{H}''$$

Beam element modeled with 2 nodes.

highest order of x

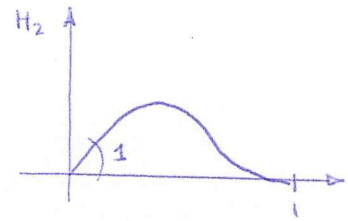
Cubic Hermitian Shape Functions

$$\overset{H_1}{\cancel{N}}_1 = 1 - 3\left[\frac{x}{\ell}\right]^2 + 2\left[\frac{x}{\ell}\right]^3$$



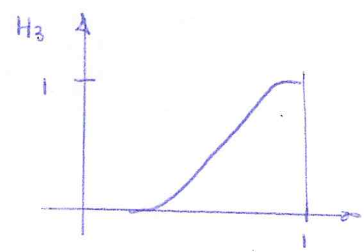
$$H_1'' = \frac{-6}{L^2} + \frac{12x}{L^3}$$

$$\overset{H_2}{\cancel{N}}_2 = x - 2\left[\frac{x^2}{\ell}\right] + \left[\frac{x^3}{\ell^2}\right]$$



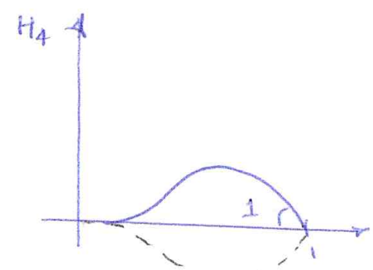
$$H_2'' = -\frac{4}{L} + \frac{6x}{L^2}$$

$$\overset{H_3}{\cancel{N}}_3 = 3\left[\frac{x}{\ell}\right]^2 - 2\left[\frac{x}{\ell}\right]^3$$



$$H_3'' = \frac{6}{L^2} - \frac{12x}{L^3}$$

$$\overset{H_4}{\cancel{N}}_4 = -\left[\frac{x^2}{\ell}\right] + \left[\frac{x^3}{\ell^2}\right]$$



$$H_4'' = \frac{-2}{L} + \frac{6x}{L^2}$$

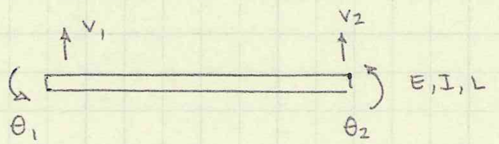
$$v(x) = N_1 v_1 + N_2 \theta_1 + N_3 v_2 + N_4 \theta_2$$

or H, not N

$$v(x) = \underset{\sim}{H}\,\underset{\sim}{v}$$

FLEXURAL MEMBERS

Example to calculate K



using 363 knowledge,

$$K_{11} = \frac{12EI}{L^3}$$

$$H_1 = 1 - 3(x/L)^2 + 2(x/L)^3$$

$$H_1'' = -6/L^2 + 12x/L^3$$

$$K = \int_0^L (\underset{\sim}{H''})^T E(x)\, I(x)\, \underset{\sim}{H''}\, dx$$

$$(H_1'')^2 = 36/L^4 - 144x/L^5 + 144x^2/L^6$$

$$= \frac{36x}{L^4} - \frac{72x^2}{L^5} + \frac{144x^3}{3L^6}\Bigg|_0^L$$

$$= \frac{36}{L^3} - \frac{72}{L^3} + \frac{144/3}{L^3} = \frac{1}{L^3}\,\frac{12}{1}\,EI = \frac{12EI}{L^3} \checkmark$$

(hey, that's kinda cool)

$$H_1'' = -6/L^2 + 12x/L^3$$
$$H_2'' = -4/L + 6x/L^2$$
$$H_3'' = 6/L^2 - 12x/L^3$$
$$H_4'' = -2/L + 6x/L^2$$

$$K = \begin{bmatrix} 12/L^3 & 6/L^2 & -12/L^3 & 6/L^2 \\ & 4/L & -6/L^2 & 2/L \\ & & 12/L^3 & -6/L^2 \\ & & & 4/L \end{bmatrix} \cdot EI$$

GENERALIZED FORCE VECTORS

Load vector

consider stress-strain relationship

$$\sigma = E(\varepsilon - \varepsilon_0) + \sigma_0$$

ε_0: initial strain, e.g. member misfit

σ_0: initial stress, e.g. residual stress in a rolled steel shape

* actual classification of ε_0 or σ_0 will be problem dependent

- for thermal change in a truss,

$\varepsilon_0 = \alpha \Delta T$
or $\sigma_0 = -E\alpha\Delta T$

use one! do not apply both into one problem!

Equivalent nodal forces

→ as in, "work equivalent"
the same amount of work occurs

Work equivalency
try to find forces / moments acting at the nodes that produce the same work as loads/moments acting at other locations

Internal virtual work

$$\delta W_{int} = \int_{Vol} \bar{\underset{\sim}{\varepsilon}}^T \underset{\sim}{\sigma}\, dvol$$ – most general way to write equation

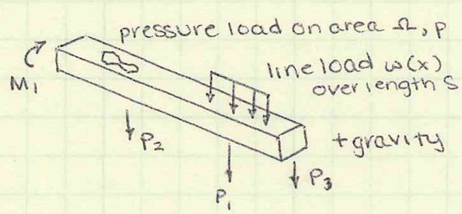
for beams and trusses,

$$\delta W_{int} = \int_{Vol} \underbrace{\bar{\varepsilon} \cdot \sigma}_{\text{non-matrix}}\, dvol$$

$$= \int_{Vol} \bar{\varepsilon}\left(E(\varepsilon - \varepsilon_0) + \sigma_0\right) dvol$$

$$= \int_V \bar{\varepsilon} \cdot E\varepsilon\, dvol - \int_V \bar{\varepsilon} E \varepsilon_0\, dvol + \int_V \bar{\varepsilon}\sigma_0\, dvol$$

External virtual work



the possibilities for loading are practically limitless

$\bar{v}(x) \equiv$ virtual displacements

$$\delta W_{ext} = P_2 \bar{v}(x_2)(-1) - P_1 \cdot \bar{v}(x_1) - M_1 \bar{v}'(x_3) - \int_\Omega p(x,z)\bar{v}(x)\, d\Omega$$

$$- \int_S \omega(x)\bar{v}(x)\, dS$$

$$- \int_V \bar{v}(x) \underbrace{\Lambda(x,y,z)}_{\text{load per unit volume}} dvol$$

$\bar{v}(x_2)$: virtual displacement at location of P_2

(-1): $v(+) = $ up

$M_1 \bar{v}'(x_3)$: moment, θ in opposite dirs.

GENERALIZED FORCE VECTORS

Using external virtual work
for example beam on last page,

$$\delta W_{ext} = P_2 \bar{v}(x_2)(-1) - P_1 \bar{v}(x_1) - M_1 \bar{v}'(x_3) - \int_{\Omega} p(x,z)\,\bar{v}(x)\,d\Omega$$

$$- \int_S w(x)\,\bar{v}(x)\,dS - \int_V \bar{v}(x)\,\Lambda(x,y,z)\,dV + P_3 \bar{v}_2$$

needs to be equal to δW_{int}

$$\bar{v}(x) = \underset{\sim}{H}\,\underset{\sim}{\bar{v}}$$

$$\bar{\varepsilon}(x) = \underset{\sim}{B}\,\underset{\sim}{\bar{v}}$$ strain-displacement matrix

$$\delta W_{ext} = \int_{vol} \bar{v}(x)\cdot\Lambda(x,y,z)\,dvol + \int_{\Omega} \bar{v}(x)\,p(x,z)\,d\Omega$$

$$+ \int_S \bar{v}(x)\cdot\omega(x)\,dS + \sum_{i=1}^{\#\text{pt. loads}} \bar{v}(x_i)\,P_i$$

$$+ \sum_{j=1}^{\#M} \bar{v}'(x_j)\,M_j + \underset{\sim}{\bar{v}}^T \underset{\sim}{R}$$

load/moment acting at the nodes

nodal virtual disp.

$$\delta W_{ext} = \int_{vol} \underset{\sim}{\bar{v}}^T \underset{\sim}{H}^T \Lambda(x,y,z)\,dvol + \int_{\Omega} \underset{\sim}{\bar{v}}^T \underset{\sim}{H}^T p(x,z)\,d\Omega$$

$$+ \int_S \underset{\sim}{\bar{v}}^T \underset{\sim}{H}^T \omega(x)\,dS + \sum_i \underset{\sim}{\bar{v}}^T \underbrace{\underset{\sim}{H}^T}_{H(x_i)} \cdot P_i + \sum_j \underset{\sim}{\bar{v}}^T \underbrace{(\underset{\sim}{H}')^T}_{H'(x_j)} M_j$$

$$+ \underset{\sim}{\bar{v}}^T \underset{\sim}{R}$$

// all terms have $\underset{\sim}{\bar{v}}^T$

remember $\delta W_{int} = \int_V \bar{\varepsilon} E \varepsilon\,dvol - \int_V \bar{\varepsilon} E \varepsilon_0\,dvol + \int_V \bar{\varepsilon}\,\sigma_0\,dvol$

also, $\bar{\varepsilon}(x) = \underset{\sim}{B}\,\underset{\sim}{\bar{v}} = \underset{\sim}{\bar{v}}^T \underset{\sim}{B}^T$

$$\delta W_{int} = \int_V \underset{\sim}{\bar{v}}^T \underset{\sim}{B}^T E \underbrace{\varepsilon}_{\underset{\sim}{B}\underset{\sim}{v}}\,dvol - \int_V \underset{\sim}{\bar{v}}^T \underset{\sim}{B}^T E \varepsilon_0\,dvol + \int_V \underset{\sim}{\bar{v}}^T \underset{\sim}{B}^T \sigma_0\,dvol$$

now all terms here have $\underset{\sim}{\bar{v}}^T$, too
either $\underset{\sim}{\bar{v}}^T = 0$ (trivial), or it can cancel out

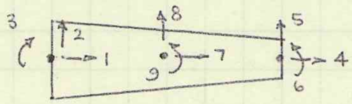
$$\delta W_{int} = \underbrace{\int_V \underset{\sim}{B}^T E \underset{\sim}{B}}_{K}\,\underset{\sim}{v}\,dvol + \ldots$$ take other terms to other side

$$\left[\underset{\sim}{F} = \int_V \underset{\sim}{B}^T E \varepsilon_0\,dvol - \int_V \underset{\sim}{B}^T \sigma_0\,dv + \int_V H^T \Lambda(\,)\,dv + \int_{\Omega} \underset{\sim}{H}^T p(\,)\,d\Omega + \int_S \underset{\sim}{H}^T \omega(x)\,dS + \sum \underset{\sim}{H}^T P_i + \sum (H'(x_j))^T M_j + \underset{\sim}{R}\right]$$

equivalent load vector

HOMEWORK STUFF

Next homework – frame element



DOF 1,7,4 are found with one set of functions,
DOF 2,3,8,9,4,5 are found using other functions

Axially:

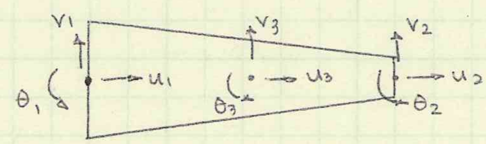
$$u(x) = \underset{\sim}{L}\,\underset{\sim}{r} = \begin{bmatrix} L_1 & 0 & 0 & L_2 & 0 & 0 & L_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{bmatrix}$$

does not influence movement in v, θ directions

Transverse:

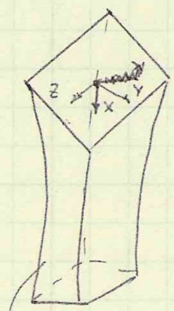
$$v(x) = \underset{\sim}{H}\,\underset{\sim}{r}, \text{ with } \underset{\sim}{H} = \begin{bmatrix} 0 & H_1 & H_2 & 0 & H_3 & H_4 & 0 & H_5 & H_6 \end{bmatrix}$$

(numbers don't match drawing)

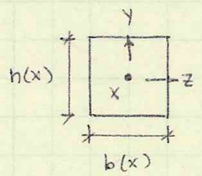


FORCE VECTORS

Example problem



γ (lb/ft³)



$b(x) = b_0(1 - x/2L)$

$h(x) = h_0(1 - x/4L)$

determine equivalent nodal forces to account for self-weight

1. Decide method of approximation

   2-noded element (linear)

2. consider equivalent load vector equation on pg 31

$$\underset{\sim}{F} = \underset{\sim}{R}eq = \int_V \underset{\sim}{L}^T \underbrace{\gamma}_{\text{constant}} \, dvol$$

$$\underset{\sim}{L} = \begin{bmatrix} 1 - x/L \\ x/L \end{bmatrix}$$

$$\underset{\sim}{R}eq = \gamma \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \underset{\sim}{L}^T \, dz\,dy\,dx$$

x y z

$$= \gamma \int_0^L \int_{-\frac{h}{2}}^{h/2} \begin{bmatrix} (1 - x/L)\, b(x) \\ x/L \; b(x) \end{bmatrix} dy\,dx$$

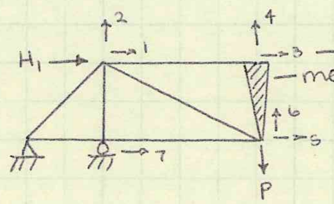
$$= \gamma \int_0^L \begin{bmatrix} (1 - x/L)\, b(x)\, h(x) \\ x/L \; b(x)\, h(x) \end{bmatrix} dx$$

$$= \gamma \begin{bmatrix} \int_0^L (1 - x/L)(1 - x/2L)(1 - x/4L)\, b_0 h_0 \, dx \\ \int_0^L x/L \,(1 - x/2L)(1 - x/4L)\, dx \cdot b_0 h_0 \end{bmatrix}$$

$$\underset{\sim}{R}eq = \gamma b_0 h_0 L \begin{bmatrix} 37/96 \\ 9/32 \end{bmatrix}$$

FORCE VECTORS

Example, con'd (or, cont'd)



$$\underset{\sim}{R_{eq}} = \gamma b_0 h_0 L \begin{bmatrix} 37/96 \\ 9/32 \end{bmatrix}$$

$$\underset{\sim}{R} = \begin{bmatrix} H_1 \\ 0 \\ H_2 \\ -\frac{37}{96}\gamma b_0 h_0 L \\ 0 \\ -P - \frac{9}{32}\gamma b_0 h_0 L \\ 0 \end{bmatrix}$$

negative because force is in the opposite direction from the assigned DOFS.

Member forces

fixed-end forces

$$Q_{taper} = \underset{\sim}{K}_{taper} \cdot \underset{\sim}{U}_{taper} + \underset{\sim}{FE}_{taper}$$

$$\underset{\sim}{U}_{taper} = \begin{bmatrix} r_4 \\ 0 \\ r_6 \\ 0 \end{bmatrix}$$

r_3, r_5 don't come into axial force in the member

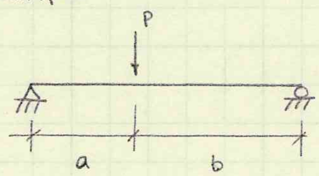
local coordinate matrices – might need to transform deformations to local from global

$$\underset{\sim}{FE} = -\underset{\sim}{R_{eq}}$$

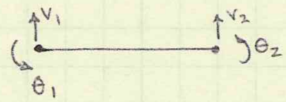
$$\underset{\sim}{Q} = \underset{\sim}{k}\,\underset{\sim}{q} + \underset{\sim}{Q_{FE}}$$

EQUIVALENT FORCE MATRIX

Beam example



model as a 2-noded beam



$$\underset{\sim}{R}eq = \sum H(x)^T P\ (-1)$$

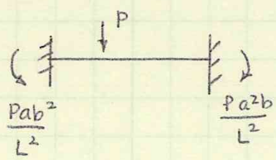
↳ load, DOF in opposite directions, negative work

$$\underset{\sim}{R}eq = H(x)^T\ (-P)$$

$$= \begin{bmatrix} 1-3(a/L)^2+2(a/L)^3 \\ a-2(a^2/L)+a^3/L^2 \\ 3(a/L)^2-2(a/L)^3 \\ -a^2/L+a^3/L^2 \end{bmatrix} (-P)$$

H evaluated at $x=a$ as that is the location of the load P

From text book:

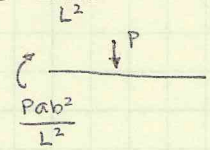


$$(\underset{\sim}{R}eq)_2 = -Pa + 2P\,a^2/L - P\,a^3/L^2$$

$$= (-PaL^2 + 2Pa^2L - Pa^3)\,1/L^2$$

$$= +\frac{Pa}{L^2}\left[L^2 + 2aL - a^2\right]$$

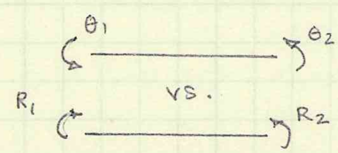
$$= \frac{-Pa}{L^2}(L-a)^2 = \frac{-Pab^2}{L^2}$$



- Negative of fixed-end forces in load vector
- Positive when added back in

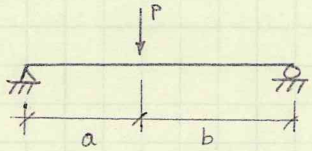
$$\underset{\sim}{R} = \frac{Pab}{L^2}\begin{bmatrix} -b \\ a \end{bmatrix}$$

opposite signs indicate not in same dir.



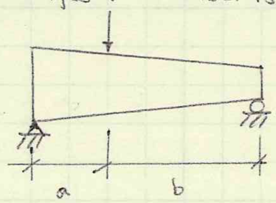
EQUIVALENT FORCES

Example, revisited



$\underset{\sim}{R}_{eq} = \underset{\sim}{H}^T(a) \cdot (-P)$

what changes if member is tapered?

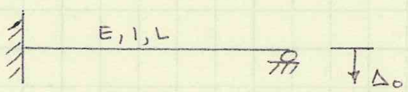


$h(x) = h_o(1 - x/2L)$

- load vector is independent of beam specifics (E, I, etc). thus, tapered beam $\underset{\sim}{F}$ is the same as prismatic beam
- deflections come out, wouldn't if ends were fixed

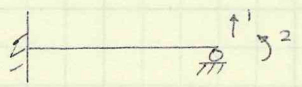
thesis: beam element aspect ratio ~ 10 (length/A)

Example #2



support settlement

- ignore axial deformations
- one DOF; apply Δ_o in $\underset{\sim}{F}$

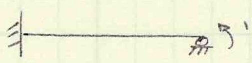


$\underset{\sim}{K}\underset{\sim}{r} = \underset{\sim}{R}$

need to partition matrix to get exact solution – boo.

$$\left[\begin{array}{c|c} & \\ \hline & \end{array}\right]\left[\begin{array}{c} -\Delta_o \\ \hline r_o \end{array}\right] = \left[\begin{array}{c} R_y \\ \hline 0 \end{array}\right]$$ ick.

- as 1 DOF:



account for Δ_o in load vector

initial strain, $\int_V \underset{\sim}{B}^T E \varepsilon_o \, dVol$

$= yH''^T$

$v_o^T = [0 \quad 0 \quad -\Delta_o \quad 0]$

initial displacements

use (+)!

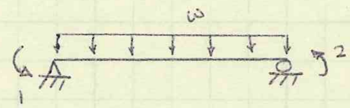
from 363

Δ_o

$F = \frac{12EI}{L^3}\Delta_o$

EQUIVALENT LOADS

Example



$$\underset{\sim}{R}eq = \int_0^L (\underset{\sim}{H}'')^T(-\omega)\,dx$$ → will result in 4 values; only 2 act in directions of declared DOFs.

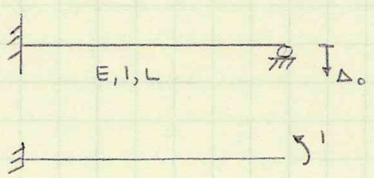
$$R_2 = \int_0^L (x - 2x^2/L + x^3/L^2)(-\omega)\,dx$$

$$R_2 = -1/12\ \omega l^2$$

$$R_4 = \omega L^2/12$$

distributed load just means integrate over the length of the load

Return to Ex. #2 on pg 36



no good! use (+)

$$v_0^T = [\,0 \quad 0 \quad -\Delta_0 \quad 0\,]$$ given to find fixed-end forces from support settlement

initial displacements

consider initial strain:

$$\underset{\sim}{\varepsilon}_0 = yH''v_0$$

$$\underset{\sim}{R}eq = \int_0^A\int_0^L y(\underset{\sim}{H}'')^T E\,\varepsilon_0\,dx\,dA$$

for a beam, $\underset{\sim}{B}^T = y\underset{\sim}{H}''^T$

$$\underset{\sim}{R}eq = \int_0^L\int_0^A y(\underset{\sim}{H}'')^T E\,y(\underset{\sim}{H}'')\,dvol \cdot \underset{\sim}{v}_0$$

pull out $\int_A y^2\,dA = I$

$$= \int_0^L (\underset{\sim}{H}'')^T E(\underset{\sim}{H}'')\,I\,dx \cdot \underset{\sim}{v}_0$$

$$= \underbrace{\int_0^L (\underset{\sim}{H}'')^T E(x)I(x)(\underset{\sim}{H}'')\,dx}_{\text{stiffness matrix}} \cdot \underset{\sim}{v}_0$$

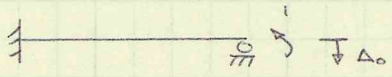
$$\underset{\sim}{R}eq = \underset{\sim}{k}\cdot\underset{\sim}{v}_0$$

in this case, we care about the fourth term now

→ goes to structural load vector

LOAD VECTOR

Example, cont'd



$\underset{\sim}{R_{eq}} = \underset{\sim}{k} \cdot \underset{\sim}{v_o}$

$$[\underset{\sim}{k}]_4 = \left[\frac{6EI}{L^2} \quad \frac{2EI}{L} \quad \frac{-6EI}{L^2} \quad \frac{4EI}{L} \right]$$

$$\times \left[0 \quad 0 \quad -\Delta_o \quad 0 \right]^T$$

$R_{eq} = \dfrac{6EI}{L^2} \Delta_o$ how applied?

structure equilibrium

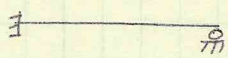
$\underset{\sim}{k} \, \underset{\sim}{r} = \underset{\sim}{R}$

$$\left[\frac{4EI}{L} \right] \theta = \left[\frac{6EI}{L^2} \Delta_o \right]$$

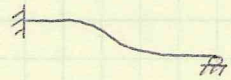
$\theta > 0$, θ is counterclockwise

but real solution should be clockwise

Δ_o term in $\underset{\sim}{v_o}$ should be positive, not negative — want to analyze as


(1)

not


(2)

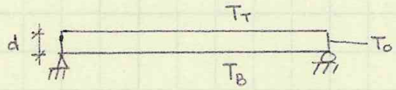
to get from (1) to (2), Δ_o goes <u>up</u>, not down → positive

this method is the only time you have to look at the DOF direction like this

and initial problem (initial strain from Δ_o or θ_o)

THERMAL LOADS

on beams



$T_B > T_o$

$T_T < T_o$

Goal: relate temperature change to beam curvature

$$\kappa(x) = \frac{(T_B - T_T)\,\alpha}{d}$$

κ: kappa

ΔT, delta between surfaces, not T_o and T_f

Assumptions:
- temperature varies linearly through the depth
- α, d are constant

More general case:

planar response

$$\kappa(x) = \frac{\alpha(x)}{I(x)} \int_{-\frac{d(x)}{2}}^{\frac{d(x)}{2}} \Delta T(x,y) \cdot y \cdot b \, dy$$

$\Delta T(x,y)$: temperature change from top to bottom

b: width (could be $b(x)$)

still does not allow for ΔT through width (z direction)

THERMAL EFFECTS

General planar response

$$K(x) = \frac{\alpha(x)}{I(x)} \int_{-\frac{d}{2}}^{\frac{d}{2}} \Delta T(x,y) \cdot y \cdot b(x) \cdot dy$$ curvature along the length of the beam (x)

Now, develop equivalent nodal forces
integrate over the volume as an initial strain

$$\int_V \underset{\sim}{B}^T E \varepsilon_o \, dvol$$

for a beam, $\varepsilon_o = y \cdot K(x)$ ← funky K = kappa, or curvature.

$$\underset{\sim}{R}_{eq} = \int_V \underset{\sim}{B}^T E \varepsilon_o \, dvol$$

for a beam, $\underset{\sim}{B}^T = y H''^T$

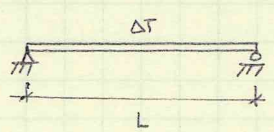
$$= \int_V y (\underset{\sim}{H}'')^T E \, y \cdot K_o \, dvol$$ — again, have $\int_A y^2 dA = I(x)$

$$= \int_0^L (\underset{\sim}{H}'')^T E I(x) K_o \, dx$$

$$\hookrightarrow K(x) = \frac{\alpha(x)}{I(x)} \int \Delta T(x,y) \, y \, b(x) \, dy$$

$$\underset{\sim}{R}_{eq} = \int_0^L (\underset{\sim}{H}'')^T E(x) \alpha(x) \int_{-\frac{d}{2}}^{\frac{d}{2}} \Delta T(x,y) \, y \, b(x) \, dy \, dx$$

Example with temperature variation



$\Delta T = T_B - T_T$

$T_B > T_T$ should get $EI\alpha \frac{\Delta T}{d}$ (I think)

$$K(x) = \frac{\alpha \Delta T}{d}$$ (both cause we know it, and integral would come out)

$$\underset{\sim}{R}_{eq} = \int_0^L (\underset{\sim}{H}'')^T EI \frac{\alpha \Delta T}{d} \, dx \longrightarrow 4 \times 1 \text{ vector}$$

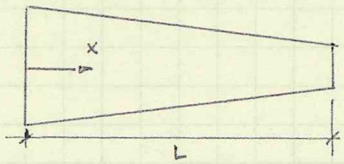
$$(R_{eq})_2 = \frac{EI \alpha \Delta T}{d} \int_0^L -4/L + 6x/L^2 \, dx$$

$$-\frac{4x}{L} + \frac{3x^2}{L^2} \Big|_0^L = -1$$

$$(R_{eq})_2 = \frac{-EI}{d} \alpha \Delta T$$ counterclockwise is ~~an~~ defined as (+)

THERMAL EFFECTS

More complex example



$b = \text{constant}$

$d = d_o\,(1 - x/2L)$

$$\Delta T(x,y) = 4\Delta T_o \left[\left(\frac{2y}{d}\right)^3 \left(\frac{x}{L} - \frac{x^2}{L^2}\right) \right]$$

actually, d(x)

varies quadratically along the length, cubically through the depth

ΔT_o = differential at midspan

1. Find initial curvature

$$K(x) = \frac{\alpha(x)}{I(x)} \int_{-\frac{d(x)}{2}}^{\frac{d(x)}{2}} 4\Delta T_o \left[\left(\frac{2y}{d(x)}\right)^3 \left(\frac{x}{L} - \frac{x^2}{L^2}\right) \right] y\,b\,dy$$

$$= \frac{\alpha(x)}{I(x)} \cdot \frac{4\Delta T_o\,(8)}{(d(x))^3} \left(\frac{x}{L} - \frac{x^2}{L^2}\right) b \int_{-\frac{d(x)}{2}}^{\frac{d(x)}{2}} y^4\,dy$$

$$\frac{1}{5} y^5 \Big|_{-d(x)/2}^{d(x)/2} = \frac{1}{10}(d(x))^5 \cdot 2 \cdot \frac{1}{32}$$

$$= \frac{\cancel{8}\;2}{\cancel{32}\;5} \frac{\alpha(x)}{I(x)} \Delta T_o \left(\frac{x}{L} - \frac{x^2}{L^2}\right) (d(x))^2\, b$$

from 1/32 term

$$K(x) = \frac{2}{5} \frac{\alpha\,\Delta T_o\, b}{I(x)} d(x)^2 \left(\frac{x}{L} - \frac{x^2}{L^2}\right)$$

2. compute equivalent loads

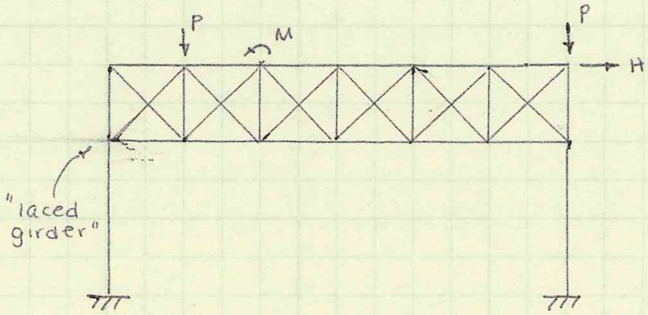
$$\underset{\sim}{R_{eq}} = \int_0^L (H'')^T E I(x) \cdot \frac{2}{5} \frac{\alpha\,\Delta T_o\, b}{I(x)} d(x)^2 \left(\frac{x}{L} - \frac{x^2}{L^2}\right) dx$$

$$= \frac{2}{5} E\alpha\,\Delta T_o\, b \int_0^L (H'')^T d_o^2 (1 - x/2L)^2 \left(\frac{x}{L} - \frac{x^2}{L^2}\right) dx$$

$$\underset{\sim}{R_{eq}} = \frac{2}{5} E\alpha\,\Delta T_o\, b\, d_o^2 \begin{bmatrix} -3/40L \\ -2/15 \\ 3/40L \\ 7/120 \end{bmatrix}$$

SPECIAL TOPICS

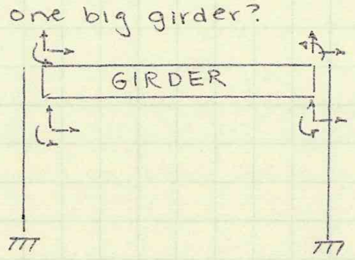
Substructure Analysis



goal: reduce number of degrees of freedom

method: divide into smaller (sub) structures

14 nodes with DOFs (x3)
= 42 DOFs. ick
wouldn't it be nicer if the entire thing were just one big girder?



$\underset{\sim}{r_i} \equiv$ DOF on the interior of substructure (not drawn)
- values not computed explicitly

$\underset{\sim}{r_b} \equiv$ DOF on the boundaries
- calculate explicitly; connect to rest of structure

loads:

$\underset{\sim}{R_i} \equiv$ loads acting on interior DOFs

$\underset{\sim}{R_b} \equiv$ loads acting on boundary DOFs

Treat girder as its own structure

$$\underset{\sim}{K}\,\underset{\sim}{r} = \underset{\sim}{R} \qquad \begin{bmatrix} \underbrace{\underset{\sim}{K_{bb}}}_{12\times 12} & \underbrace{\underset{\sim}{K_{bi}}}_{12\times 30} \\ \underbrace{\underset{\sim}{K_{ib}}}_{30\times 12} & \underbrace{\underset{\sim}{K_{ii}}}_{30\times 30} \end{bmatrix} \begin{bmatrix} \underbrace{\underset{\sim}{r_b}}_{12\times 1} \\ \underbrace{\underset{\sim}{r_{ai}}}_{30\times 1} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{R_B} \\ \underset{\sim}{R_i} \end{bmatrix}$$

(1) $\underset{\sim}{K_{bb}} \cdot \underset{\sim}{r_b} + \underset{\sim}{K_{bi}} \cdot \underset{\sim}{r_i} = \underset{\sim}{R_B}$

(2) $\underset{\sim}{K_{ib}}\,\underset{\sim}{r_b} + \underset{\sim}{K_{ii}} \cdot \underset{\sim}{r_i} = \underset{\sim}{R_i}$

goal: remove i terms from equations (using bs)

$$\underset{\sim}{r_i} = \underset{\sim}{K_{ii}}^{-1}\left[\underset{\sim}{R_i} - \underset{\sim}{K_{ib}}\,r_b\right] \quad \text{from (2)}$$

into (1):

$$\underset{\sim}{K_{bb}} \cdot \underset{\sim}{r_b} + \underset{\sim}{K_{bi}} \cdot \underset{\sim}{K_{ii}}^{-1}\left[\underset{\sim}{R_i} - \underset{\sim}{K_{ib}}\,\underset{\sim}{r_b}\right] = \underset{\sim}{R_B}$$

$$\underbrace{(K_{bb} - K_{bi} \cdot K_{ii}^{-1} K_{ib})}_{\text{stiffness matrix of substructure}}\, r_b = \underbrace{R_B - K_{bi} K_{ii}^{-1} R_i}_{\text{equivalent load vector}}$$

SUBSTRUCTURE ANALYSIS

Resulting Equation:

$$\underset{\sim}{K} = K_{bb} - K_{bi} K_{ii}^{-1} K_{ib}$$

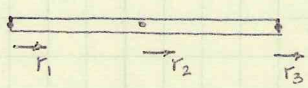
$$F = R_B - \underbrace{K_{bi} K_{ii}^{-1} R_i}$$

similar to fixed-end forces → gives the effect of the interior forces if they were located at the boundaries

$$\underset{\sim}{K}\, \underset{\sim}{r_b} = \underset{\sim}{F}$$

Advantages to substructure method
- with repeated substructures
- problem (without substructure) is too large for computer memory
- dynamic analyses (esp. when interior nodes are idealized as massless)

Example



3-noded axial bar
but... want to ignore center DOF
treat bar as substructure

(E, A, L)

$$r_i = r_2, \quad r_b = [r_1 \;\; r_3]^T$$

$$\underset{\sim}{K} = \begin{bmatrix} 7/3 & -8/3 & 1/3 \\ -8/3 & 16/3 & -8/3 \\ 1/3 & -8/3 & 7/3 \end{bmatrix} \frac{EA}{L}$$

bad news: terms aren't collected together

$$K_{bb} = \begin{bmatrix} 7 & 1 \\ 1 & 7 \end{bmatrix}, \quad K_{ib} = \begin{bmatrix} -8 & -8 \end{bmatrix}$$

$$K_{bi} = \begin{bmatrix} -8 & -8 \end{bmatrix}^T$$

$$K_{ii} = \begin{bmatrix} 16 \end{bmatrix} \qquad \times \frac{EA}{3L}$$

$$\underset{\sim}{K'} = \frac{EA}{3L} \begin{array}{c} \begin{bmatrix} 7 & 1 & -8 \\ \cancel{-8}\,1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \end{array} \begin{array}{c} r_1 \\ r_3 \\ r_2 \end{array}$$

$$\frac{3L}{EA} \underset{\sim}{K} = \begin{bmatrix} 7 & 1 \\ 1 & 7 \end{bmatrix} - \begin{bmatrix} -8 \\ -8 \end{bmatrix} [16]^{-1} \begin{bmatrix} -8 & -8 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 1/3 \end{bmatrix} \cdot 9 \leftarrow \text{dammit}$$

$$\underset{\sim}{K} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

that's the same as a 2-noded element

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3-Noded Axial Member

we tried making 3-noded into 2-noded, ended up with the same thing we had before.

What about a more complex (non-prismatic) beam?

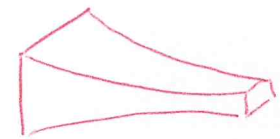
$$\text{vector}(x,L) := \begin{pmatrix} \frac{-3}{L} + 4\cdot\frac{x}{L^2} \\ \frac{4}{L} - 8\cdot\frac{x}{L^2} \\ \frac{-1}{L} + 4\cdot\frac{x}{L^2} \end{pmatrix}$$

Transpose of "**B**" vector for the element

$$\text{vector}(x,L)\cdot\text{vector}(x,L)^T \rightarrow \begin{bmatrix} \left(\frac{-3}{L} + 4\cdot\frac{x}{L^2}\right)^2 & \left(\frac{-3}{L} + 4\cdot\frac{x}{L^2}\right)\cdot\left(\frac{4}{L} - 8\cdot\frac{x}{L^2}\right) & \left(\frac{-3}{L} + 4\cdot\frac{x}{L^2}\right)\cdot\left(\frac{-1}{L} + 4\cdot\frac{x}{L^2}\right) \\ \left(\frac{-3}{L} + 4\cdot\frac{x}{L^2}\right)\cdot\left(\frac{4}{L} - 8\cdot\frac{x}{L^2}\right) & \left(\frac{4}{L} - 8\cdot\frac{x}{L^2}\right)^2 & \left(\frac{4}{L} - 8\cdot\frac{x}{L^2}\right)\cdot\left(\frac{-1}{L} + 4\cdot\frac{x}{L^2}\right) \\ \left(\frac{-3}{L} + 4\cdot\frac{x}{L^2}\right)\cdot\left(\frac{-1}{L} + 4\cdot\frac{x}{L^2}\right) & \left(\frac{4}{L} - 8\cdot\frac{x}{L^2}\right)\cdot\left(\frac{-1}{L} + 4\cdot\frac{x}{L^2}\right) & \left(\frac{-1}{L} + 4\cdot\frac{x}{L^2}\right)^2 \end{bmatrix}$$

Two-Point Gauss Quadrature

$x_1 := 0.21132 \cdot L \qquad x_2 := 0.788675 \cdot L$



$A = A_o \left(1 - x/2L\right)^2 \left(1 - \sin \pi x/4L\right)$

$$k_1(E,A,L) := \overset{B^T}{\mathrm{vector}(x_1,L)} \cdot \left(\overset{B}{\mathrm{vector}(x_1,L)}\right)^T \cdot \overset{E}{E} \cdot \overset{A}{A \cdot \left(1 - \frac{x_1}{2 \cdot L}\right)^2 \cdot \left(1 - \sin\left(\frac{\pi \cdot x_1}{4 \cdot L}\right)\right)} \cdot \frac{L}{2}$$

$$k_2(E,A,L) := \mathrm{vector}(x_2,L) \cdot \left(\mathrm{vector}(x_2,L)\right)^T \cdot E \cdot A \cdot \left(1 - \frac{x_2}{2 \cdot L}\right)^2 \cdot \left(1 - \sin\left(\frac{\pi \cdot x_2}{4 \cdot L}\right)\right) \cdot \frac{L}{2}$$

$$K(E,A,L) := k_1(E,A,L) + k_2(E,A,L)$$

$$K(E,A,L)\ \mathrm{float}, 9 \rightarrow \begin{pmatrix} \frac{1.55185124}{L} \cdot E \cdot A & \frac{-1.68879315}{L} \cdot E \cdot A & \frac{.136941900}{L} \cdot E \cdot A \\ \frac{-1.68879315}{L} \cdot E \cdot A & \frac{2.19089149}{L} \cdot E \cdot A & \frac{-.502098345}{L} \cdot E \cdot A \\ \frac{.136941900}{L} \cdot E \cdot A & \frac{-.502098345}{L} \cdot E \cdot A & \frac{.365156445}{L} \cdot E \cdot A \end{pmatrix}$$

Three-Point Gauss Quadrature

$x_1 := \frac{L}{2}\cdot\left(1 - \sqrt{\frac{3}{5}}\right)$ $\quad w_1 := \frac{5}{9}$ $\quad x_2 := \frac{L}{2}\cdot(1-0)$ $\quad w_2 := \frac{8}{9}$ $\quad x_3 := \frac{L}{2}\cdot\left(1 + \sqrt{\frac{3}{5}}\right)$ $\quad w_3 := \frac{5}{9}$

$$k_1(E,A,L) := \mathrm{vector}(x_1,L)\cdot\left(\mathrm{vector}(x_1,L)\right)^T\cdot E\cdot A\cdot\left(1 - \frac{x_1}{2\cdot L}\right)^2\cdot\left(1 - \sin\left(\frac{\pi\cdot x_1}{4\cdot L}\right)\right)\cdot w_1\cdot\frac{L}{2}$$

$$k_2(E,A,L) := \mathrm{vector}(x_2,L)\cdot\left(\mathrm{vector}(x_2,L)\right)^T\cdot E\cdot A\cdot\left(1 - \frac{x_2}{2\cdot L}\right)^2\cdot\left(1 - \sin\left(\frac{\pi\cdot x_2}{4\cdot L}\right)\right)\cdot w_2\cdot\frac{L}{2}$$

$$k_3(E,A,L) := \mathrm{vector}(x_3,L)\cdot\left(\mathrm{vector}(x_3,L)\right)^T\cdot E\cdot A\cdot\left(1 - \frac{x_3}{2\cdot L}\right)^2\cdot\left(1 - \sin\left(\frac{\pi\cdot x_3}{4\cdot L}\right)\right)\cdot w_3\cdot\frac{L}{2}$$

$$K(E,A,L) := k_1(E,A,L) + k_2(E,A,L) + k_3(E,A,L)$$

$$K(E,A,L)\ \mathrm{float},9 \rightarrow \begin{pmatrix} \frac{1.62892372}{L}\cdot E\cdot A & \frac{-1.83339076}{L}\cdot E\cdot A & \frac{.204467029}{L}\cdot E\cdot A \\ \frac{-1.83339076}{L}\cdot E\cdot A & \frac{2.46031659}{L}\cdot E\cdot A & \frac{-.626925836}{L}\cdot E\cdot A \\ \frac{.204467029}{L}\cdot E\cdot A & \frac{-.626925836}{L}\cdot E\cdot A & \frac{.422458807}{L}\cdot E\cdot A \end{pmatrix}$$

Condense Middle Displacement DOF

$$P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

permutation matrix – switches rows & columns

$$K(E,A,L) := \begin{pmatrix} \frac{1.62892372}{L}\cdot E\cdot A & \frac{-1.83339075}{L}\cdot E\cdot A & \frac{.204467027}{L}\cdot E\cdot A \\ \frac{-1.83339075}{L}\cdot E\cdot A & \frac{2.46031659}{L}\cdot E\cdot A & \frac{-.626925837}{L}\cdot E\cdot A \\ \frac{.204467027}{L}\cdot E\cdot A & \frac{-.626925837}{L}\cdot E\cdot A & \frac{.422458807}{L}\cdot E\cdot A \end{pmatrix}$$

K_{bb} $\qquad$ K_{bi}

$$K_{permuted}(E,A,L) := P^T \cdot K(E,A,L)\cdot P \rightarrow \left(\begin{array}{cc|c} \frac{1.62892372}{L}\cdot E\cdot A & \frac{.204467027}{L}\cdot E\cdot A & \frac{-1.83339075}{L}\cdot E\cdot A \\ \frac{.204467027}{L}\cdot E\cdot A & \frac{.422458807}{L}\cdot E\cdot A & \frac{-.626925837}{L}\cdot E\cdot A \\ \hline \frac{-1.83339075}{L}\cdot E\cdot A & \frac{-.626925837}{L}\cdot E\cdot A & \frac{2.46031659}{L}\cdot E\cdot A \end{array}\right)$$

K_{ib} $\qquad$ K_{ii}

$K_{ii}(E,A,L) := \text{submatrix}(K_{permuted}(E,A,L),2,2,2,2)$

$K_{bb}(E,A,L) := \text{submatrix}(K_{permuted}(E,A,L),0,1,0,1)$

from row 0 to row 1, to column 0 to column 1

$K_{bi}(E,A,L) := \text{submatrix}(K_{permuted}(E,A,L),0,1,2,2)$

$K_{ib}(E,A,L) := \text{submatrix}(K_{permuted}(E,A,L),0,1,2,2)^T$

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$$K_{condensed}(E,A,L) := K_{bb}(E,A,L) - K_{bi}(E,A,L)\cdot K_{ii}(E,A,L)^{-1}\cdot K_{ib}(E,A,L)$$

$$K_{bi}(E,A,L) := \mathrm{submatrix}(K_{permuted}(E,A,L),0,1,2,2) \rightarrow \begin{pmatrix} \frac{-1.83339075}{L}\cdot E\cdot A \\ \frac{-.626925837}{L}\cdot E\cdot A \end{pmatrix}$$

$$K_{bb}(E,A,L) := \mathrm{submatrix}(K_{permuted}(E,A,L),0,1,0,1) \rightarrow \begin{pmatrix} \frac{1.62892372}{L}\cdot E\cdot A & \frac{.204467027}{L}\cdot E\cdot A \\ \frac{.204467027}{L}\cdot E\cdot A & \frac{.422458807}{L}\cdot E\cdot A \end{pmatrix}$$

$$K_{ib}(E,A,L) := \mathrm{submatrix}(K_{permuted}(E,A,L),0,1,2,2)^{T} \rightarrow \begin{pmatrix} \frac{-1.83339075}{L}\cdot E\cdot A & \frac{-.626925837}{L}\cdot E\cdot A \end{pmatrix}$$

$$K_{ii.inverse}(E,A,L) := \mathrm{submatrix}(K_{permuted}(E,A,L),2,2,2,2)^{-1} \rightarrow .40645175668225689605\cdot\frac{L}{E\cdot A}\cdot(.9999999999999999999)$$

$$K_{ii.inverse}(E,A,L) := .40645175668225689605\cdot\frac{L}{(E\cdot A)}$$

$$K_{temp}(E,A,L) := K_{bi}(E,A,L)\cdot K_{ii.inverse}(E,A,L)\cdot K_{ib}(E,A,L) \rightarrow \begin{pmatrix} \frac{1.3662150862404104262}{L}\cdot E\cdot A & \frac{.46717566152403490072}{L}\cdot E\cdot A \\ \frac{.46717566152403490073}{L}\cdot E\cdot A & \frac{.15975017471151977600}{L}\cdot E\cdot A \end{pmatrix}$$

$$K_{condensed}(E,A,L) := K_{bb}(E,A,L) - K_{temp}(E,A,L) \text{ float}, 5 \rightarrow \begin{pmatrix} \frac{.26271}{L}\cdot E\cdot A & \frac{-.26271}{L}\cdot E\cdot A \\ \frac{-.26271}{L}\cdot E\cdot A & \frac{.26271}{L}\cdot E\cdot A \end{pmatrix}$$

$\alpha = 0.26271$

What if, rather than carrying out the condensation, we simply used the standard two-node truss element with a constant cross-sectional area to represent the doubly tapered member?

The results below are for the case where the representative area is chosen to correspond with the centroid.

$$A(A_o, x, L) := A_o \cdot \left[\left(1 - \frac{x}{2\cdot L}\right)^2 \cdot \left(1 - \sin\left(\frac{\pi\cdot x}{4\cdot L}\right)\right)\right]$$

$$A_{tot}(x,L) := \left[\int_0^L \left(1 - \frac{x}{2\cdot L}\right)^2 \cdot \left(1 - \sin\left(\frac{\pi\cdot x}{4\cdot L}\right)\right) dx\right] \qquad A_{tot}(x,L) \text{ float}, 5 \rightarrow .41064\cdot L$$

$$x_{bar.temp} := \int_0^L x\cdot\left(1 - \frac{x}{2\cdot L}\right)^2 \cdot \left(1 - \sin\left(\frac{\pi\cdot x}{4\cdot L}\right)\right) dx \qquad x_{bar.temp} \text{ float}, 5 \rightarrow .1296\cdot L^2$$

$x_{bar} := \dfrac{x_{bar.temp}}{A_{tot}(x,L)} \,\Big|\, \begin{array}{l}\text{simplify} \\ \text{float}, 6\end{array} \rightarrow .316375 \cdot L$ $\qquad A(A_o, x_{bar}, L) \text{ float}, 5 \rightarrow .53437 \cdot A_o$

% difference = 103% from results obtained from condensation

(0.26271)

What if, rather than carrying out the condensation, we simply derived the stiffness matrix for a two-noded truss with linear shape functions?

$$vector_2(L) := \begin{pmatrix} \dfrac{-1}{L} \\ \dfrac{1}{L} \end{pmatrix}$$

<u>**Three-Point Gauss Quadrature**</u>

$x_1 := \dfrac{L}{2} \cdot \left(1 - \sqrt{\dfrac{3}{5}}\right) \qquad w_1 := \dfrac{5}{9} \qquad x_2 := \dfrac{L}{2} \cdot (1 - 0) \qquad w_2 := \dfrac{8}{9} \qquad x_3 := \dfrac{L}{2} \cdot \left(1 + \sqrt{\dfrac{3}{5}}\right) \qquad w_3 := \dfrac{5}{9}$

$$k_{1.2}(E, A_o, L) := vector_2(L) \cdot (vector_2(L))^T \cdot E \cdot A_o \cdot \left(1 - \frac{x_1}{2 \cdot L}\right)^2 \cdot \left(1 - \sin\left(\frac{\pi \cdot x_1}{4 \cdot L}\right)\right) \cdot w_1 \cdot \frac{L}{2}$$

$$k_{2.2}(E, A_o, L) := vector_2(L) \cdot (vector_2(L))^T \cdot E \cdot A_o \cdot \left(1 - \frac{x_2}{2 \cdot L}\right)^2 \cdot \left(1 - \sin\left(\frac{\pi \cdot x_2}{4 \cdot L}\right)\right) \cdot w_2 \cdot \frac{L}{2}$$

$$k_{3.2}(E, A_o, L) := vector_2(L) \cdot (vector_2(L))^T \cdot E \cdot A_o \cdot \left(1 - \frac{x_3}{2 \cdot L}\right)^2 \cdot \left(1 - \sin\left(\frac{\pi \cdot x_3}{4 \cdot L}\right)\right) \cdot w_3 \cdot \frac{L}{2}$$

$$K(E,A_o,L) := k_{1.2}(E,A_o,L) + k_{2.2}(E,A_o,L) + k_{3.2}(E,A_o,L)$$

$$K(E,A_o,L)\ \text{float},4 \rightarrow \begin{pmatrix} \frac{.4106}{L}\cdot E\cdot A_o & \frac{-.4106}{L}\cdot E\cdot A_o \\ \frac{-.4106}{L}\cdot E\cdot A_o & \frac{.4106}{L}\cdot E\cdot A_o \end{pmatrix}$$

A, total area

Same Answer as if evaluated so that the total area is preserved

% difference = 30% from results at centroid
(56% from results obtained from condensation)

What about the case for a linearly tapered element?

Exact solution = 0.7213 Ao – have a value to compare versions to

$\alpha = 0.7213$

$$A3(A_o,x,L) := A_o\cdot\left(1 - \frac{x}{2\cdot L}\right)$$

$$A_{tot}(A_o,x,L) := \left(\int_0^L A3(A_o,x,L)\,dx\right) \qquad A_{tot}(A_o,x,L)\ \text{float},5 \rightarrow .75000\cdot L\cdot A_o$$

How does this result compare to case when original element is approximated by 2-noded truss?

$$x_{bar.temp}(A_o,L) := \int_0^L x\cdot A3(A_o,x,L)\,dx \qquad x_{bar.temp}(A_o,L)\ \text{float},5 \rightarrow .33333\cdot L^2\cdot A_o$$

$$x_{bar} := \frac{x_{bar.temp}(A_o,L)}{A_{tot}(A_o,x,L)}\ \Bigg|\begin{array}{l}\text{simplify}\\ \text{float},6\end{array} \rightarrow .444444\cdot L$$

centroid location

at centroid:

$$A3(A_o,x_{bar},L)\ \text{float},5 \rightarrow .77778\cdot A_o$$

vs. 0.7213
not bad, but not exact

2-noded
linear approx.

Three-Point Gauss Quadrature

$x_1 := \frac{L}{2}\cdot\left(1 - \sqrt{\frac{3}{5}}\right)$ $\quad w_1 := \frac{5}{9}$ $\quad x_2 := \frac{L}{2}\cdot(1-0)$ $\quad w_2 := \frac{8}{9}$ $\quad x_3 := \frac{L}{2}\cdot\left(1 + \sqrt{\frac{3}{5}}\right)$ $\quad w_3 := \frac{5}{9}$ $\quad \text{vector}_2(L) := \begin{pmatrix} \frac{-1}{L} \\ \frac{1}{L} \end{pmatrix}$

$$k_{1.2}(E, A_o, L) := \text{vector}_2(L)\cdot(\text{vector}_2(L))^T\cdot E\cdot A_3(A_o, x_1, L)\cdot w_1\cdot\frac{L}{2}$$

$$k_{2.2}(E, A_o, L) := \text{vector}_2(L)\cdot(\text{vector}_2(L))^T\cdot E\cdot A_3(A_o, x_2, L)\cdot w_2\cdot\frac{L}{2}$$

$$k_{3.2}(E, A_o, L) := \text{vector}_2(L)\cdot(\text{vector}_2(L))^T\cdot E\cdot A_3(A_o, x_3, L)\cdot w_3\cdot\frac{L}{2}$$

$$K(E, A_o, L) := k_{1.2}(E, A_o, L) + k_{2.2}(E, A_o, L) + k_{3.2}(E, A_o, L)$$

$$K(E, A_o, L) \text{ float}, 4 \rightarrow \begin{pmatrix} \frac{.7500}{L}\cdot E\cdot A_o & \frac{-.7500}{L}\cdot E\cdot A_o \\ \frac{-.7500}{L}\cdot E\cdot A_o & \frac{.7500}{L}\cdot E\cdot A_o \end{pmatrix}$$

A_{tot}

vs. 0.7213

<u>Same Answer as if evaluated so that the total area is preserved</u>

% difference = 3% difference from guess at centroid

Compute results from condensed matrix for solution first approximated by 3-noded truss

Three-Point Gauss Quadrature

$x_1 := \frac{L}{2}\cdot\left(1 - \sqrt{\frac{3}{5}}\right)$ $\quad w_1 := \frac{5}{9}$ $\quad x_2 := \frac{L}{2}\cdot(1 - 0)$ $\quad w_2 := \frac{8}{9}$ $\quad x_3 := \frac{L}{2}\cdot\left(1 + \sqrt{\frac{3}{5}}\right)$ $\quad w_3 := \frac{5}{9}$

$$k_{1.ex}(E, A_o, L) := \text{vector}(x_1, L)\cdot(\text{vector}(x_1, L))^T\cdot E\cdot A_o\cdot\left(1 - \frac{x_1}{2\cdot L}\right)\cdot w_1\cdot\frac{L}{2}$$

3-noded
stiffness matrix

$$k_{2.ex}(E, A_o, L) := \text{vector}(x_2, L)\cdot(\text{vector}(x_2, L))^T\cdot E\cdot A_o\cdot\left(1 - \frac{x_2}{2\cdot L}\right)\cdot w_2\cdot\frac{L}{2}$$

$$k_{3.ex}(E, A_o, L) := \text{vector}(x_3, L)\cdot((\text{vector}(x_3, L)))^T\cdot E\cdot A_o\cdot\left(1 - \frac{x_3}{2\cdot L}\right)\cdot w_3\cdot\frac{L}{2}$$

$$K_{ex}(E, A_o, L) := k_{1.ex}(E, A_o, L) + k_{2.ex}(E, A_o, L) + k_{3.ex}(E, A_o, L)$$

$$K_{ex}(E, A_o, L)\ \text{float}, 9 \rightarrow \begin{pmatrix} \frac{2.08333334}{L}\cdot E\cdot A_o & \frac{-2.33333334}{L}\cdot E\cdot A_o & \frac{.250000004}{L}\cdot E\cdot A_o \\ \frac{-2.33333334}{L}\cdot E\cdot A_o & \frac{4.00000000}{L}\cdot E\cdot A_o & \frac{-1.66666667}{L}\cdot E\cdot A_o \\ \frac{.250000004}{L}\cdot E\cdot A_o & \frac{-1.66666667}{L}\cdot E\cdot A_o & \frac{1.41666666}{L}\cdot E\cdot A_o \end{pmatrix}$$

Condense Middle Displacement DOF

$$P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad K_{ex2}(E, A_o, L) := \begin{pmatrix} \frac{2.08333334}{L} \cdot E \cdot A_o & \frac{-2.33333334}{L} \cdot E \cdot A_o & \frac{.250000002}{L} \cdot E \cdot A_o \\ \frac{-2.33333334}{L} \cdot E \cdot A_o & \frac{4.00000001}{L} \cdot E \cdot A_o & \frac{-1.66666667}{L} \cdot E \cdot A_o \\ \frac{.250000002}{L} \cdot E \cdot A_o & \frac{-1.66666667}{L} \cdot E \cdot A_o & \frac{1.41666666}{L} \cdot E \cdot A_o \end{pmatrix}$$

$$K_{permuted2}(E, A_o, L) := P^T \cdot K_{ex2}(E, A_o, L) \cdot P \rightarrow \begin{pmatrix} \frac{2.08333334}{L} \cdot E \cdot A_o & \frac{.250000002}{L} \cdot E \cdot A_o & \frac{-2.33333334}{L} \cdot E \cdot A_o \\ \frac{.250000002}{L} \cdot E \cdot A_o & \frac{1.41666666}{L} \cdot E \cdot A_o & \frac{-1.66666667}{L} \cdot E \cdot A_o \\ \frac{-2.33333334}{L} \cdot E \cdot A_o & \frac{-1.66666667}{L} \cdot E \cdot A_o & \frac{4.00000001}{L} \cdot E \cdot A_o \end{pmatrix}$$

$K_{ii}(E, A_o, L) := \text{submatrix}(K_{permuted2}(E, A_o, L), 2, 2, 2, 2)$ $\qquad K_{bb}(E, A_o, L) := \text{submatrix}(K_{permuted2}(E, A_o, L), 0, 1, 0, 1)$

$K_{bi}(E, A_o, L) := \text{submatrix}(K_{permuted2}(E, A_o, L), 0, 1, 2, 2)$ $\qquad K_{ib}(E, A_o, L) := \text{submatrix}(K_{permuted2}(E, A_o, L), 0, 1, 2, 2)^T$

$$K_{condensed}(E, A_o, L) := K_{bb}(E, A_o, L) - K_{bi}(E, A_o, L) \cdot K_{ii}(E, A_o, L)^{-1} \cdot K_{ib}(E, A_o, L)$$

$$K_{bi}(E, A_o, L) \rightarrow \begin{pmatrix} \frac{-2.33333334}{L} \cdot E \cdot A_o \\ \frac{-1.66666667}{L} \cdot E \cdot A_o \end{pmatrix}$$

$$K_{bb}(E, A_o, L) \rightarrow \begin{pmatrix} \frac{2.08333334}{L} \cdot E \cdot A_o & \frac{.250000002}{L} \cdot E \cdot A_o \\ \frac{.250000002}{L} \cdot E \cdot A_o & \frac{1.41666666}{L} \cdot E \cdot A_o \end{pmatrix}$$

$$K_{ib}(E, A_o, L) \rightarrow \begin{pmatrix} \frac{-2.33333334}{L} \cdot E \cdot A_o & \frac{-1.66666667}{L} \cdot E \cdot A_o \end{pmatrix}$$

$$K_{ii.inverse}(E, A_o, L) := \mathrm{submatrix}(K_{permuted}(E, A_o, L), 2, 2, 2, 2)^{-1} \rightarrow .40645175668225689605 \cdot \frac{L}{E \cdot A_o} \cdot (.99999999999999999999)$$

$$K_{ii.inverse}(E, A_o, L) := .24999999937500000156 \cdot \frac{L}{(E \cdot A_o)}$$

$$K_{temp}(E, A_o, L) := K_{bi}(E, A_o, L) \cdot K_{ii.inverse}(E, A_o, L) \cdot K_{ib}(E, A_o, L) \rightarrow \begin{pmatrix} \frac{1.3611111154861111113}{L} \cdot E \cdot A_o & \frac{.9722222245138888871}{L} \cdot E \cdot A_o \\ \frac{.9722222245138888871}{L} \cdot E \cdot A_o & \frac{.6944444454861111128}{L} \cdot E \cdot A_o \end{pmatrix}$$

$$K_{condensed}(E, A, L) := K_{bb}(E, A, L) - K_{temp}(E, A, L) \text{ float}, 5 \rightarrow \begin{pmatrix} \frac{.72222}{L} \cdot E \cdot A & \frac{-.72222}{L} \cdot E \cdot A \\ \frac{-.72222}{L} \cdot E \cdot A & \frac{.72222}{L} \cdot E \cdot A \end{pmatrix}$$

vs. 0.7213

% difference = .1% from exact solution!! woo!

Condensation technique is beneficial!!

Find good value for A, then get rid of (annoying) center nodes

SUBSTRUCTURE ANALYSIS

Two-noded / three-noded trusses

$$\underset{\sim}{k} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

all the same number – "best" equivalent area for varying cross-section

$$A = A_o(1 - x/2L)$$

$$\underset{\sim}{k} = \frac{E\alpha A_o}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

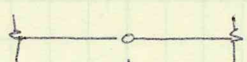
$\alpha \equiv$ unknown constant unique to cross-section properties

$\alpha = 3/4$ for area equation above

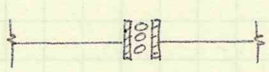
The key of using 2-noded elements is in guessing an "accurate" area. When many elements are similar, study one closely, then get rid of all middle nodes.

Fewer DOF, woo!

Releases



a hinge – releases moment at that point



shear release – halves can slide, but moments / rotation must still match

Typically, the "released" stress resultant is equal to zero

moment at a hinge, e.g.

– utilize this information for element stiffness matrix

$$\underset{\sim}{k}\,\underset{\sim}{q} = \underset{\sim}{Q}$$

element stiffness matrix

element displacement quantities

element force quantities

RELEASE MECHANISMS

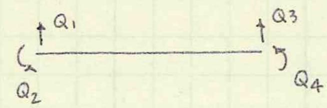
Calculations in / from releases

$$\underset{\sim}{k}\,\underset{\sim}{q} = \underset{\sim}{Q}$$

$$\left[\begin{array}{c|c} k_{aa} & k_{ab} \\ \hline k_{ba} & k_{bb} \end{array}\right]\left[\begin{array}{c} q_a \\ \hline q_b \end{array}\right] = \left[\begin{array}{c} Q_a \\ \hline Q_b \end{array}\right]$$

let's assume Q_b corresponds to release force quantities

$$\underset{\sim}{Q_b} = \underset{\sim}{0}$$



could release two, but must remain stable (no: Q_1 & Q_3)

(1) $k_{aa} q_a + k_{ab} q_b = Q_a$

(2) $k_{ba} q_a + k_{bb} q_b = Q_b = 0$

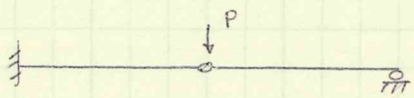
(2) $\longrightarrow$ $q_b = -k_{bb}^{-1} k_{ba} q_a$

sub into (1): $\underbrace{(k_{aa} - k_{ab} k_{bb}^{-1} k_{ba})}_{\text{modified stiffness matrix}} q_a = Q_a$

Disadvantages:

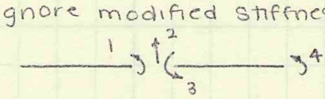
- only tracks non-released DOF terms directly
  i.e. if rotation 2 is released (or, moment), rotation can't be calculated in first step (only by returning to (1)).

Example



E, I, L, 2-noded elements

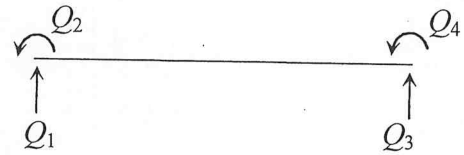
case 1 – ignore modified stiffness (classical approach)



Development of Element Stiffness Matrices to Account for Releases

General Case - Ignoring Axial Response

$$k(EI,L) := \begin{pmatrix} \frac{12\cdot EI}{L^3} & \frac{6\cdot EI}{L^2} & \frac{-12\cdot EI}{L^3} & \frac{6\cdot EI}{L^2} \\ \frac{6\cdot EI}{L^2} & \frac{4\cdot EI}{L} & \frac{-6\cdot EI}{L^2} & \frac{2\cdot EI}{L} \\ \frac{-12\cdot EI}{L^3} & \frac{-6\cdot EI}{L^2} & \frac{12\cdot EI}{L^3} & \frac{-6\cdot EI}{L^2} \\ \frac{6\cdot EI}{L^2} & \frac{2\cdot EI}{L} & \frac{-6\cdot EI}{L^2} & \frac{4\cdot EI}{L} \end{pmatrix}$$



Assumptions
- prismatic
- plane sections, etc.

Case 1 - Assume Left End is hinged -> Q2 = 0

Step 1: Permute rows and columns to isolate force quantity that is = 0

$$P := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

switch row 2 to row 1 to group known, unknown values

$$k_{new}(EI,L) := P^T\cdot k(EI,L)\cdot P \rightarrow \begin{array}{c} K_{bb} \qquad\qquad K_{ba} \\ \left(\begin{array}{c|ccc} 4\cdot\frac{EI}{L} & 6\cdot\frac{EI}{L^2} & -6\cdot\frac{EI}{L^2} & 2\cdot\frac{EI}{L} \\ \hline 6\cdot\frac{EI}{L^2} & 12\cdot\frac{EI}{L^3} & -12\cdot\frac{EI}{L^3} & 6\cdot\frac{EI}{L^2} \\ -6\cdot\frac{EI}{L^2} & -12\cdot\frac{EI}{L^3} & 12\cdot\frac{EI}{L^3} & -6\cdot\frac{EI}{L^2} \\ 2\cdot\frac{EI}{L} & 6\cdot\frac{EI}{L^2} & -6\cdot\frac{EI}{L^2} & 4\cdot\frac{EI}{L} \end{array}\right) \\ K_{ab} \qquad\qquad K_{aa} \end{array}$$

$$Q = \begin{bmatrix} Q_1 \\ 0 \\ Q_3 \\ Q_4 \end{bmatrix}$$

Step 2: Compute modified stiffness matrix with formula developed in class

$$k_{mod}(EI,L) := \text{submatrix}\left(k_{new}(EI,L),1,3,1,3\right) - \begin{pmatrix} 6\cdot\frac{EI}{L^2} \\ -6\cdot\frac{EI}{L^2} \\ 2\cdot\frac{EI}{L} \end{pmatrix} \cdot \frac{L}{4\cdot EI} \cdot \begin{pmatrix} 6\cdot\frac{EI}{L^2} & -6\cdot\frac{EI}{L^2} & 2\cdot\frac{EI}{L} \end{pmatrix} \rightarrow \begin{pmatrix} 3\cdot\frac{EI}{L^3} & -3\cdot\frac{EI}{L^3} & 3\cdot\frac{EI}{L^2} \\ -3\cdot\frac{EI}{L^3} & 3\cdot\frac{EI}{L^3} & -3\cdot\frac{EI}{L^2} \\ 3\cdot\frac{EI}{L^2} & -3\cdot\frac{EI}{L^2} & 3\cdot\frac{EI}{L} \end{pmatrix}$$

Step 3: Expand back to a 4x4 matrix

$$k_{mod1}(EI,L) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3\cdot\frac{EI}{L^3} & -3\cdot\frac{EI}{L^3} & 3\cdot\frac{EI}{L^2} \\ 0 & -3\cdot\frac{EI}{L^3} & 3\cdot\frac{EI}{L^3} & -3\cdot\frac{EI}{L^2} \\ 0 & 3\cdot\frac{EI}{L^2} & -3\cdot\frac{EI}{L^2} & 3\cdot\frac{EI}{L} \end{pmatrix}$$

– needed to mesh with all other elements in structure

Step 4: Re-permute rows/columns to correspond to original DOF order

$$k_{near.hinge}(EI,L) := P^T \cdot k_{mod1}(EI,L) \cdot P \rightarrow \begin{pmatrix} 3\cdot\frac{EI}{L^3} & 0 & -3\cdot\frac{EI}{L^3} & 3\cdot\frac{EI}{L^2} \\ 0 & 0 & 0 & 0 \\ -3\cdot\frac{EI}{L^3} & 0 & 3\cdot\frac{EI}{L^3} & -3\cdot\frac{EI}{L^2} \\ 3\cdot\frac{EI}{L^2} & 0 & -3\cdot\frac{EI}{L^2} & 3\cdot\frac{EI}{L} \end{pmatrix}$$

switched back to 1-2 order

DOF 2 was released; row/column 2 are all zeroes.

now can be mapped to global system

Case 2: far end is hinged -> Q4 = 0

If the far end is hinged, we simply need to modify ***P*** *(the permutation matrix) because released moment quantity has already been computed*

only works because Q2 and Q4 are both moments — NOT forces!

$$P_{far} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

switches R/C 2 and 4

$$k_{far.hinge}(EI,L) := P_{far}{}^T \cdot k_{near.hinge}(EI,L) \cdot P_{far} \rightarrow \begin{pmatrix} 3\cdot\frac{EI}{L^3} & 3\cdot\frac{EI}{L^2} & -3\cdot\frac{EI}{L^3} & 0 \\ 3\cdot\frac{EI}{L^2} & 3\cdot\frac{EI}{L} & -3\cdot\frac{EI}{L^2} & 0 \\ -3\cdot\frac{EI}{L^3} & -3\cdot\frac{EI}{L^2} & 3\cdot\frac{EI}{L^3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

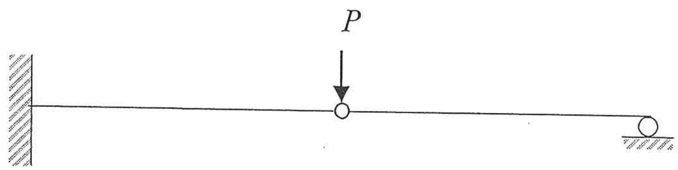
released

45c

07March07

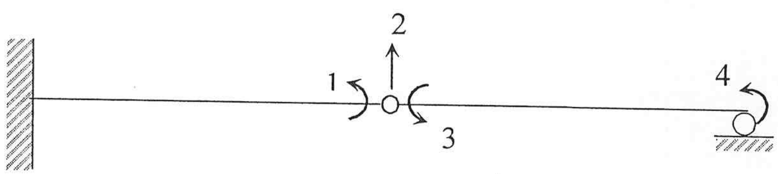
45d

Example Problem



Both members have length L, bending modulus EI, and are modeled as 2-noded beams.

Case 1 - Add an additional DOF to account for the hinge (assemble element stiffness matrices using "standard" formulation.)



$$K(EI,L) := \begin{pmatrix} \frac{4\cdot EI}{L} & \frac{-6\cdot EI}{L^2} & 0 & 0 \\ \frac{-6\cdot EI}{L^2} & \frac{24\cdot EI}{L^3} & \frac{6\cdot EI}{L^2} & \frac{6\cdot EI}{L^2} \\ 0 & \frac{6\cdot EI}{L^2} & \frac{4\cdot EI}{L} & \frac{2\cdot EI}{L} \\ 0 & \frac{6\cdot EI}{L^2} & \frac{2\cdot EI}{L} & \frac{4\cdot EI}{L} \end{pmatrix} \qquad R(P) := \begin{pmatrix} 0 \\ -P \\ 0 \\ 0 \end{pmatrix}$$

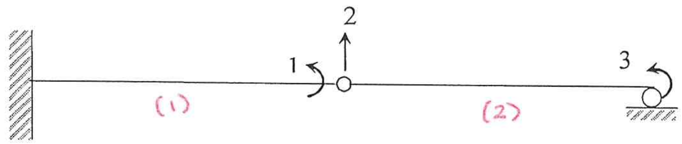
$$r(EI,L,P) := K(EI,L)^{-1}\cdot R(P) \text{ simplify } \rightarrow \begin{pmatrix} \frac{-1}{2\cdot EI}\cdot L^2\cdot P \\ \frac{-1}{3\cdot EI}\cdot L^3\cdot P \\ \frac{1}{3\cdot EI}\cdot L^2\cdot P \\ \frac{1}{3\cdot EI}\cdot L^2\cdot P \end{pmatrix}$$

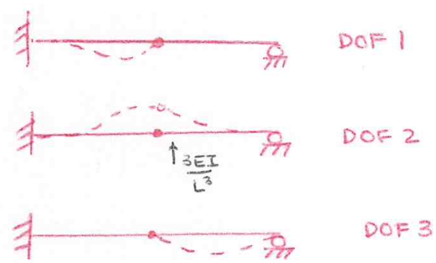
solution without modifications for hinge

Case 2 - Use a modified stiffness matrix and "release" one DOF corresponding to Q2 for the beam on the right (i.e., assemble the modified stiffness matrix for the right-hand beam with the "standard" stiffness matrix for the left-hand beam).

(2) (2) (1)

comparable to original element – K on pg 3





$$K_{release}(EI,L) := \begin{pmatrix} \frac{4\cdot EI}{L} & \frac{-6\cdot EI}{L^2} & 0 \\ \frac{-6\cdot EI}{L^2} & \frac{15\cdot EI}{L^3} & \frac{3\cdot EI}{L^2} \\ 0 & \frac{3\cdot EI}{L^2} & \frac{3\cdot EI}{L} \end{pmatrix} \qquad R_{release}(P) := \begin{pmatrix} 0 \\ -P \\ 0 \end{pmatrix}$$

$K^{14}_{(2)}$ $K^{33}_{(2)}$ $K^{41}_{(2)}$

$K^{33}_{(1)} + K^{11}_{(2)}$ individual stiffness matrix values

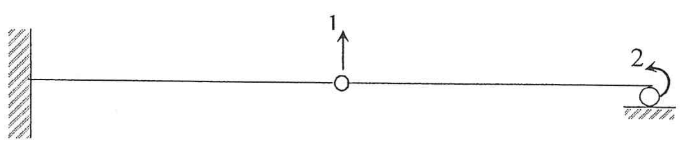
$$r_{release}(EI,L,P) := K_{release}(EI,L)^{-1}\cdot R_{release}(P) \rightarrow \begin{array}{c} \begin{pmatrix} \frac{-1}{2\cdot EI}\cdot L^2\cdot P \\ \frac{-1}{3\cdot EI}\cdot L^3\cdot P \\ \frac{1}{3\cdot EI}\cdot L^2\cdot P \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 4 \end{array}$$

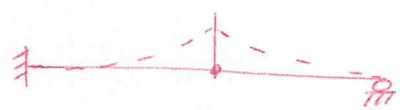
same values

Note: This approach results in the same solution as Case 1 (i.e., the displacement quantities at the retained DOF are the same for both cases). What we lose with this approach is the ability to know the rotation at the left end of the beam on the right. Be sure you know how to find this quantity using the modified stiffness approach.

$q_B = -K_{BB}^{-1} K_{BA} q_A$

Case 3 - Use a modified stiffness matrix and release two DOF -- one DOF corresponding to Q4 for the beam on the left and one DOF corresponding to Q2 for the beam on the right (i.e., assemble the structural stiffness matrix K using modified element stiffness matrices for both beams).





$K_{(1)}^{33} + K_{(2)}^{"}$ – matrices from pg 45c

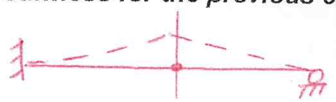
$$K_{release2}(EI,L) := \begin{pmatrix} \frac{6\cdot EI}{L^3} & \frac{3\cdot EI}{L^2} \\ \frac{3\cdot EI}{L^2} & \frac{3\cdot EI}{L} \end{pmatrix}$$

$K_{(2)}^{34}$ $K_{(2)}^{41}$ $K_{(2)}^{44}$

$$R_{release2}(P) := \begin{pmatrix} -P \\ 0 \end{pmatrix}$$

$$r_{release2}(EI,L,P) := K_{release2}(EI,L)^{-1}\cdot R_{release2}(P) \rightarrow \begin{pmatrix} \frac{-1}{3\cdot EI}\cdot L^3\cdot P \\ \frac{1}{3\cdot EI}\cdot L^2\cdot P \end{pmatrix} \begin{matrix} 2 \\ 4 \end{matrix}$$

Note: *This approach results in the same solution as Case 1 (i.e., the displacement quantities at the retained DOF are the same for both cases). Further note that it would be possible to carry this methodology further by releasing the rotation at both ends of the beam on the right. Doing so would result in the right-hand beam deforming as a rigid body when DOF 1 in Case 3 is displaced by a unit amount. Because a rigid body displacement causes no deformation, the right-hand beam does not contribute any force in displacing the hinge vertically by a unit amount. Thus, using the modified stiffness for the previous case for the beam on the left, we arrive at the conclusion that the displacement is equal to*



$$\frac{-1}{(3\cdot EI)}\cdot L^3\cdot P$$

which is the same value we calculated for all previous cases.

In general, when we apply this approach to solve for the response of structures with releases, our modified stiffness matrix and force vector must account for any fixed-end forces or moments that may be acting.

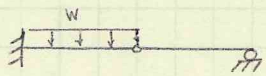
$\underset{\sim}{Q} = \underset{\sim}{k}\underset{\sim}{q} + \underset{\sim}{Q}_{FE}$

needed when forces are not directly at the node

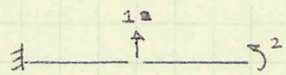
$\underset{\sim}{Q}' = \underset{\sim}{Q} - \underset{\sim}{Q}_{FE}$

RELEASED MOMENTS

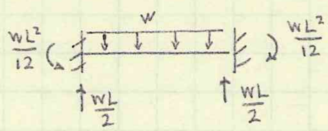
Example with fixed-end forces

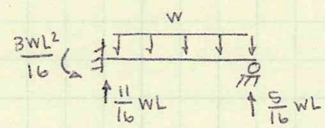


degrees of freedom:



fixed-end forces:





need to use these values to recognize that moment has been released

with all active DOFs,

$$R = \begin{bmatrix} \frac{wL^2}{12} \\ -\frac{wL}{2} \\ 0 \\ 0 \end{bmatrix}$$

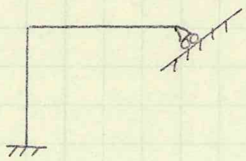
with 2 active DOFs,

$$R = \begin{bmatrix} -5/16\, wL \\ 0 \end{bmatrix}$$

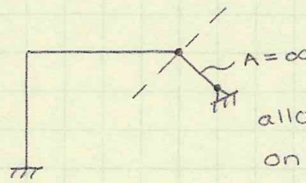
because stiffness matrix has changed, load vector must change as well to result in the same displacements at end

OBLIQUE SUPPORTS

constraints, etc.



model as:



allows beam end to rotate on that line

- length of strut should be equal to longest member in structure
- A should be very large

or, use constraints –

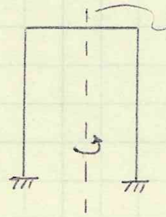
- relate unconstrained equations / DOFs to constrained DOFs; modify equations as necessary
- use gamma / constraint matrix
  (mpcs in ABAQUS)

SYMMETRY

For planar structures

symmetric structures:

- with respect to a line (or plane)

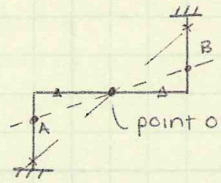


plane or line of symmetry

think of a 180° rotation around line of symmetry – is it the same?

- columns must be the same (I, A, E...)
- boundaries must be the same

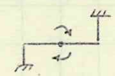
- with respect to a point



point of symmetry

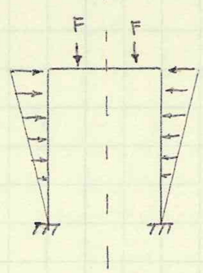
- points A and B must be the same (for any A, B such that a ~~xxxxx~~ line fits)

think of a planar 180° rotation of structure around point



Loading conditions

- symmetric loading conditions



symmetric loading

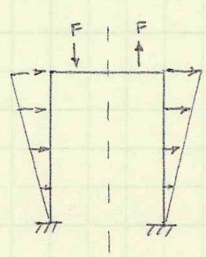
consider a structure with symmetry WRT a plane (as drawn)

symmetric loading means loads line up when structure is flipped

- will respond symmetrically

- antisymmetric loading

loads are in the same position, but in the opposite direction



antisymmetric loading

flip loads, then reverse direction

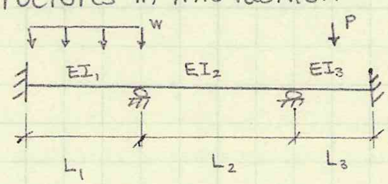
- will respond anti-symmetrically

SYMMETRY

Two main principles:

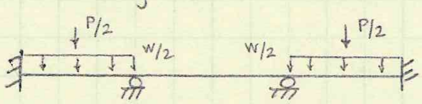
- symmetric structure with symmetric loading
  → symmetric response
- symmetric structure, anti-symmetric loading
  → anti-symmetric response

Modeling structures in this fashion



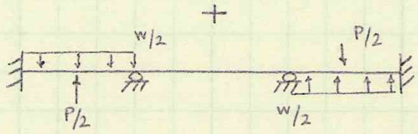
for symmetry, $EI_1 = EI_3$, $L_1 = L_3$

If the structure is linear and elastic, ~~can~~ we can modify loads to make them (anti)symmetric



symmetric

+

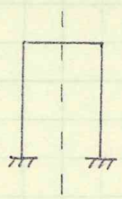


antisymmetric

= original loading

ONLY APPLIES FOR LINEAR ELASTIC PROBLEMS!

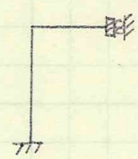
Boundary conditions to account for symmetry

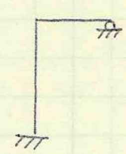


"symmetric" vs "anti-symmetric" response

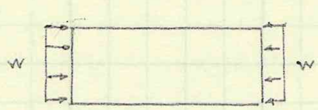
$\theta = 0$, $u = 0$, $V = 0$ at plane of symmetry

$V = 0$, $M = 0$, $P = 0$

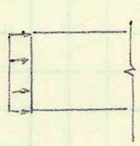


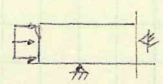


Example – closed cell structure



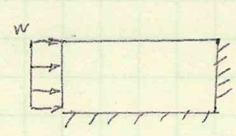
how to analyze?



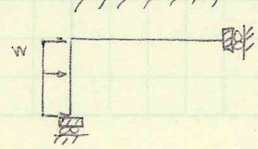


fix both sides

so,

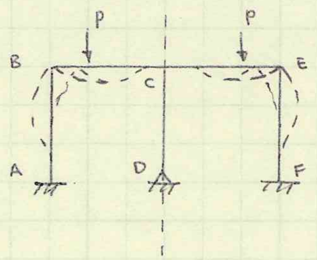


or,



SYMMETRY

Members on the plane of symmetry



how are boundaries on line of symmetry handled?

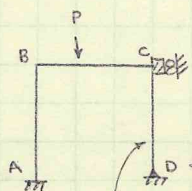
symmetric loading case:

central point can move ↕

cannot move ⟷

has zero slope

model half of structure



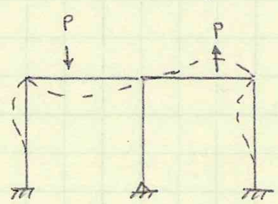
no rotation, no horizontal movement

D ← becomes a fixed support — member CD cannot rotate

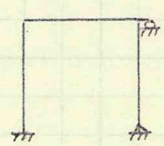
geometric properties are split — $\frac{EA}{2}$, $\frac{EI}{2}$ — as only half of column is in this half

however — results / reactions / etc. calculated are half the total value for CD

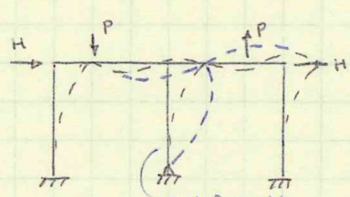
antisymmetric loading



boundaries:

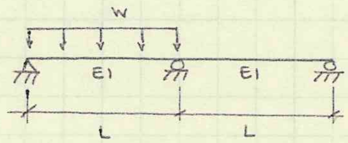


or



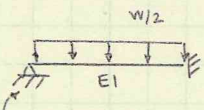
SYMMETRY

Example problem



loads can be represented in two models, one symmetric, one anti-.

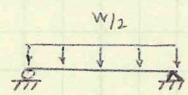
symmetric half:



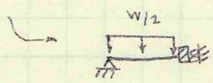
1 DOF

can be considered as a roller

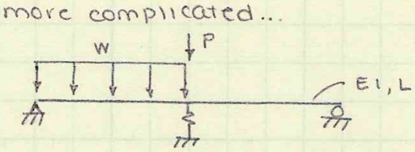
Anti-symmetric half:



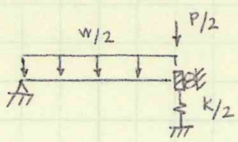
2 DOF (but, it's symmetric, so, 1 DOF)



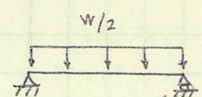
A bit more complicated...



symmetric:



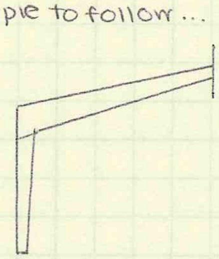
anti-symmetric



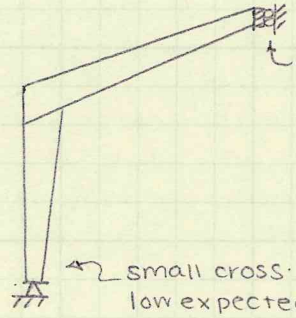
spring can't deflect, as point doesn't move up or down at all

no P/2 on this one – already symmetric, accounted for in that case

Big example to follow...

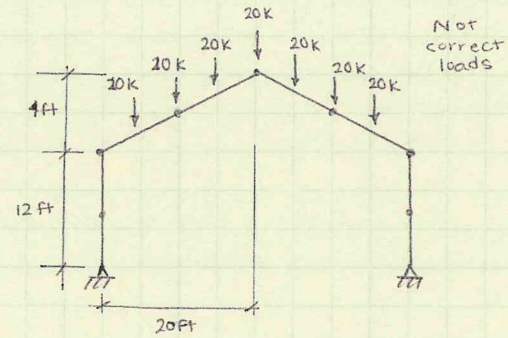


BIG EXAMPLE PROBLEM



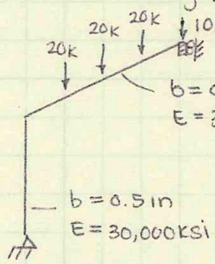
removes 2 more DOF
24 to 13 to 11

small cross-section,
low expected moments –
very pin-like



– 3-noded elements
(23 DOFs / frame)

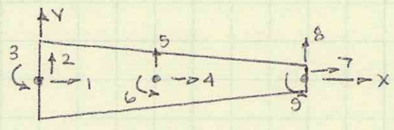
– symmetric : 13 DOFs



b = 0.5 in $\quad d_o = 30$ in
E = 30,000 ksi $\quad \alpha = 0.25$

b = 0.5 in $\quad \alpha = 0.5$
E = 30,000 ksi $\quad d_o = 20$ in

General case



$d(x) = d_o(1 - \alpha\, x/L)$

depends on particular geometries

$$\alpha = \frac{d_o - d_F}{d_o}$$

assume width is constant

1) Develop shape functions $\underset{\sim}{H}$ for axial and flexural responses
$\to \underset{\sim}{L}$

gather terms, as we did in HW

$\underset{\sim}{H}$ uses six DOFs
$\underset{\sim}{L}$ uses remaining 3

2) calculate stiffness terms

$$K_A = \int_0^L (dL/dx)^T (dL/dx)\, EA(x)\, dx \longrightarrow \text{DOFs } 1, 4, 7$$

$$K_F = \int_0^L (d^2H/dx^2)^T (d^2H/dx^2)\, EI(x)\, dx \longrightarrow \text{DOFs } 2,3,5,6,8,9$$

combine into one giant K; use zeros between terms

3) consider global coordinates

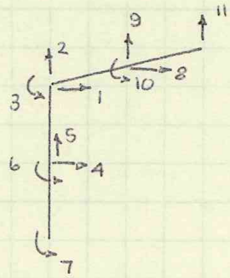
– transformation matrix T

remember: moment transformation value = 1 on diagonal

$T^T K_{local} \cdot T = K_{global}$

BIG EXAMPLE PROBLEM

Global stiffness matrix



rotations:

column $\theta = 270°$

actually ~268°, to match vertical edge of column

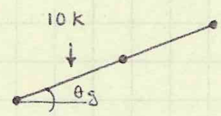
beam $\theta = \tan^{-1}(4/20)$

4) Assemble global stiffness matrix

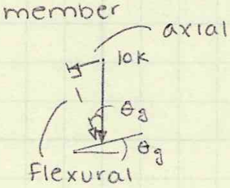
use mathcad, excel

BIG EXAMPLE PROBLEM

Load vector formulation



- load is not orthogonal or perpendicular to the member



$P_F = (10\text{ k})\cos\theta_g$

$P_A = (10\text{ k})\sin\theta_g$

see handout

- apply to nodes:

$\underset{\sim}{H}^T \big|_{0.25L} \cdot (-P_F)$

six Hermetian, three zeros

$\underset{\sim}{L}^T \big|_{0.25L} \cdot (-P_A)$

three axial, six zeros

Derive the General Equation for a 3-noded frame member with a linear taper assuming linear response (flexural and axial modes are uncoupled)

Consider the flexural response first

3-Noded Beam

Find shape functions for 3-noded beam (6 DOF)

$$A_3(L) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & \frac{L}{2} & \frac{L^2}{4} & \frac{L^3}{8} & \frac{L^4}{16} & \frac{L^5}{32} \\ 0 & 1 & L & \frac{3 \cdot L^2}{4} & \frac{L^3}{2} & \frac{5 \cdot L^4}{16} \\ 1 & L & L^2 & L^3 & L^4 & L^5 \\ 0 & 1 & 2 \cdot L & 3 \cdot L^2 & 4 \cdot L^3 & 5 \cdot L^4 \end{pmatrix} \qquad b_3(v_1,\theta_1,v_2,\theta_2,v_3,\theta_3) := \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{pmatrix}$$

← flexural

$$A_3(L)^{-1} \cdot b_3(v_1,\theta_1,v_2,\theta_2,v_3,\theta_3) \rightarrow \begin{pmatrix} v_1 \\ \theta_1 \\ \frac{-23}{L^2} \cdot v_1 - \frac{6}{L} \cdot \theta_1 + \frac{16}{L^2} \cdot v_2 - \frac{8}{L} \cdot \theta_2 + \frac{7}{L^2} \cdot v_3 - \frac{1}{L} \cdot \theta_3 \\ \frac{66}{L^3} \cdot v_1 + \frac{13}{L^2} \cdot \theta_1 - \frac{32}{L^3} \cdot v_2 + \frac{32}{L^2} \cdot \theta_2 - \frac{34}{L^3} \cdot v_3 + \frac{5}{L^2} \cdot \theta_3 \\ \frac{-68}{L^4} \cdot v_1 - \frac{12}{L^3} \cdot \theta_1 + \frac{16}{L^4} \cdot v_2 - \frac{40}{L^3} \cdot \theta_2 + \frac{52}{L^4} \cdot v_3 - \frac{8}{L^3} \cdot \theta_3 \\ \frac{24}{L^5} \cdot v_1 + \frac{4}{L^4} \cdot \theta_1 + \frac{16}{L^4} \cdot \theta_2 - \frac{24}{L^5} \cdot v_3 + \frac{4}{L^4} \cdot \theta_3 \end{pmatrix}$$

21 March 07

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Flexural shape functions:

$$H_1(x,L) := 1 - 23\cdot\left(\frac{x}{L}\right)^2 + 66\left(\frac{x}{L}\right)^3 - 68\left(\frac{x}{L}\right)^4 + 24\left(\frac{x}{L}\right)^5$$

$$H_3(x,L) := 16\cdot\left(\frac{x}{L}\right)^2 - 32\left(\frac{x}{L}\right)^3 + 16\left(\frac{x}{L}\right)^4$$

$$H_5(x,L) := 7\cdot\left(\frac{x}{L}\right)^2 - 34\left(\frac{x}{L}\right)^3 + 52\left(\frac{x}{L}\right)^4 - 24\left(\frac{x}{L}\right)^5$$

$$H_2(x,L) := x - 6\cdot\frac{x^2}{L} + 13\cdot\frac{x^3}{L^2} - 12\cdot\frac{x^4}{L^3} + 4\cdot\frac{x^5}{L^4}$$

$$H_4(x,L) := -8\cdot\frac{x^2}{L} + 32\cdot\frac{x^3}{L^2} - 40\cdot\frac{x^4}{L^3} + 16\cdot\frac{x^5}{L^4}$$

$$H_6(x,L) := -1\cdot\frac{x^2}{L} + 5\cdot\frac{x^3}{L^2} - 8\cdot\frac{x^4}{L^3} + 4\cdot\frac{x^5}{L^4}$$

Now consider the axial response

$x_1(L) := 0 \quad x_2(L) := \frac{L}{2} \quad x_3(L) := L$

$$L_1(x,L) := \frac{(x_2(L) - x)\cdot(x_3(L) - x)}{(x_2(L) - x_1(L))\cdot(x_3(L) - x_1(L))}$$

$$L_1(x,L) \,\Big|\, \begin{array}{l}\text{simplify}\\ \text{expand}, 3\end{array} \rightarrow 1 - \frac{3}{L}\cdot x + \frac{2}{L^2}\cdot x^2$$

$$L_2(x,L) := \frac{(x_1(L) - x)\cdot(x_3(L) - x)}{(x_1(L) - x_2(L))\cdot(x_3(L) - x_2(L))} \qquad L_2(x,L) \left| \begin{array}{l} \text{simplify} \\ \text{expand}, 3 \end{array} \right. \rightarrow \frac{4}{L}\cdot x - \frac{4}{L^2}\cdot x^2$$

$$L_3(x,L) := \frac{(x_1(L) - x)\cdot(x_2(L) - x)}{(x_1(L) - x_3(L))\cdot(x_2(L) - x_3(L))} \qquad L_3(x,L) \left| \begin{array}{l} \text{simplify} \\ \text{expand}, 3 \end{array} \right. \rightarrow \frac{-1}{L}\cdot x + \frac{2}{L^2}\cdot x^2$$

axial shape functions

Numerically evaluate the individual terms in the 9x9 local element stiffness matrix

$$k_{11}(E,A_o,\alpha,L) := \int_0^L \left(\frac{d}{dx}L_1(x,L)\right)\cdot\left(\frac{d}{dx}L_1(x,L)\right)\cdot E\cdot A_o\cdot\left(1 - \alpha\cdot\frac{x}{L}\right) dx \text{ simplify } \rightarrow \frac{-1}{6\cdot L}\cdot(3\cdot\alpha - 14)\cdot E\cdot A_o$$

1st derivative for axial

$\frac{d^2H}{dx^2}$ (2nd) for flexure

$$k_{14}(E,A_o,\alpha,L) := \int_0^L \left(\frac{d}{dx}L_1(x,L)\right)\cdot\left(\frac{d}{dx}L_2(x,L)\right)\cdot E\cdot A_o\cdot\left(1 - \alpha\cdot\frac{x}{L}\right) dx \text{ simplify } \rightarrow \frac{2}{3\cdot L}\cdot(\alpha - 4)\cdot E\cdot A_o$$

General Stiffness Matrix in Local Coordinates

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} A & F1 & F2 & A & F1 & F2 & A & F1 & F2 \rightarrow \end{array} \\
k33_{local}(E, I_o, A_o, L, \alpha) := & \begin{pmatrix}
k_{11}(E,A_o,\alpha,L) & 0 & 0 & k_{14}(E,A_o,\alpha,L) & 0 & 0 & k_{17}(E,A_o,\alpha,L) & 0 & \\
0 & k_{22}(E,I_o,\alpha,L) & k_{23}(E,I_o,\alpha,L) & 0 & k_{25}(E,I_o,\alpha,L) & k_{26}(E,I_o,\alpha,L) & 0 & k_{28}(E,I_o,\alpha,L) & k_{29} \\
0 & k_{23}(E,I_o,\alpha,L) & k_{33}(E,I_o,\alpha,L) & 0 & k_{35}(E,I_o,\alpha,L) & k_{36}(E,I_o,\alpha,L) & 0 & k_{38}(E,I_o,\alpha,L) & k_{39} \\
k_{14}(E,A_o,\alpha,L) & 0 & 0 & k_{44}(E,A_o,\alpha,L) & 0 & 0 & k_{47}(E,A_o,\alpha,L) & 0 & \\
0 & k_{25}(E,I_o,\alpha,L) & k_{35}(E,I_o,\alpha,L) & 0 & k_{55}(E,I_o,\alpha,L) & k_{56}(E,I_o,\alpha,L) & 0 & k_{58}(E,I_o,\alpha,L) & k_{59} \\
0 & k_{26}(E,I_o,\alpha,L) & k_{36}(E,I_o,\alpha,L) & 0 & k_{56}(E,I_o,\alpha,L) & k_{66}(E,I_o,\alpha,L) & 0 & k_{68}(E,I_o,\alpha,L) & k_{69} \\
k_{17}(E,A_o,\alpha,L) & 0 & 0 & k_{47}(E,A_o,\alpha,L) & 0 & 0 & k_{77}(E,A_o,\alpha,L) & 0 & \\
0 & k_{28}(E,I_o,\alpha,L) & k_{38}(E,I_o,\alpha,L) & 0 & k_{58}(E,I_o,\alpha,L) & k_{68}(E,I_o,\alpha,L) & 0 & k_{88}(E,I_o,\alpha,L) & k_{89} \\
0 & k_{29}(E,I_o,\alpha,L) & k_{39}(E,I_o,\alpha,L) & 0 & k_{59}(E,I_o,\alpha,L) & k_{69}(E,I_o,\alpha,L) & 0 & k_{89}(E,I_o,\alpha,L) & k_{99}
\end{pmatrix}
\end{array}
$$

Transformation from Local to Global

$$
T(\theta) := \begin{pmatrix}
\cos(\theta) & \sin(\theta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\sin(\theta) & \cos(\theta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos(\theta) & \sin(\theta) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\sin(\theta) & \cos(\theta) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cos(\theta) & \sin(\theta) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

Compute column stiffness matrix in global coordinates

$$\theta_c := \frac{3\cdot\pi}{2}$$

$$T(\theta_c)^T \cdot k33_{local}(\overset{E}{30000}, \overset{I_o}{333.333}, \overset{A_o}{10}, \overset{L}{144}, \overset{\alpha}{0.5}) \cdot T(\theta_c) \text{ float}, 5 \rightarrow \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & - & - & 7 \end{array} \\ \begin{pmatrix} 382.13 & 0 & 13618. & -302.94 & 0 & 18027. & -79.191 & 0 & 1570.2 \\ 0 & 4340.3 & 0 & 0 & -4861.1 & 0 & 0 & 520.83 & 0 \\ 13618. & 0 & 5.9011\cdot10^5 & -11390. & 0 & 5.0661\cdot10^5 & -2227.5 & 0 & 44147. \\ -302.94 & 0 & -11390. & 339.87 & 0 & -9185.8 & -36.934 & 0 & 1423.8 \\ 0 & -4861.1 & 0 & 0 & 8333.3 & 0 & 0 & -3472.2 & 0 \\ 18027. & 0 & 5.0661\cdot10^5 & -9185.8 & 0 & 1.2540\cdot10^6 & -8841.3 & 0 & 1.7394\cdot10^5 \\ -79.191 & 0 & -2227.5 & -36.934 & 0 & -8841.3 & 116.13 & 0 & -2994.0 \\ 0 & 520.83 & 0 & 0 & -3472.2 & 0 & 0 & 2951.4 & 0 \\ 1570.2 & 0 & 44147. & 1423.8 & 0 & 1.7394\cdot10^5 & -2994.0 & 0 & 1.1053\cdot10^5 \end{pmatrix} \end{array}$$

Compute Girder Stiffness Matrix in Global Coordinates

$$\theta_g := \operatorname{atan}\left(\frac{4}{20}\right)$$

$$T(\theta_g)^T \cdot k33_{local}(30000, 1125, 15, 244.752937, 0.25) \cdot T(\theta_g) \text{ float}, 5 \rightarrow \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 8 & 9 & 10 & - & 11 & - \end{array} \\ \begin{pmatrix} 3915.4 & 724.30 & -3327.4 & -4428.3 & -840.91 & -4912.3 & 512.93 & 116.61 & -495.32 \\ 724.30 & 438.71 & 16637. & -840.91 & -391.91 & 24561. & 116.61 & -46.793 & 2476.6 \\ -3327.4 & 16637. & 1.2344\cdot10^6 & 2733.4 & -13667. & 1.1051\cdot10^6 & 594.03 & -2970.3 & 1.0744\cdot10^5 \\ -4428.3 & -840.91 & 2733.4 & 8262.6 & 1587.3 & 1399.6 & -3834.3 & -746.42 & -1216.7 \\ -840.91 & -391.91 & -13667. & 1587.3 & 643.42 & -6997.9 & -746.42 & -251.51 & 6083.6 \\ -4912.3 & 24561. & 1.1051\cdot10^6 & 1399.6 & -6997.9 & 3.4841\cdot10^6 & 3512.7 & -17564. & 6.6797\cdot10^5 \\ 512.93 & 116.61 & 594.03 & -3834.3 & -746.42 & 3512.7 & 3321.4 & 629.81 & 1712.0 \\ 116.61 & -46.793 & -2970.3 & -746.42 & -251.51 & -17564. & 629.81 & 298.30 & -8560.2 \\ -495.32 & 2476.6 & 1.0744\cdot10^5 & -1216.7 & 6083.6 & 6.6797\cdot10^5 & 1712.0 & -8560.2 & 6.0199\cdot10^5 \end{pmatrix} \end{array}$$

MathCAD matrices limited to 10x10. Boo.

Use Excel

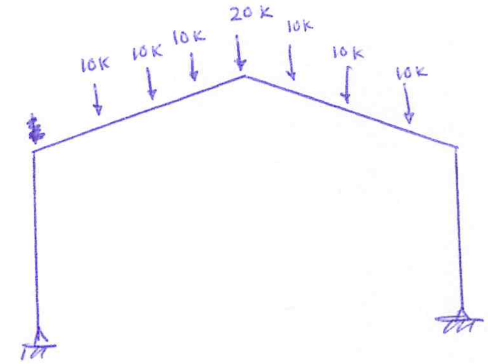
Consider equal loads acting at the quarter points for the 3-noded beam

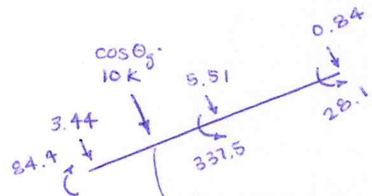
$$L_g := 244.752937$$

$$Q_{eq.1}(P,\theta,x,L) := \begin{pmatrix} 0 \\ H_1(x,L) \\ H_2(x,L) \\ 0 \\ H_3(x,L) \\ H_4(x,L) \\ 0 \\ H_5(x,L) \\ H_6(x,L) \end{pmatrix} \cdot -P \cdot \cos(\theta)$$

Flexural equivalent

$$Q_{eq.1}(10,\theta_g,0.25 \cdot L_g,L_g) \text{ float},6 \rightarrow \begin{pmatrix} 0 \\ -3.44736 \\ -84.3750 \\ 0 \\ -5.51576 \\ 337.500 \\ 0 \\ -.842684 \\ 28.1250 \end{pmatrix}$$





equivalent to all the others

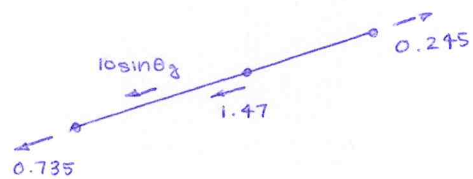
do not switch signs!

$$Q_{eq.2}(P,\theta,x,L) := \begin{pmatrix} L_1(x,L) \\ 0 \\ 0 \\ L_2(x,L) \\ 0 \\ 0 \\ L_3(x,L) \\ 0 \\ 0 \end{pmatrix} \cdot -P \cdot \sin(\theta)$$

Axial

$$Q_{eq.2}(10,\theta_g,0.25 \cdot L_g,L_g) \text{ float},6 \rightarrow \begin{pmatrix} -.735437 \\ 0 \\ 0 \\ -1.47087 \\ 0 \\ 0 \\ .245145 \\ 0 \\ 0 \end{pmatrix}$$

equivalent loads for quarter point



$$Q_{eq.1}(10,\theta_g,0.5\cdot L_g,L_g)\ \text{float},6\ \rightarrow \begin{pmatrix} 0\\0\\0\\0\\-9.80582\\0\\0\\0\\0 \end{pmatrix}$$

$= 10\cos\theta_g$

half·point

(as expected)

$$Q_{eq.2}(10,\theta_g,0.5\cdot L_g,L_g)\ \text{float},6\ \rightarrow \begin{pmatrix} 0\\0\\0\\-1.96116\\0\\0\\0\\0\\0 \end{pmatrix}$$

$= 10\sin\theta_g$

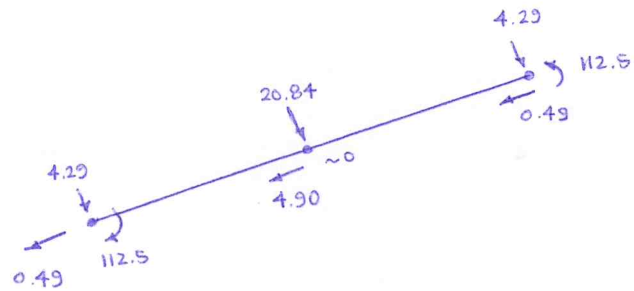
$$Q_{eq.1}(10,\theta_g,0.75\cdot L_g,L_g)\ \text{float},6\ \rightarrow \begin{pmatrix} 0\\-.842684\\-28.1250\\0\\-5.51576\\-337.500\\0\\-3.44736\\84.3750 \end{pmatrix}$$

three·quarter point

$$Q_{eq.2}(10,\theta_g,0.75\cdot L_g,L_g)\ \text{float},6\ \rightarrow \begin{pmatrix} .245145\\0\\0\\-1.47087\\0\\0\\-.735437\\0\\0 \end{pmatrix}$$

$$Q_{structure\_load\_vector} := Q_{eq.1}(10,\theta_g,0.75\cdot L_g,L_g) + Q_{eq.2}(10,\theta_g,0.75\cdot L_g,L_g) + Q_{eq.1}(10,\theta_g,0.5\cdot L_g,L_g) + Q_{eq.2}(10,\theta_g,0.5\cdot L_g,L_g) + Q_{eq.1}(10,\theta_g,0.25\cdot L_g,L_g) + Q_{e}$$

$$Q_{structure\_load\_vector} = \begin{pmatrix} -0.49029 \\ -4.29004 \\ -112.5 \\ -4.902903 \\ -20.837339 \\ 6.252776 \times 10^{-12} \\ -0.49029 \\ -4.29004 \\ 112.5 \end{pmatrix}$$



does not include load at ~~ ~~ top point — either add in parts (F and A) in local $\underset{\sim}{Q}$, or add in total (along DOF) in global $\underset{\sim}{R}$

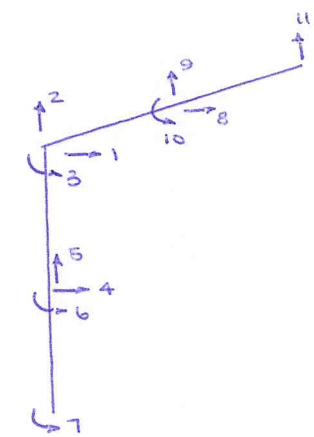
These values are oriented in member local coordinates and must be transformed to global coordinates

local loads from single girder

$$R_{structure\_load\_vector} := T(\theta_g)^T \cdot Q_{structure\_load\_vector}$$

No $\cdot$ T only columns / only rows

$$R_{structure\_load\_vector} = \begin{array}{c} \begin{pmatrix} 0.360577 \\ -4.302885 \\ -112.5 \\ -0.721154 \\ -21.394231 \\ 6.252776 \times 10^{-12} \\ 0.360577 \\ -4.302885 \\ 112.5 \end{pmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 8 \\ 9 \\ 10 \\ — \\ 11 \\ — \end{array}$$



From Excel, insert the displacement vector in global coordinates for the girder

$$\begin{bmatrix} R1 \\ R2 \\ R3 \\ 0 \\ 0 \\ 0 \\ 0 \\ R4 \\ R5 \\ R6 \end{bmatrix} \qquad \begin{bmatrix} R8 \cdot -10 \end{bmatrix}$$

$$r_{girder} := \begin{pmatrix} -0.59893243 \\ -0.02660667 \\ -0.01451351 \\ -0.2126594 \\ -1.9934787 \\ -0.01555616 \\ 0 \\ -3.09137547 \\ 0 \end{pmatrix}$$

boundaries

$$\left.\begin{array}{r} -0.59893243 \\ -0.02660667 \\ -0.01451351 \\ -0.98256045 \\ -0.01552059 \\ 0.00445334 \\ 0.01973653 \\ -0.2126594 \\ -1.9934787 \\ -0.01555616 \\ -3.09137547 \end{array}\right\}$$

from excel,

$$r_{col} = \begin{bmatrix} E1 \\ E2 \\ E3 \\ E4 \\ E5 \\ E6 \\ \\ 0 \\ 0 \\ E7 \end{bmatrix}$$

$$q_{girder} := T(\theta_g) \cdot r_{girder} \text{ simplify } \rightarrow \begin{pmatrix} -.59251956419190180709 \\ 9.1370326934035613342 \cdot 10^{-2} \\ -1.451351 \cdot 10^{-2} \\ -.59948303626831709093 \\ -1.9130607509926519882 \\ -1.555616 \cdot 10^{-2} \\ -.60626860943738717662 \\ -3.0313430471869358831 \\ 0 \end{pmatrix}$$

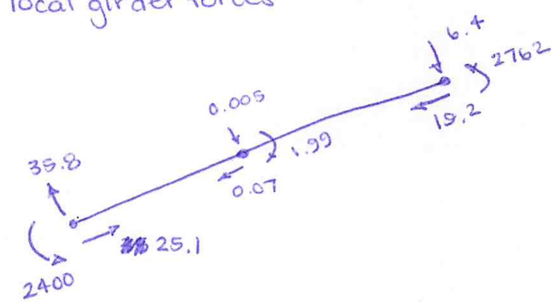
$$Q_{girder.FEM} := \begin{pmatrix} 0.49029 \\ 4.29004 \\ 112.5 \\ 4.902903 \\ 20.837339 \\ 0 \\ 0.49029 \\ 4.29004 \\ -112.5 \end{pmatrix}$$

= − Q structure_load_vector

$$Q_{girder} := k33_{local}(30000, 1125, 15, 244.752937, 0.25) \cdot q_{girder} + Q_{girder.FEM}$$

$$Q_{girder} = \begin{array}{c} \begin{pmatrix} 25.124699 \\ 35.796789 \\ 2400.762105 \\ -0.073431 \\ -0.004544 \\ -1.985284 \\ -19.167785 \\ -6.374826 \\ 2762.036503 \end{pmatrix} \end{array} \begin{array}{l} \text{A} \\ \text{S} \\ \text{M} \\ \\ \\ \\ \\ \\ \\ \end{array}$$

local girder forces



does not include load at top point (?)

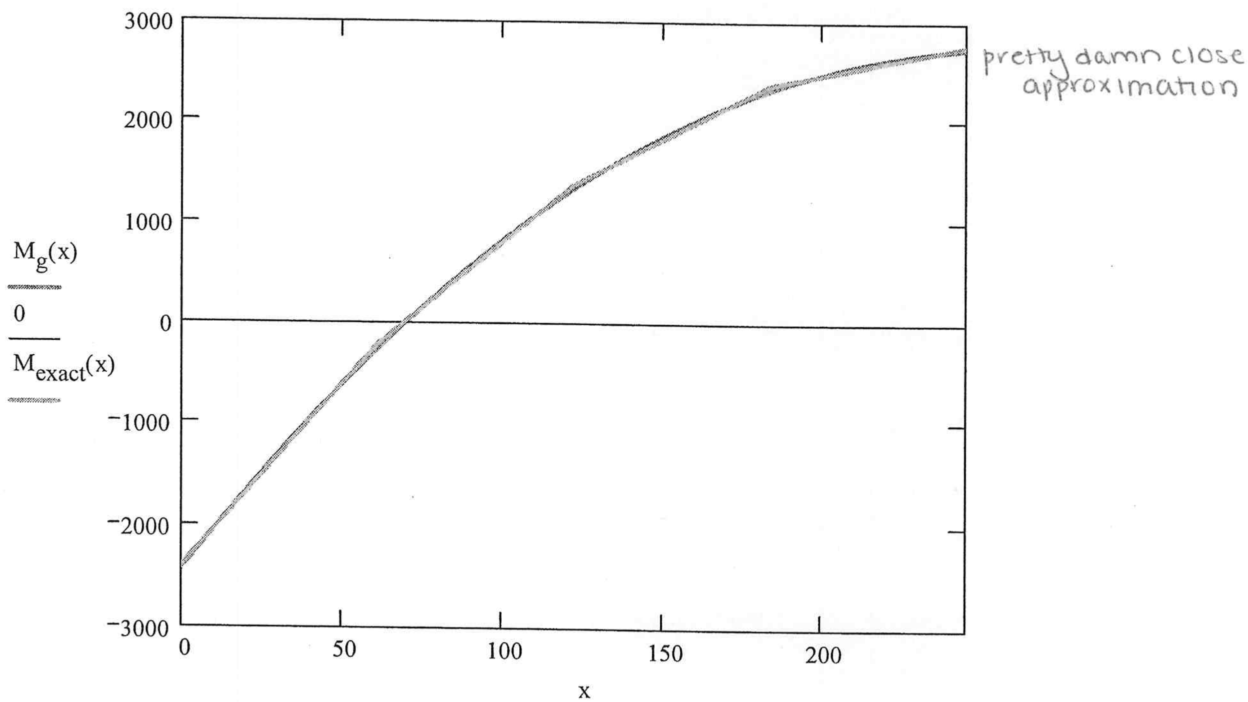
Compute Moment from second derivative of displacement function

$E := 30000 \qquad I_{og} := 1125$

$$M_g(x) := \left[\begin{bmatrix} 0 \\ \frac{d^2}{dx^2}\left(H_1\left(x,L_g\right)\right) \\ \frac{d^2}{dx^2}\left(H_2\left(x,L_g\right)\right) \\ 0 \\ \frac{d^2}{dx^2}\left(H_3\left(x,L_g\right)\right) \\ \frac{d^2}{dx^2}\left(H_4\left(x,L_g\right)\right) \\ 0 \\ \frac{d^2}{dx^2}\left(H_5\left(x,L_g\right)\right) \\ \frac{d^2}{dx^2}\left(H_6\left(x,L_g\right)\right) \end{bmatrix}^T \cdot E \cdot I_{og} \cdot \left(1 - 0.25 \cdot \frac{x}{L_g}\right)^3 \cdot q_{girder}\right] \Bigg| \begin{matrix} \text{simplify} \\ \text{float}, 3 \end{matrix} \rightarrow 39.2 \cdot x - 2.01 \cdot 10^{-4} \cdot x^3 + 5.62 \cdot 10^{-7} \cdot x^4 - 4.96 \cdot 10^{-10} \cdot x^5 - 5.17 \cdot 10^{-2} \cdot x^2 - 2.43 \cdot 10^3 + 1.49 \cdot 10^{-13} \cdot x^6$$

calculate moment approximation—
sixth degree

$$M_{exact}(x) := \begin{cases} -2400 + 35.8 \cdot x & \text{if } \left(0 \le x < \dfrac{L_g}{4}\right) \\ -2400 + 35.8 \cdot \dfrac{L_g}{4} + 26 \cdot \left(x - \dfrac{L_g}{4}\right) & \text{if } \left(\dfrac{L_g}{4} \le x < \dfrac{L_g}{2}\right) \\ -2400 + 35.8 \cdot \dfrac{L_g}{4} + 26 \cdot \left(\dfrac{L_g}{4}\right) + 16.18 \cdot \left(x - \dfrac{L_g}{2}\right) & \text{if } \left(\dfrac{L_g}{2} \le x < \dfrac{3L_g}{4}\right) \\ -2400 + 35.8 \cdot \dfrac{L_g}{4} + 26 \cdot \left(\dfrac{L_g}{4}\right) + 16.18 \cdot \left(\dfrac{L_g}{4}\right) + 6.37 \cdot \left(x - \dfrac{3 \cdot L_g}{4}\right) & \text{if } \left(x \ge \dfrac{3L_g}{4}\right) \end{cases}$$



Flexibility-Based Methods of Matrix Analysis

As discussed many times throughout the semester, all structural analysis problems must address three basic issues: equilibrium, constitution, and kinematics. To date, we have assumed elastic material response. Thus, we have focused most of our attention on the concepts of kinematics and equilibrium. You saw earlier in the semester that, when using virtual work principles, either the displacement terms or the force terms could be selected as the "virtual" quantity. With the principal of virtual displacements, the displacement terms are selected to be the virtual quantities. Equating the external virtual work to the internal virtual work, for all admissible virtual displacements, gives us an alternative way to express the requirement of equilibrium for our structural system. With the principal of virtual forces, the force terms are selected as the "virtual" quantities. The *principle of complimentary virtual work*, which applies to the case in which forces are selected as the virtual quantities, states that the strains and displacements in a deformable system are compatible and consistent with the constraints if and only if the external complimentary virtual work is equal to the complimentary internal virtual work for every system of virtual forces and stresses that satisfies the conditions of equilibrium. From this description, it is clear that the principle of virtual forces and the principle of virtual displacements are closely related. In fact, the principle of virtual forces is often called the "work conjugate dual" of the principle of of virtual displacements. Table 1 below shows a comparison of the two principles.

Table 1: Comparison of Virtual Work Methods

	Principle of Virtual Displacements	**Principle of Virtual Forces**
Requirements	Virtual displacements must be admissible	Virtual force system must satisfy equilibrium (as does the real force system)
Equating external to internal work or complimentary work	Enforces equilibrium	Enforces compatibility
Accuracy	When the real displaced shape corresponds to the case in which equilibrium is satisfied, the solution is exact.	When the real force state corresponds to a state of strain which meets the compatibility conditions, the solution is exact

Perhaps without realizing it, you likely are very familiar with the principle of virtual forces. In fact, it is this principle that allows us to compute displacements for a structure using the "unit dummy load" method.

In order to derive the expressions we will need to compute the response of frame structures, we begin by first considering the general expression that relates external complimentary virtual work (δW^*_{ext}) to internal complimentary virtual work (δW^*_{int}). Thus, Be sure that you can derive this formula on your own and that you can apply it to compute deflections/rotations in frame structures.

Derivation of Element Flexibility Matrices - Generalizing the development above, we can derive expressions for the complimentary internal virtual work for both axial and flexural members. For axial members, we have the following relationship:

$$\delta W^*_{int} = \int_0^{\ell} \frac{\overline{N}(x)N(x)}{E(x)A(x)}dx \tag{5}$$

where we note that, in general, it is possible for all quantities under the integral to vary over the length of the member. Likewise, for a flexural element,

$$\delta W^*_{int} = \int_0^{\ell} \frac{\overline{M}(x)M(x)}{E(x)I(x)}dx \tag{6}$$

Note the similarities between the two formulas.

For analysis of structures using matrix methods, it is preferable to express all unknown quantities in terms of nodal values. Just as we did before with the principle of virtual displacements, we expressed the variation in displacement over the length of the element in terms of nodal values by using displacement "shape functions." Likewise, we can express the variation of axial force or bending moment over the length of an element in terms of nodal values by defining appropriate force "shape functions." To avoid confusion, we will not refer to the variation in forces with the term "shape functions", but, rather, we will call them "force distribution functions." Assuming that we can develop such a relationship for any stress resultant of interest, we can define the relationship

$$Q(x) = \boldsymbol{D}\boldsymbol{F}_f \tag{7}$$

where $\boldsymbol{D}$ is a vector of force distribution functions, and $\boldsymbol{F}_f$ is a vector of unknown force quantities at the nodes. In general, the vector $\boldsymbol{F}_f$ does not contain all terms of $\boldsymbol{Q}$ (i.e., it does not contain all nodal force quantities at each node) because the nodal forces are not independent of each other if equilibrium is enforced. For the principle of virtual forces, we must establish equilibrium from the outset of the problem. Inserting Eq. (7) into Eqs. (5) or (6), we note that it is possible to develop a general expression for internal complimentary virtual work for any structural element. Thus,

$$\delta W^*_{int} = \overline{F}_f^T\left[\int_0^{\ell} \boldsymbol{D}^T\left(\frac{1}{E(x)}\left(\frac{1}{\beta(x)}\right)\right)\boldsymbol{D}dx\right]F_f \tag{8}$$

where $\beta(x)$ is a geometric parameter that depends upon the type of element under consideration. For an axial member, $\beta(x) = A(x)$, and for a flexural member, $\beta(x) = I(x)$. For a member that translate, the right end is free to displace, and we have an acceptable arrangement. Thus, for the purposes of developing the flexibility matrix for the member above, we will investigate the response for the case shown in Fig. 2 below.

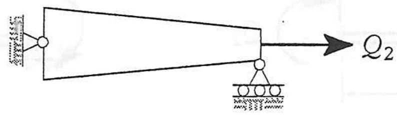


Fig. 2 Tapered Axial Member with Boundary Conditions Applied

For this case, we can determine the axial force over the length of the member in terms of the independent nodal value Q_2. Thus, based on the definitions given above, we have the following:

$$N(x) = Q(x) = Q_2 \text{ and} \tag{10}$$

$$Q(x) = \boldsymbol{D}\boldsymbol{F}_f \tag{11}$$

where $N(x) = Q(x)$ is the axial force along the length of the element, $\boldsymbol{D} = [1]$ and $\boldsymbol{F}_f = [Q_2]$. Recall that we are using the notation that $Q(x)$ is the stress resultant of interest and can be the axial force, shear force, or bending moment. Because the only stress resultant we are considering for this member is the axial force, $\boldsymbol{E}(\beta(x)) = [EA(x)]$. According to Eq. (9),

$$f = \frac{1}{EA_o}\int_0^{\ell}\frac{dx}{1-\frac{x}{2\ell}} = -\frac{2\ell}{EA_o}\ln\left(\frac{1}{2}\right) \tag{12}$$

Note that if the member had a constant cross-sectional area of A, then $\boldsymbol{f} = [\ell/EA]$. If we solve for the inverse, $\boldsymbol{f}^{-1} = [EA/\ell]$, we observe that it is equal to the stiffness matrix corresponding to the axial element with the boundary conditions specified. We will see that this observation holds in general for all types of structural members. Thus, *the inverse of the flexibility matrix is equal to the stiffness matrix provided the same boundary terms are retained in the formulation of both.*

Example 2 - In this example, we will develop the flexibility matrix for a prismatic flexural member. A schematic of the element is given in Fig. 3. As with the axial element above, we must apply suitable boundary conditions in order to eliminate rigid body modes of displacement. There are several possible choices, but one acceptable case is shown in the right-hand side of the figure. With this system, we can derive an expression for the bending moment along the length of the element in terms of the independent nodal quantities Q_1 and Q_2. In order to do so, we first compute the shear at the left end to be $(Q_1 + Q_2)/\ell$. Next, we develop an expression for the bending moment along the length of the beam to be

$$M(x) = Q(x) = \frac{(Q_1 + Q_2)}{\ell}\cdot x - Q_1 \tag{13}$$

where $M(x) = Q(x)$ is the bending moment in the element. In terms of Eq. (7), To overcome the inherent limitations of both methods, we will take a combined approach. That is, under certain circumstances, we will find it more convenient to form the flexibility matrix for an element rather than form the stiffness matrix directly. Once the flexibility matrix is known, we can invert it to get the stiffness matrix corresponding to the selected boundary conditions. Then, for computational purposes, we can expand this matrix to obtain all nodal quantities so that we form the stiffness matrix for the complete set of nodal displacements and/or rotations. In order to accomplish this task, we will make use of the concepts associated with partitioned matrices. As you have seen with the topics of substructuring and constraints, partitioned matrices play a big role in developing computational methods for structural analysis.

To begin the formulation, we first need to define how the independent nodal quantities are related to the dependent, reaction force quantities for the member under consideration. As such, let $\boldsymbol{\Phi}$ be a matrix that relates the dependent nodal force values to the independent nodal force values (i.e., $\boldsymbol{Q}_1 = \boldsymbol{\Phi}\boldsymbol{Q}_0$). Let us also define the equilibrium equations for the element being studied through use of a partitioned stiffness matrix. Thus,

$$\left[\begin{array}{c|c} \boldsymbol{k}_{00} & \boldsymbol{k}_{01} \\ \hline \boldsymbol{k}_{10} & \boldsymbol{k}_{11} \end{array}\right] \left[\begin{array}{c} \boldsymbol{q}_0 \\ \hline \boldsymbol{q}_1 \end{array}\right] = \left[\begin{array}{c} \boldsymbol{Q}_0 \\ \hline \boldsymbol{Q}_1 \end{array}\right] \tag{14}$$

where the subscript "0" on the force and displacement terms correspond to the independent variables, and a subscript of "1" corresponds to the dependent variables. Expressing the partitioned system of equations independently,

$$\boldsymbol{k}_{00}\,\boldsymbol{q}_0 + \boldsymbol{k}_{01}\,\boldsymbol{q}_1 = \boldsymbol{Q}_0 \tag{15}$$

$$\boldsymbol{k}_{10}\,\boldsymbol{q}_0 + \boldsymbol{k}_{11}\,\boldsymbol{q}_1 = \boldsymbol{Q}_1 \tag{16}$$

In general, the boundary conditions that correspond to the dependent nodal quantities are selected such that $\boldsymbol{q}_1 = \boldsymbol{0}$. From our discussion above, we saw that the flexibility matrix is related to the stiffness matrix corresponding to the selected boundary conditions. Thus, it is expressed in terms of the independent nodal values. In terms of the notation in Eq. (14), this statement implies that $\boldsymbol{k}_{00} = \boldsymbol{f}^{-1}$. We can now find the other portions of the partitioned stiffness matrix in terms of the flexibility matrix, $\boldsymbol{f}$, and the matrix relating dependent and independent nodal quantities, $\boldsymbol{\Phi}$.

Considering Eq. (16) for the case in which $\boldsymbol{q}_1 = \boldsymbol{0}$, we can solve for the stiffness term $\boldsymbol{k}_{10}$. To wit,

$$\boldsymbol{k}_{10}\,\boldsymbol{q}_0 = \boldsymbol{Q}_1 = \boldsymbol{\Phi}\,\boldsymbol{Q}_0 = \boldsymbol{\Phi}\,\boldsymbol{k}_{00}\,\boldsymbol{q}_0$$

Thus, if both sides of the equation are equal, this implies that $\boldsymbol{k}_{10} = \boldsymbol{\Phi}\,\boldsymbol{k}_{00} = \boldsymbol{\Phi}\boldsymbol{f}^{-1}$. By the symmetry of the stiffness matrix (i.e., Maxwell's Reciprocity Theorem), we know that $\boldsymbol{k}_{01}^T = \boldsymbol{k}_{10}$. Therefore, we have $\boldsymbol{k}_{01} = \boldsymbol{f}^{-1}\boldsymbol{\Phi}^T$ because of the symmetry of the flexibility matrix (the transpose of the inverse of a matrix is equivalent to the inverse of the transpose of a matrix). Finally, we must express $\boldsymbol{k}_{11}$ in terms of $\boldsymbol{f}$ and $\boldsymbol{\Phi}$. To do so, we will return to Eq. (16) and note that

With both f and Φ known, the stiffness matrix for the complete set of nodal displacement quantities can be computed using Eq. (20). The result of this computation gives

$$\boldsymbol{k} = \frac{EI}{\ell^3}\begin{bmatrix} 4\ell^2 & 2\ell^2 & 6\ell & -6\ell \\ 2\ell^2 & 4\ell^2 & 6\ell & -6\ell \\ 6\ell & 6\ell & 12 & -12 \\ -6\ell & -6\ell & -12 & 12 \end{bmatrix}$$

which is the same result we computed before using a displacement-based approach. While this example serves to demonstrate the essential features of developing member stiffness matrices using a flexibility-based (sometimes referred to as a force-based) approach, it does not capture the strength of the method. As stated earlier, the flexibility-based approach is ideally suited for those cases in which approximating the displaced shape of an element may be difficult. Examples of such a situation include the analysis of curved or tapered frame members.

Analysis of Local Effects

In the development above, we only considered the analysis of nodally loaded structures. In this section, the ideas presented thus far will be extended to accommodate local effects such as might occur due to thermal straining or loads and couples distributed along the length of a member. Previously, in accounting for these effects using a displacement-based approach, we were able to derive a general expression for equivalent nodal loads by equating the work done by intermediate or distributed loadings, initial strains or stresses, support settlements, and thermal effects to the work done by the "equivalent" loads acting at the nodes. Thus, the loads applied at the degrees of freedom for our analysis with the stiffness method were based on the concept of work equivalency. Recall that the general formula to compute these loads utilized the displacement shape functions. When using a force-based approach (i.e., the member flexibility matrix) to determine element stiffness matrices, displacement shape functions are not utilized. Hence, we must develop and alternative formula to the one used previously for determining equivalent nodal loads.

Flexibility Analysis of Local Effects - The principle of superposition can be utilized to carry out an exact analysis of local effects using the flexibility method. Thus, the total response is computed using two separate load cases — one due to end loads acting alone and the other due to all local effects acting alone. The final solution is simply the sum of these cases. The stress resultants $\boldsymbol{Q}(x)$ (designated as a vector because it may include axial, bending, and possibly shear, stress resultants) acting at a section can be expressed as the sum of the components as

$$\boldsymbol{Q}(x) = \boldsymbol{Q}_{\mathrm{H}}(x) + \boldsymbol{Q}_{\mathrm{P}}(x) \tag{21}$$

where $\boldsymbol{Q}_{\mathrm{H}}$ (homogeneous) are the stress resultants that occur when only end forces and/or end moments are acting, and $\boldsymbol{Q}_{\mathrm{P}}$ (particular) are the stress resultants induced in the member by the

$$\delta W^*_{int} = \int_{Vol} \bar{\sigma}^T \epsilon \, dVol \tag{24}$$

In terms of stress resultants and the total strain at a cross section (Eq. (23)), Eq. (24) can be expressed as follows:

$$\delta W^*_{int} = \int_0^{\ell} \left[\bar{F}_f^T D^T \left(\epsilon_T(x) + [E(\beta(x))]^{-1} (Q_H(x) + Q_P(x)) \right) \right] dx \tag{25}$$

It is important to note that the only virtual forces acting on the system are applied at the nodes through the virtual end forces $\bar{F}_f$. Thus, we have assumed that no distributed virtual force quantities are acting on the member under consideration. This restriction is perfectly acceptable because the only requirement on the virtual force system is that it be in equilibrium. Hence, our choice is valid and is made because it simplifies the analysis. Because we are dealing only with end forces in the virtual system, the complimentary virtual work done by these forces acting through the nodal displacements is simply equal to $\bar{F}_f^T q_f$, where q_f is the vector of displacement quantities that are work-conjugate to the force quantities in $\bar{F}_f$. Compatibility of the solution for the nodal displacements is ensured by equating the complimentary virtual work by the nodal quantities with that given in Eq. (25). Thus,

$$\bar{F}_f^T q_f = \bar{F}_f^T \left\{ \int_0^{\ell} D^T [E(\beta(x))]^{-1} D \, dx \, F_f + \int_0^{\ell} D^T \left[\epsilon_T(x) + [E(\beta(x))]^{-1} Q_P(x) \right] dx \right\} \tag{26}$$

Since the $\bar{F}_f$ are arbitrary, independent virtual forces, Eq. (26) yields the flexibility relationship

$$q_f = f F_f + q_{f_0} \tag{27}$$

where the flexibility matrix f is defined as before (see Eq. 9), and the discrete member deformations q_{f_0} due to thermal effects and local loading effects are defined through the relationship

$$q_{f_0} = \int_0^{\ell} D^T \left[\epsilon_T(x) + [E(\beta(x))]^{-1} Q_P(x) \right] dx \tag{28}$$

The *fixed end forces* are easily computed using Eq. (27) by noting that these are the forces associated with the condition that the total deformation, q_f, is zero:

$$Q^{FE} \equiv -f^{-1} q_{f_0} \tag{29}$$

The fixed end forces are traditionally used in matrix structural analysis to find a suitable set of nodal forces, having an effect on the structure equivalent to the distributed loading and/or temperature change. Using this approach, a structure with $-Q^{FE}$ (i.e., the *negative* of the fixed end

Eq. (28) is now solved as follows:

$$q_{f_0} = \begin{bmatrix} q_{1_P} \\ q_{2_P} \end{bmatrix} = \frac{1}{EI}\int_0^{\ell} D^T\left(\frac{w\ell}{2}x - \frac{w}{2}x^2\right)dx = \begin{bmatrix} -\frac{w\ell^3}{24EI} \\ \frac{w\ell^3}{24EI} \end{bmatrix}$$

Eq. (29) can now be used to determine the fixed end forces

$$Q^{FE} = -\frac{2EI}{\ell}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -\frac{w\ell^3}{24EI} \\ \frac{w\ell^3}{24EI} \end{bmatrix} = \begin{bmatrix} \frac{w\ell^2}{12} \\ -\frac{w\ell^2}{12} \end{bmatrix}$$

as expected. Once these values have been determined, the equivalent forces acting at the degrees of freedom corresponding to the dependent force quantities can be determined from equilibrium.

References

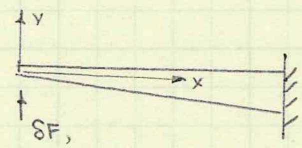
McGuire, W., Gallagher, R. H., and Ziemian, R. D. (2000). *Matrix Structural Analysis: Second Edition.* John Wiley & Sons, Inc., New York.

FLEXIBILITY METHOD

overview and basics

- "work·dual" of the stiffness method
  - uses force quantities as virtual values
  - relies on the Principle of Virtual Forces
- from stiffness method:
  - virtual displacements must be admissible
    - ↳ dealt with continuity of function
  - for flexibility, force quantities (real and virtual) must satisfy equilibrium

Example

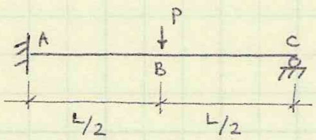


δF, virtual force

$\delta M(x) = \delta F \cdot x$

can't just randomly assume relationship; must match structure, meet equilibrium

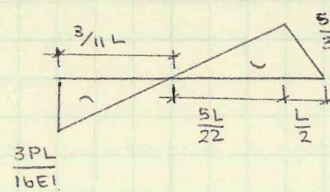
Example x2



$\Delta B = ?$

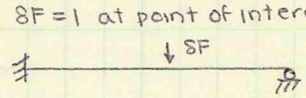
using moment·area

- need curvature (M/EI) diagram



or, virtual work

- apply $\delta F = 1$ at point of interest

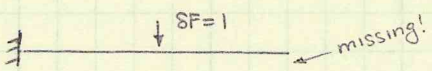


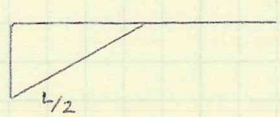
scale moment diagram by P

- calculate ΔB

$$\Delta B = \frac{1}{2} \cdot \frac{3L}{11} \cdot \frac{-3LP}{16EI} \cdot \frac{-3L}{16} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{5L}{22} \cdot \frac{5PL}{32EI} \cdot \frac{2}{3} \cdot \frac{5L}{32} + \frac{1}{2} \cdot \frac{L}{2} \cdot \frac{5PL}{32EI} \cdot \frac{2}{3} \cdot \frac{5L}{32} = 0.009115 \frac{PL^3}{EI}$$

Removing a boundary:





(Area under real M)·(value of virtual at centroid of real M)

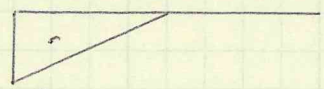
$$= \frac{1}{2}\left(-\frac{3PL}{EI}\right)\left(\frac{3}{11}L\right) \cdot \frac{9-L}{22} + \frac{1}{2}\left(\frac{5PL}{32EI}\right)\left(\frac{5L}{22}\right) \cdot \frac{1}{3}\left(\frac{5}{22}\right)\frac{L}{2}$$

$$= 0.009115\, PL^3/EI$$

FLEXIBILITY METHOD

Examples, cont'd

Assume real moment, virtual moment look like this:



$\frac{L}{2}\left(\frac{P}{EI}\right)$

$$\Delta B = \frac{1}{2}\left(\frac{-PL}{2EI}\right)\frac{L}{2}\cdot\frac{2}{3}\left(\frac{-L}{2}\right) = \frac{PL^3}{24EI} = 0.0416\,\frac{PL^3}{EI}$$

not the same answer; not good

original solution was a (not good) approximation

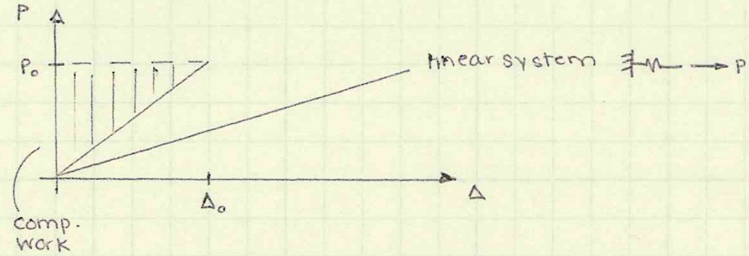
With the principle of virtual displacements, if assumed displaced shape is correct, we'll get the right answer every time.

With the principle of virtual forces, if assumed force matches compatibility, exact solution will result

Benefit of flexibility method: easier to enforce equilibrium than to guess deformed shape (consider tapered logrithmic beam)

FLEXIBILITY METHOD

complimentary work



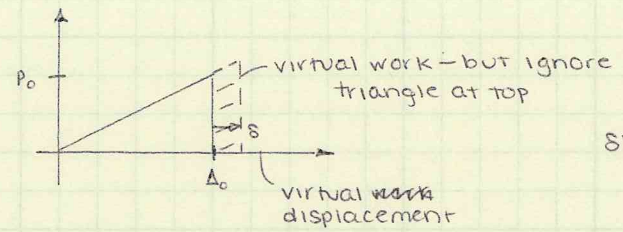
work = $\frac{1}{2}\Delta_o P_o$

complimentary work = area above the curve

for linear system, they're the same

$= \frac{1}{2}\Delta_o P_o$

virtual work



$\delta P = k\delta$

$\delta W = P_o\delta + \frac{1}{2}\delta P\delta$

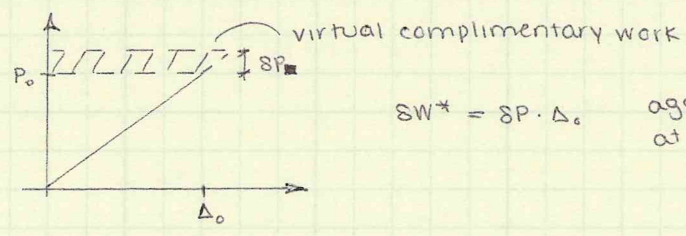
$= P_o\delta + \frac{1}{2}k\delta^2$

because δ is small, $\delta^2 \sim 0$, so ignore this term

$\delta W = P_o\delta$ – external work done by system

Now, consider that force is virtual

use complimentary work



$\delta W^* = \delta P \cdot \Delta_o$ again, ignore triangle at end, as it is small

internal virtual work

$$\delta W_{int} = \int_{vol} \sigma \cdot \delta\varepsilon \, dvol \quad \text{or} \quad \delta\underset{\sim}{\varepsilon}^T \cdot \underset{\sim}{\sigma}$$ – each one is a vector

$$= \int_{vol} \delta\underset{\sim}{\varepsilon}^T \cdot \underset{\sim}{E} \cdot \underset{\sim}{\varepsilon} \, dvol \longleftarrow$$

internal complimentary virtual work

$$\delta W^*_{int} = \int_{vol} \delta\underset{\sim}{\sigma}^T \cdot \underset{\sim}{E}^{-1} \cdot \underset{\sim}{\sigma} \, dvol \longleftarrow$$

FLEXIBILITY METHOD

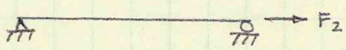
Apply internal complimentary virtual work to a truss member

- establish a statically determinate system
- for equilibrium, F_1 and F_2 are not independent
- select F_2 as independent force quantity;

(1) F_1 is dependent (arbitrary choice)

$F_1 = -F_2$

physically, it's like the left end is now pinned



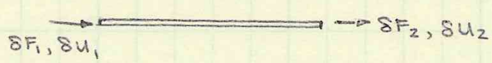
boundaries are established

(2) Develop an expression for variation of stress resultant of interest (axial force, here)

$N(x) = F_2$ — does not change with member geometry — big advantage of this method over stiffness

(3) Develop virtual force system — it's smart to have it match real system (don't make F_1 independent, etc.)

$\delta N(x) = \delta F_2$



(4) Establish $\delta W^*_{ext} = \delta W^*_{int}$, which deals with compatibility of the system (equilibrium established in $N(x)$ eqs.)

$$\delta W^*_{int} = \int_{vol} \underset{\sim}{\delta\sigma}^T \cdot \underset{\sim}{E}^{-1} \cdot \underset{\sim}{\sigma} \, dvol$$

$\underset{\sim}{\delta\sigma} = \frac{\delta N(x)}{A(x)}$, $\underset{\sim}{E}^{-1} = \frac{1}{E}$ constant, isotropic, etc., $\underset{\sim}{\sigma} = \frac{F}{A} = \frac{N(x)}{A(x)}$

$$= \int_V \frac{\delta N(x) \cdot N(x)}{A(x)^2 \cdot E} \, dvol = \int_V \frac{\delta F_2}{A} \cdot \frac{1}{E} \cdot \frac{F_2}{A} \, dvol$$

$$= \int_{vol} \frac{\delta F_2}{A} \cdot \frac{1}{E} \cdot \frac{F_2}{A} \, dA \, dx$$ — all constants, so:

$$= \delta F_2 F_2 \, L/EA$$

FLEXIBILITY METHOD

Truss member formulation

(4) cont'd

$$\delta W^*_{int} = \frac{\delta F_2 \cdot F_2 \cdot L}{EA}$$

$$\delta W^*_{ext} = \delta F_2 \cdot u_2$$ — opposite from before, as is compilmentary external virtual work

$$\delta F_2 \cdot u_2 = \frac{\delta F_2 \cdot F_2}{E \cdot A} L$$

either $\delta F_2 = 0$ (arbitrary), or

$$\underline{\underline{F_2 \frac{L}{EA} = u_2}}$$

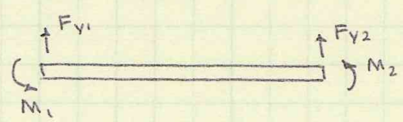
$\underset{\sim}{f} \equiv$ flexibility matrix

1x1 for truss member

— develops a portion of the stiffness matrix, when inverted

— can get the rest, we'll see later

Flexural member (beam)



(1) Pick independent, dependent — many, many choices

etc.

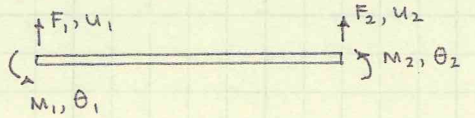
Assume M_1, M_2 are independent
thus, F_1, F_2 are dependent

> pin support situation

$$F_1 = \frac{M_1 + M_2}{L}, \quad F_2 = \frac{M_1 + M_2}{-L}$$

FLEXIBILITY METHOD

consider a flexural member



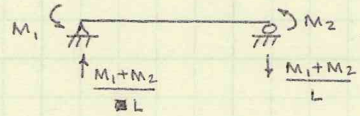
two equilibrium equations – ignore axial, as it is uncoupled in small displacement problems

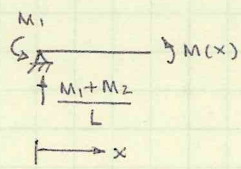
(1) select independent force quantities (two)
many choices
M_1, M_2

$$F_1 = \frac{M_1 + M_2}{L}, \quad F_2 = \frac{-(M_1+M_2)}{L}$$

(2) Develop an expression for equilibrium
satisfy moment, shear or both equilibriums?
here, use moment (ignore shear deformations)

$$M(x) = \left(\frac{M_1+M_2}{L}\right)x - M_1$$





$$M(x) = \left(\frac{x}{L} - 1\right)M_1 + \frac{x}{L}M_2$$

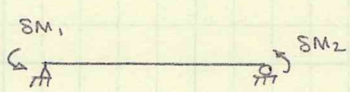
$$= \begin{bmatrix} x/L - 1 & x/L \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

$$\boxed{M(x) = \underset{\sim}{D}^{*}\, \underset{\sim}{F}_F}$$

nodal force values

variation in bending moments – similar to shape functions

(3) Develop a virtual force system



match geometries for best results

$$\delta M(x) = \underset{\sim}{D}^{*}\, \delta\underset{\sim}{F}_F$$

EBW declares D a row vector

FLEXIBILITY METHOD

Flexural member (cont'd)

(4) complimentary internal virtual work / external virtual work

$$\delta W^*_{int} = \int_{vol} \delta\sigma \cdot \frac{1}{E} \cdot \sigma \, dvol$$

scalar value in this case

can vary with volume

$$\delta\sigma = \delta M(x) \cdot \frac{y}{I}$$

$$\sigma = M(x) \cdot y/I$$

$$\delta W^*_{int} = \int_{vol} \delta M(x) \cdot \frac{M(x)}{E(x) I(x)} \cdot \frac{y^2 dA}{I(x)} \, dx$$

yay, equals one!

$$= \int_0^L \delta M(x) \frac{M(x)}{E(x) I(x)} \, dx$$

look at form — that's moment·area / virtual work method:

integrate M/EI graph x δM ...

$$\delta W^*_{int} = \int_0^L \underset{\sim}{D} \, \underset{\sim}{\delta F_F} \cdot \frac{\underset{\sim}{D} \, \underset{\sim}{F_F}}{E(x) I(x)} \, dx$$

$$= \int_0^L \delta F_F^T D^T \frac{1}{E} \cdot \frac{1}{I} D F_F \, dx$$

what varies with x? all but $\underset{\sim}{F_F}$

$$= \delta F_F \underbrace{\int_0^L \underset{\sim}{D}^T \frac{1}{E} \frac{1}{I} \underset{\sim}{D} \, dx}_{\text{flexibility matrix}} \cdot \underset{\sim}{F_F}$$

(5) complimentary external virtual work

$$\delta W^*_{ext} = \underset{\sim}{\delta F_F^T} \cdot \underset{\sim}{r_o}$$

displacement quantities corresponding to independent force quantities

$$\underset{\sim}{r_o} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

$$= \delta M_1 \cdot \theta_1 + \delta M_2 \cdot \theta_2$$

(6) Enforce compatibility

$$\delta W^*_{int} = \delta W^*_{ext}$$

either $\delta F_F = 0$ or

in terms of (x)

$$\underline{\underline{\underset{\sim}{r_o} = \int_0^L \underset{\sim}{D}^T \frac{1}{E} \cdot \frac{1}{I} \underset{\sim}{D} \, dx \cdot \underset{\sim}{F_F}}} \quad , \quad \underset{\sim}{f} = \int_0^L \underset{\sim}{D}^T \frac{\underset{\sim}{D}}{E(x) I(x)} \, dx$$

FLEXIBILITY METHOD

General form:

$$\underset{\sim}{f} = \int_0^L \underset{\sim}{D}^T \frac{1}{E(x)} \cdot \frac{1}{\beta(x)} \cdot \underset{\sim}{D}\, dx$$

- geometric parameter as a function of the cross-section
- $A(x)$, $I(x)$, $A_s(x)$ — axial, bending, shear

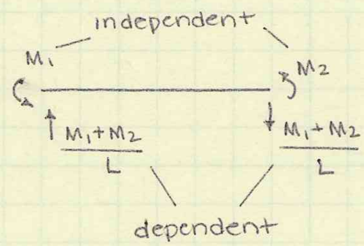
Variation of forces over the length of the member

material parameter — could be $G(x)$, for shear considerations

Does not form full matrix because loads are not necessarily independent of each other

Stiffness-flexibility transformations
... using partitioned matrices

$$\left[\begin{array}{c|c} k_{00} & k_{01} \\ \hline k_{10} & k_{11} \end{array}\right] \left[\begin{array}{c} q_0 \\ \hline q_1 \end{array}\right] = \left[\begin{array}{c} Q_0 \\ \hline Q_1 \end{array}\right] = \left[\begin{array}{c} Q_0 \\ \phi Q_0 \end{array}\right]$$



declare: $\underset{\sim}{Q_0}$ = independent force quantities

$\underset{\sim}{Q_1}$ = dependent force quantities

$$\underset{\sim}{Q_1} = \underset{\sim}{\phi}\, \underset{\sim}{Q_0}$$

$$\underbrace{\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}}_{Q_1} = \underbrace{\begin{bmatrix} 1/L & 1/L \\ -1/L & -1/L \end{bmatrix}}_{\phi} \underbrace{\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}}_{Q_0}$$

"boundary conditions" = dependent force quantities
(must be enough to leave stable, statically determinate structure

Reorganizing matrices...

(1) $k_{00} \cdot q_0 + k_{01}\, q_1 = Q_0$

(2) $k_{10} \cdot q_0 + k_{11} \cdot q_1 = \phi Q_0$

FLEXIBILITY METHOD

Load, stiffness matrices

$$\underset{\sim}{f}\,\underset{\sim}{Q_0} = \underset{\sim}{q_0}\ , \text{ or } \quad \underset{\sim}{Q_0} = \underset{\sim}{f}^{-1}\underset{\sim}{q_0}$$

$$\left.\begin{aligned} k_{00}q_0 + k_{01}q_1 &= Q_0 \\ k_{10}q_0 + k_{11}q_1 &= \phi Q_0 \end{aligned}\right\rangle$$ but, $q_1 = 0$ – making values dependent was like adding boundaries

$$k_{00}q_0 = Q_0 = f^{-1}q_0$$

$$\boxed{\underset{\sim}{f}^{-1} = \underset{\sim}{k_{00}}}$$

Now, $q_1 = 0$, $\quad k_{10}q_0 = \phi Q_0 = \phi q_0 f^{-1}$

$$\boxed{\underset{\sim}{k_{10}} = \underset{\sim}{\phi}\,\underset{\sim}{f}^{-1}}\ , \quad \underset{\sim}{k_{01}} = \underset{\sim}{k_{10}}^T$$

$$k_{01} = (\phi f^{-1})^T$$

since f^{-1} is symmetric,

$$\boxed{\underset{\sim}{k_{01}} = \underset{\sim}{f}^{-1}\cdot\underset{\sim}{\phi}^T}$$

k_{11} remains:

– multiply eq. (1) by ϕ

$$\phi k_{00}q_0 + \phi k_{01}q_1 = \phi Q_0$$

– subtract (2) from (1)$\cdot\phi$

$$\begin{aligned} \phi f^{-1}q_0 + \phi f^{-1}\cdot\phi^T q_1 &= \phi Q_0 \\ -\ \phi f^{-1}q_0 + \quad k_{11}q_1 &= \phi Q_0 \\ \hline \phi f^{-1}\phi^T q_1 - k_{11}q_1 &= 0 \end{aligned}$$

either $q_1 = 0$, or

$$\boxed{\underset{\sim}{k_{11}} = \underset{\sim}{\phi}\,\underset{\sim}{f}^{-1}\,\underset{\sim}{\phi}^T}$$

Stiffness matrix:

$$\underset{\sim}{K} = \begin{bmatrix} f^{-1} & f^{-1}\phi^T \\ \phi f^{-1} & \phi f^{-1}\phi^T \end{bmatrix}$$

FLEXIBILITY METHOD

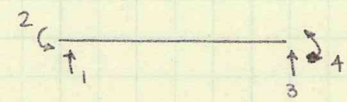
combing (or, combining), considering:

$$f^{-1} = \frac{EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}, \quad \phi = \frac{1}{L}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$K = \left[\begin{array}{cc|cc} \frac{4EI}{L} & \frac{2EI}{L} & \frac{6EI}{L^2} & \frac{-6EI}{L^2} \\ \frac{2EI}{L} & \frac{4EI}{L} & \frac{6EI}{L^2} & \frac{-6EI}{L^2} \\ \hline \frac{6EI}{L^2} & \frac{6EI}{L^2} & & \\ -\frac{6EI}{L^2} & -\frac{6EI}{L^2} & \multicolumn{2}{c}{\phi f^{-1} \phi^T} \end{array}\right]$$

Problem: not in our standard order for DOFS

instead of



Benefits:

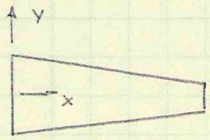
- particularly good in inelastic situations – plastic hinge forms, forces stay constant for stiffness, assumed shape becomes bad

Drawbacks

- can't compute values between nodes

FLEXIBILITY METHOD

Tapered beam example

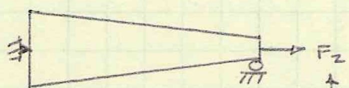


$A(x) = A_0(1 - x/2L)$

E, L

Find flexibility matrix

(1) choose independent



independent

(2) develop expression for equilibrium

$N(x) = F_2$

$Q \equiv$ stress resultant $= \underset{\sim}{D}\,\underset{\sim}{F}_F$

$\underset{\sim}{D} = [1]$

(3) use integral equation:

$$\underset{\sim}{f} = \int_0^L [1]\,\frac{1}{E}\,\frac{1}{A_0(1-x/2L)}\,[1]\,dx$$

$$= \frac{1}{EA_0}\ln(1-x/2L)(-2L)\Big|_0^L$$

$$= \frac{-2L}{EA_0}\ln 1/2$$

hey look, solution is logrithmic! exact solution.

Now, get to $\underset{\sim}{K}$

$\phi: \quad F_1 = \phi F_2 \;, \quad \phi = -1$

$$\underset{\sim}{K} = \begin{bmatrix} \frac{-EA_0}{2L}\cdot\frac{1}{\ln(1/2)} & -f_\phi \\ -f_\phi & f \end{bmatrix}$$

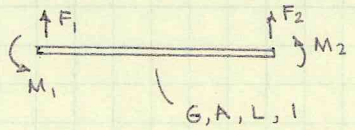
all values are the same because $\phi = -1$

$$\underset{\sim}{K} = \frac{-EA_0}{L}\cdot \text{const.} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

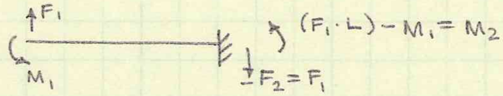
interesting, eh?

EXAMPLE PROBLEMS

Flexure and shear deformations

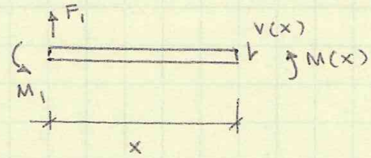


Pick independent variables



write equilibrium equations

$\Sigma F_y = 0: \quad F_2 = -F_1$



$$\left| \begin{array}{l} V(x) = F_1 \\ M(x) = F_1 x - M_1 \end{array} \right.$$

2 equations, 2 $\underset{\sim}{D}$ vectors

$$V(x) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\underset{\sim}{D}_V} \begin{bmatrix} F_1 \\ M_1 \end{bmatrix}$$

$$M(x) = \underbrace{\begin{bmatrix} x & -1 \end{bmatrix}}_{\underset{\sim}{D}_M} \begin{bmatrix} F_1 \\ M_1 \end{bmatrix}$$

consider complimentary internal virtual work

$$\delta W^{*}_{int} = \int_{vol} \delta\sigma \cdot \varepsilon \, dvol$$ — scalar value, combining shear, flexure contributions

$$= \underbrace{\int_{vol} \delta\sigma \cdot \varepsilon \, dvol}_{\text{flexure}} + \underbrace{\int_{vol} \delta\tau \cdot \gamma \, dvol}_{\text{shear}}$$

$$\delta \underset{\sim}{F}^T \int_0^L \underset{\sim}{D}_V^T \, \frac{1}{G} \cdot \frac{1}{A_s} \, \underset{\sim}{D}_V \, dx \cdot \underset{\sim}{F} = \delta F^T \cdot q$$

$A_s \equiv$ equivalent shear area
considers parabolic stress distribution from shear
e.g. rectangular x-section:
$A_s \sim 2/3 A$
W-shape:
$A_s \sim A_w$ (woo!)
circle:
$A_s \sim 0.9A$

EXAMPLE PROBLEMS

Flexure / shear considerations (cont'd)

complimentary work, flexibility matrices:

shear

$$f_v = \int_0^L \underset{\sim}{D}_v^T \frac{1}{G} \cdot \frac{1}{A_s} \cdot \underset{\sim}{D}_v \, dx \quad , \quad \underset{\sim}{D}_v = [\,1 \quad 0\,]$$

Flexure

$$f_f = \int_0^L \underset{\sim}{D}_M^T \frac{1}{E} \cdot \frac{1}{I} \cdot \underset{\sim}{D}_M \, dx \qquad \underset{\sim}{D}_M = [\,x \quad -1\,]$$

for this problem, or this assignment of independent & dependent forces

combining:

$$(\underset{\sim}{f}_V + \underset{\sim}{f}_M)\underset{\sim}{F}_F = \underset{\sim}{q}$$

or f_f, above

$$f = f_V + f_M = \begin{bmatrix} \frac{L^3}{3EI} + \frac{L}{A_sG} & \frac{-L^2}{2EI} \\ \frac{-L^2}{2EI} & \frac{L}{EI} \end{bmatrix}$$

only shear term because $D_V^T \cdot D_V = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

very small contribution here, it seems

Get to the stiffness matrix

- develop ϕ

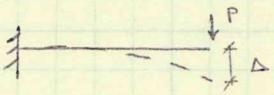
$$\begin{bmatrix} F_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ L & -1 \end{bmatrix} \begin{bmatrix} F_1 \\ M_1 \end{bmatrix}, \text{ as } \begin{aligned} F_2 &= -F_1 \\ M_2 &= F_1 L - M_1 \end{aligned}$$

so much easier than forming a deflected shape for shear deformations

- use

$$\left.\begin{aligned} k_{00} &= f^{-1} \\ k_{10} &= \phi f^{-1}, \quad k_{01} = f^{-1}\phi^T \\ k_{11} &= \phi f^{-1} \phi^T \end{aligned}\right\} \text{combine to form } \underset{\sim}{k}$$

How important are shear deformations?



$$\Delta = \frac{PL^3}{3EI} + \frac{PL}{A_sG}$$ (hey, look, that's $f_{11} \cdot P$)

- assuming $\nu = 0.3$

$A_s = 2/3 A_g$

$$\Delta = \frac{PL^3}{3EI}\left(1 + 0.975\frac{h^2}{L^2}\right)$$

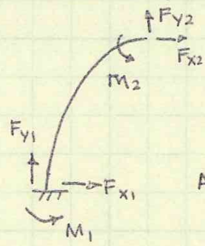
thus, shear impact depends on depth / length ratio

- generally, $L \gg h$, $(h/L)^2 \to 0$
- if not true, shear matters

ACI "deep beam", $h/L = 1/5$

EXAMPLE PROBLEMS

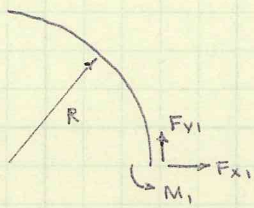
Curved beams (in plane)



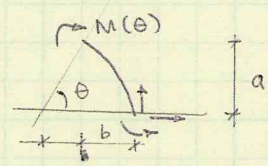
e.g. the St. Louis arch

Assumptions:

- basic flexural beam theory holds
- ignore curved beam theory
  arch must be somewhat gradual
  small displacements, etc.
- shape is circular : constant radius of curvature



(1) choose independent force quantities
F_{x1}, F_{y1}, M_1



(2) $M(\theta) = M_1 + F_{x1}a + F_{y1}b$

$$= \begin{bmatrix} a & b & 1 \end{bmatrix} \begin{bmatrix} F_{x1} \\ F_{y1} \\ M \end{bmatrix} \qquad \begin{array}{l} a = R\sin\theta \\ b = R(1-\cos\theta) \end{array}$$

$$\underset{\sim}{D} = \begin{bmatrix} R\sin\theta & R(1-\cos\theta) & 1 \end{bmatrix}$$

General formula for f

$$f = \int_0^L \underset{\sim}{D}^T \frac{1}{E}\frac{1}{I}\underset{\sim}{D}\, dx$$ — but, D is a function of R, length is ψR (ψ = total rotation)

$x = R\cos\theta$, $y = R\sin\theta$

$dx = -R\sin\theta\, d\theta$, $dy = R\cos\theta\, d\theta$

$ds = (dx^2 + dy^2)^{1/2} = R\, d\theta$

$$f = \int_0^{\psi} \underset{\sim}{D}^T \frac{1}{E}\frac{1}{I}\underset{\sim}{D} \cdot R\, d\theta$$

integrate.

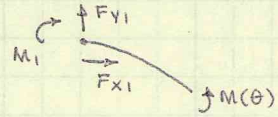
if R varies, make it a function of θ

same thing if E, I vary along length – must write equations expressing variation

EXAMPLE PROBLEMS

Curved beams (cont'd)

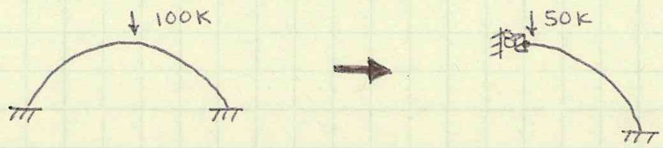
using different independents, ordering (the EBW way):



$$\underset{\sim}{D} = \begin{bmatrix} 1 & R(1-\sin\theta) & R\cos\theta \end{bmatrix} \begin{bmatrix} M_1 \\ F_{x1} \\ F_{y1} \end{bmatrix}$$

$$f = \frac{R}{EI} \begin{bmatrix} \frac{\pi}{2} & R(\pi/2 - 1) & R \\ & R^2(3\pi - 8)/4 & \frac{R^2}{2} \\ & & \frac{\pi}{4} R^2 \end{bmatrix}$$

Second example of curved beam



one DOF, whoop whoop!

$v = \frac{\pi}{4} R^2 \cdot P$ — 33 term from above × load

FLEXIBILITY METHOD

Equivalent nodal forces

$$\sigma = E(\varepsilon - \varepsilon_o) + \sigma_o$$

initial stress, strain

$$\delta W^*_{int} = \int_{vol} \delta\sigma \cdot \varepsilon \, dvol$$

rearrange first equation to be for ε

$$\varepsilon = \frac{1}{E}(\sigma - \sigma_o) + \varepsilon_o$$

$$\delta W^*_{int} = \int_{vol} \delta\sigma \cdot \frac{1}{E}\left[(\sigma - \sigma_o) + \varepsilon_o\right] dvol$$

$$= \int_{vol} \delta\sigma \cdot \frac{1}{E}\sigma \, dvol - \int_{vol} \delta\sigma \frac{1}{E}\sigma_o \, dvol + \int_{vol} \delta\sigma \, \varepsilon_o \, dvol$$

minus sign missing in here

$$= \delta F_F^T f F_F + \delta F_F^T \int_0^L D^T \frac{1}{E}\frac{1}{I} M_o \, dx + \delta F_F^T \int_0^L D^T K_o \, dx$$

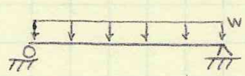
moment due to distributed loads (varies with x)

initial curvature (varies with x)

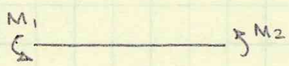
considering compatibility,

$$\underset{\sim}{q} = \underset{\sim}{f}\,\underset{\sim}{F_F} + \int_0^L \underset{\sim}{D^T} \frac{1}{E}\frac{1}{I} M_o(x)\, dx + \int_0^L \underset{\sim}{D^T} K_o(x)\, dx$$

Example – beam with uniform load



(1)

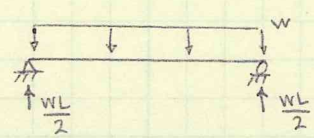


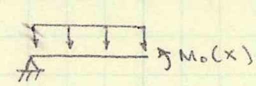
$$M(x) = \begin{bmatrix} x/L - 1 & \frac{x}{L} \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}, \quad f = \int_0^L \underset{\sim}{D^T} \frac{1}{EI} \underset{\sim}{D} \, dx$$

use ϕ to get $\underset{\sim}{k}$

... equivalent nodal loads

$$M_o(x) = \frac{wL}{2} \cdot x - \frac{wx^2}{2}$$





EQUIVALENT NODAL LOADS

Example (cont'd)

$$q \propto \int_0^L D^T \frac{1}{EI} M_o(x)\,dx$$

$$\hookrightarrow = \frac{wL}{2}x - \frac{wx^2}{2}$$

multiply, integrate

$$= \frac{wL^3}{24EI}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$ — should have been +/−, not −/+ result should be fixed-end

Fixed-end forces:

$\underset{\sim}{q} = 0$ – not displacements at ends of member

$$\underset{\sim}{f}\,\underset{\sim}{F_F} - \frac{wL^3}{24EI}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0$$

$$\underset{\sim}{f}\,\underset{\sim}{F_F} = \frac{wL^3}{24EI}\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad F_F = \underset{\sim}{f}^{-1} \cdot \frac{\cancel{2/4}\, wL^3}{24EI}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$k_{oo} = \frac{EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\underset{\sim}{F_F} = \begin{bmatrix} -\frac{wL^2}{12} \\ \frac{wL^2}{12} \end{bmatrix}$$

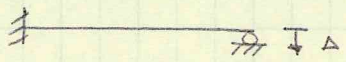
– not fixed-end, but the actual values you need to apply to the structure

– incorrect signage before; should get fixed-end values

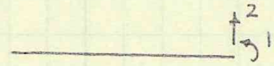
- Error exists somewhere in here
- EBW may find it later. Maybe.

REVIEW SESSION

Propped cantilever with support settlement



DOFS:



Using the stiffness method:

$$K = \frac{EI}{L}\left[\begin{array}{c|c} 4 & -6/L \\ \hline -6/L & 12/L^2 \end{array}\right]$$

2-node beam
prismatic, EI constant...

$$R = \left[\begin{array}{c} 0 \\ \hline R_{pin} \end{array}\right], \quad r = \left[\begin{array}{c} \theta \\ \hline -\Delta \end{array}\right]$$

$$K_{11} = 4\frac{EI}{L}$$

$$K_{12} = \frac{-6}{L^2}EI$$

$$K_{22} = 12\frac{EI}{L^3}$$

$$(K_{22} - K_{21}K_{11}^{-1}K_{12})\, r_2 = R_2 - K_{21}K_{11}^{-1}R_1$$

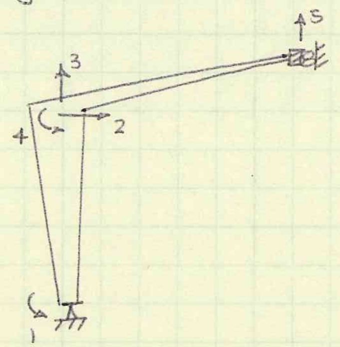
watch EIs, etc.

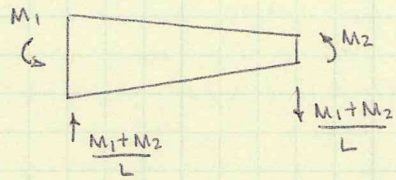
$$\frac{EI}{L}\left[\,{}^{12}/_{L^2} - {}^{-6}/_{L}\cdot{}^{1}/_{4}\cdot{}^{-6}/_{L}\right](-\Delta) = R_y$$

$$\frac{3EI}{L^3}(-\Delta) = R_y \quad , \quad \theta = -\frac{3}{2}\frac{\Delta}{L}$$

REVIEW SESSION

Flexibility example





(should include axial response, too)

– $f = \int_0^L \underset{\sim}{D}^T \frac{1}{E} \cdot \frac{1}{I(x)} \underset{\sim}{D} \, dx$

$\underset{\sim}{D} = \left[x/L - 1 \quad x/L \right]$

$M(x) = \underset{\sim}{D}(x) \underset{\sim}{F}_F$

– convert $\underset{\sim}{f}$ to $\underset{\sim}{k}$ using ϕ matrix

– transform $\underset{\sim}{k}_2$ into global coordinates

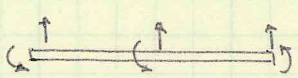
$\underset{\sim}{K}_2 = \underset{\sim}{T}_2^T \underset{\sim}{k}_2 \underset{\sim}{T}_2$

cannot map to global matrix from f because it depends on what was chosen as redundant

what's leftover must be statically determinate and stable

- one must be axial
- flexural has options

3-Noded beam with ~~flexural~~ flexure method



4 independent forces

M(x) would have multiple parts

not so awesome –

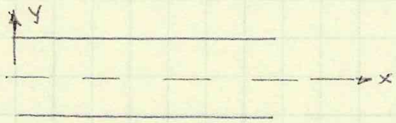
- 2 $\underset{\sim}{D}$ vectors
- integral must be split

Not so awesome. Avoid.

REVIEW SESSION

Negative problem

$$\delta W^*_{int} = \int_{vol} \delta\sigma \cdot \frac{\sigma}{E} \, dvol - \int_{vol} \delta\sigma \, \frac{1}{E} \sigma_o \, dvol + \int_{vol} \delta\sigma \, \varepsilon_o \, dvol$$



positive curvature, negative strain

$$\delta\sigma = \frac{\delta M \cdot y}{I}, \quad \sigma = \frac{My}{I}$$

→ I is a scalar
y has +/− that goes against $\delta\sigma$

$$\int \delta M \cdot \frac{M_o}{EI} dx \quad \text{or} \quad \int \delta M \cdot K_o \, dx$$

So,

$$\delta\sigma = \frac{-\delta M \cdot y}{I}$$

$$\sigma = \frac{-M \cdot y}{I}$$

Normally, $\delta\sigma \cdot \sigma$ meant negatives cancel

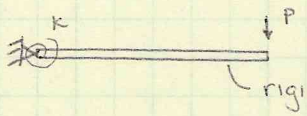
in $\int_{vol} \delta\sigma \cdot \varepsilon_o \, dvol$

not necessarily negative

NON-LINEAR ANALYSIS

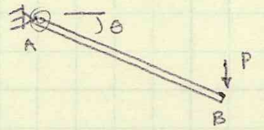
Sources of nonlinearity
- material behavior (yielding)
- kinematics (geometric nonlinearity), e.g. P-Δ effects
  Δ - structure, δ - member
- change of constraint
  or, contact problems

Example



k - rotational spring stiffness

deformed position:



establish equilibrium

$\sum M_A = 0$: $PL\cos\theta = +k\theta$

normally assumed to be $= 1$

$$\frac{PL}{k} = \frac{\theta}{\cos\theta}$$

Solution:

1. establish equilibrium in deformed position

Linearized Analysis

"linearize" the nonlinear equilibrium equation(s)

$$\cos\theta \sim 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$$

even function — $\cos\theta = \cos(-\theta)$

consider up to 1st order terms of a function
for $\cos\theta$, $= 1$

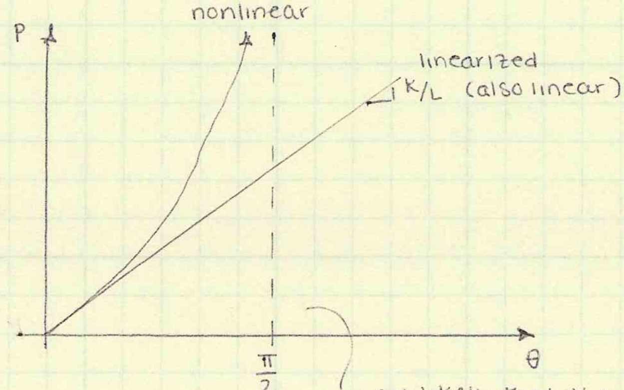
linearized equilibrium equation becomes:

$$PL(1) = k\theta$$

$$\theta = \frac{k}{PL}, \quad P = \frac{k}{L}\theta$$

compare to:

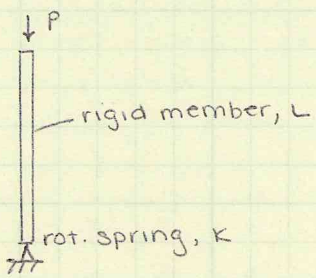
$$P = \frac{k}{L}\frac{\theta}{\cos\theta}$$



not likely that the beam can actually have $\theta > \pi/2$

NONLINEAR ANALYSIS

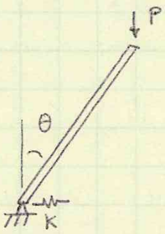
Example modification



- if P is small, structure can be stable, will return to original orientation

- undeformed position:

$A_y = P$ equilibrium



- deformed position; establish equilibrium

$A_y = P$, $K\theta = PL\sin\theta$

$$P = \frac{k}{L}\frac{\theta}{\sin\theta}$$

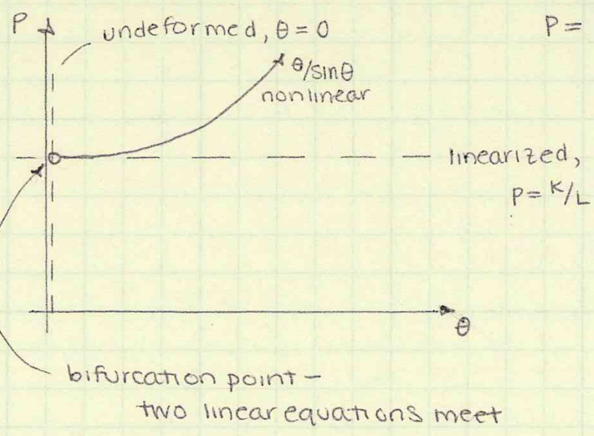
- linearize:

$$\sin\theta \sim \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots$$

only term equal or below first order

$$P = \frac{k}{L} \cdot \frac{\theta}{\theta} = \frac{k}{L}$$ — P is constant

$$P = k/L$$



non-linear response

$\theta \to 0$, $P \to k/L$

as θ increases,

$P \propto \theta/\sin\theta$

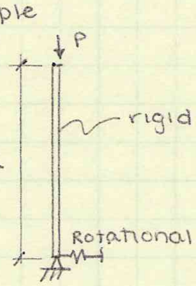
"stable post-buckling behavior"

- even above buckling load, the column can hold load, θ will not go straight to $\pi/2$
- column will ~~be~~ not immediately fall down

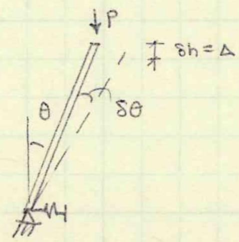
NONLINEAR ANALYSIS

Various rules
- superposition is no longer valid
- generally difficult to solve in closed form
  - possibly no solution (closed-form)
  - requires incremental or numerical progression
  - extra source of error in solution

Example



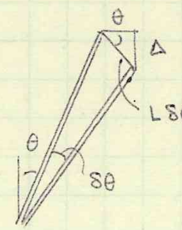
Virtual work-based approach



- assume structure is in equilibrium at this point
- now, structure undergoes small virtual displacement
- consider virtual work terms

External work:

$$\delta W_{ext} = P\delta h = P\Delta$$
$$= PL \cdot \delta\theta \cdot \sin\theta$$
$$= PL\sin\theta \cdot \delta\theta$$



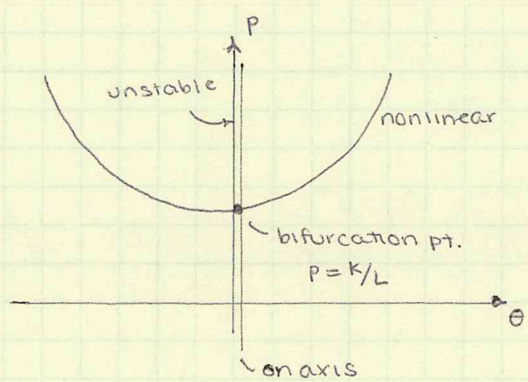
$\Delta = L\delta\theta \cdot \sin\theta$

Internal work:

$$\delta W_{int} = M \cdot \delta\theta = k\theta \cdot \delta\theta$$

moment in the spring at original deformed equilibrium position

Solving $\delta W_{int} = \delta W_{ext}$



$$k\theta \cdot \delta\theta = PL\sin\theta \cdot \delta\theta$$

$$(PL\sin\theta - k\theta)\delta\theta = 0$$

$\delta\theta = 0$, trivial

or,

$$PL\sin\theta = k\theta$$

$$P(\theta) = \frac{k}{L}\frac{\theta}{\sin\theta}$$ – same as equation from using equilibrium

NONLINEAR ANALYSIS

Bifurcation point

add load to that point (intersection with nonlinear response), then additional load will cause displacement

Stability:

when disturbed from equilibrium, if it comes back, it's stable
if it does not, it is unstable

Relationship between work and energy

$$W = \int_a^b F \cdot dx$$

work — force — displacement through which force moves

$$dx = \frac{dx}{dt} \cdot dt = v\,dt$$

v → velocity

$$F = ma = m\frac{dv}{dt}$$

assuming mass is constant

$$W = \int_a^b m \cdot \frac{dv}{dt} \cdot v\,dt = \int_{v_a}^{v_b} m \cdot v\,dv$$

v_b ← velocity at b

if mass is constant, $W = \frac{1}{2}mv^2 \Big|_{v \text{ at } a}^{v \text{ at } b}$

change in kinetic ~~work~~ energy between two points

Apply to stability

we would like velocity to be decreasing

$$\frac{d}{dt}(KE) < 0$$

losing kinetic energy

$\frac{d}{d\theta}(\text{work}) < 0$ but, we have internal and external work

$$\frac{d}{d\theta}\delta W_{int} > \frac{d}{d\theta}\delta W_{ext}$$

for stability

stabilizes structure — causes instability

NONLINEAR ANALYSIS

Apply stability law to example

$$\delta W_{int} = k\theta \cdot \delta\theta$$
$$\delta W_{ext} = PL\sin\theta \cdot \delta\theta$$

for stability,

$$\frac{d}{d\theta} k\theta \cdot \delta\theta > \frac{d}{d\theta} PL\sin\theta \cdot \delta\theta$$

using the chain rule,

$$k\theta \cdot \frac{d}{d\theta}\delta\theta + k \underbrace{\frac{d\theta}{d\theta}}_{1} \delta\theta > PL\left[\sin\theta \frac{d}{d\theta}\delta\theta + \cos\theta \cdot \delta\theta\right]$$

according to EBW, $\frac{d}{d\theta}\delta\theta \to 0$ (proofs exist)

$$k\delta\theta > PL\cos\theta \cdot \delta\theta$$

$$P < \frac{k}{L}\frac{1}{\cos\theta}, \text{ or } \delta\theta = 0$$

for stability to occur

Stability
- structure must first be in equilibrium
- consider all configurations:

(1) $\theta = 0$

$$P < \frac{k}{L}$$

(2) $P(\theta) = \frac{k}{L}\frac{\theta}{\sin\theta}$

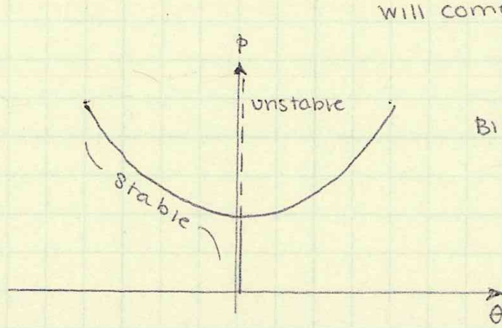
$$\theta < \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$1 - \frac{\theta}{\tan\theta} > 0 \text{ for stability}$$

true for all values of θ ✓

structure doesn't fall down;

will come to rest on nonlinear line



Bifurcation diagram

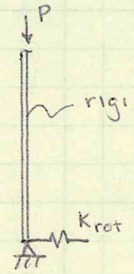
—— stable

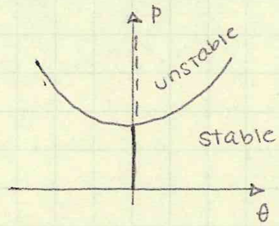
- - - unstable

dashed lines

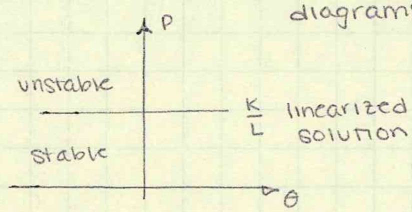
NONLINEAR ANALYSIS

STABILITY





Bifurcation diagrams



Example

column drawn in deformed shape

$\delta W_{int} = kx, \quad x = L\sin\theta$

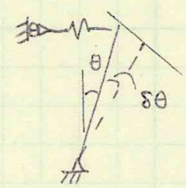
$= kL\sin\theta\,\delta\theta \cdot \cos\theta \cdot L$

$= kL^2 \sin\theta\cos\theta\,\delta\theta$

$\delta W_{ext} = PL\sin\theta \cdot \delta\theta$

For stability, $\frac{d}{d\theta}\delta W_{int} > \frac{d}{d\theta}\delta W_{ext}$

→ force in spring at equilibrium
× additional force from
further rotation = δθ



Establish equilibrium:

$\delta W_{int} = \delta W_{ext}$

$kL^2\sin\theta\cos\theta\,\delta\theta = PL\sin\theta\,\delta\theta$

↓

$kL^2\cos\theta = PL$

$\underline{P = kL\cos\theta}$

→ don't delete sines

$kL^2\sin\theta\cos\theta\,\delta\theta - PL\sin\theta\,\delta\theta = 0$

$(kL^2\sin\theta\cos\theta - PL\sin\theta)\,\delta\theta = 0$

either δθ or ( ) equal zero

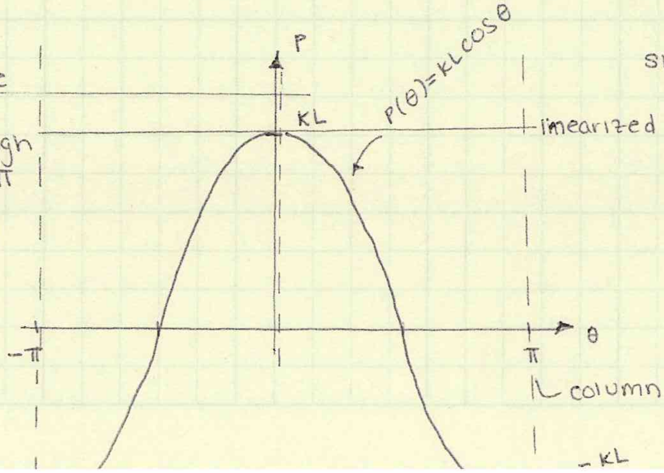
$\sin\theta\,(kL^2\cos\theta - PL) = 0$

$\sin\theta = 0$ or $kL^2\cos\theta - PL = 0$

→ $\underline{\theta = 0, \pm\pi \ldots}$

load above kL would snapthrough to $\theta = \pi, -\pi$

UNSTABLE!



NONLINEAR ANALYSIS

Stability

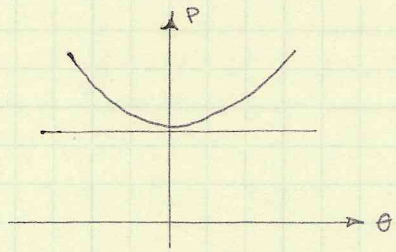
Example, cont'd

Linearizing equation: $\cos\theta = 1 + \frac{\theta^2}{2!} \cdots$

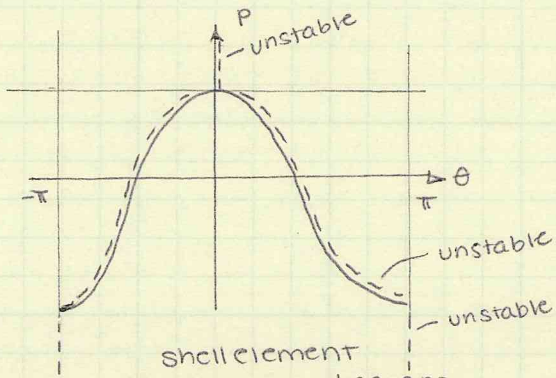
$$P = KL$$

same equation as rotational spring

⟨ Doing a nonlinear analysis allows us to know what happens near stability point ⟩



pin·pin column bifurcation diagram



shell element bifurcation diagram

(SNAP.THROUGH BUCKLING)

Stability?

$$\frac{d\,\delta W_{int}}{d\theta} = \frac{d}{d\theta}\left(KL^2 \sin\theta \cos\theta\, \delta\theta\right)$$

$$= KL^2(\cos^2\theta - \sin^2\theta)\cdot\delta\theta$$

$$\frac{d\,\delta W_{ext}}{d\theta} = \frac{d}{d\theta}(PL\sin\theta\cdot\delta\theta)$$

$$= PL\cos\theta\cdot\delta\theta\ ,\ \frac{d}{d\theta}\delta\theta \rightarrow 0$$

$$= PL\cos\theta\cdot\delta\theta$$

$$\frac{d}{d\theta}\delta W_{int} > \frac{d}{d\theta}\delta W_{ext}$$

$$KL^2(\cos^2\theta - \sin^2\theta) > PL\cos\theta$$

$$P < LK\frac{\cos^2\theta - \sin^2\theta}{\cos\theta}$$

consider equilibrium points:

(1) $\theta = 0$

$P < KL$

(2) $\theta = \pi$

$P > -KL$

(3)

$P(\theta) = KL\cos\theta$ ← $PL\cos\theta$

$KL^2(\cos^2\theta - \sin^2\theta) > KL^2\cos\theta\cdot\cos\theta$

$KL^2\cos^2\theta - KL^2\sin^2\theta > KL^2\cos^2\theta$

$-KL^2\sin^2\theta > 0$ → never true $K, L, \sin^2\theta$ → all positive

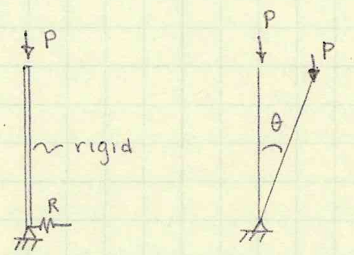
all values of θ are unstable

NONLINEAR ANALYSIS

Stability – energy based procedure

- system is conservative: only works for elastic behavior, no damping, no friction...
- material can still be nonlinear

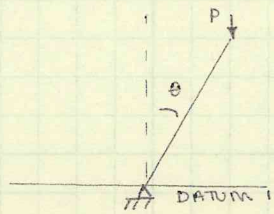
Example



conservation of energy

Potential energy of the load

– select a datum from which to consider loads, energies



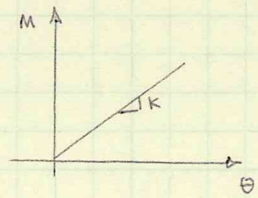
$V_1 = P \cdot h_{load} = PL\cos\theta$

If the top of the column was the datum (2)

$V_2 = -PL(1-\cos\theta)$

$= PL\cos\theta - PL$

Internal strain energy of the spring



$U = \frac{1}{2}k\theta^2$

in general,

$$U = \int_0^{\theta} M(\theta)\,d\theta$$

area under M-θ curve, with $M = k\theta$ in this case

translational spring

$U = \frac{1}{2}kx^2$

For a conservative system, total energy is constant

for equilibrium, rate of change of energy = 0

$$\frac{d\Pi}{d\theta} = \frac{d}{d\theta}(U+V) = 0$$

Π = total energy

rate of change of energy

derivative is the same for D1, D2

$$\frac{d\Pi}{d\theta} = k\theta + (-PL)\sin\theta = 0$$

$k\theta - PL\sin\theta = 0$ — same equation as previous methods

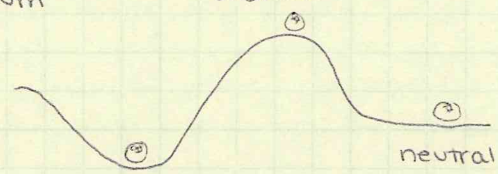
NONLINEAR ANALYSIS

Stability of equilibrium

first derivative – rate of change of energy

second derivative – stability calculation

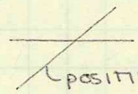
total energy must be at a relative minimum



$$\frac{d^2\pi}{d\theta^2} = k - PL\cos\theta > 0$$

relative minimum

slope (–) to (+)



$k - PL\cos\theta > 0$ – same as established using other methods

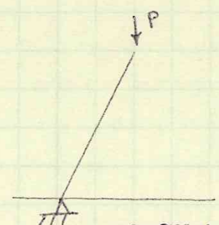
comparison

	virtual work	Energy
Equilibrium	$\delta W_{ext} = \delta W_{int}$ $\delta W_{int} - \delta W_{ext} = 0$	$\frac{d}{d\theta}U + \frac{d}{d\theta}V = 0$

$$\frac{d}{d\theta}U = \delta W_{int}$$

$$\frac{d}{d\theta}V = -\delta W_{ext}$$

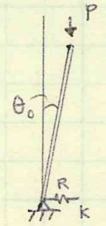
very important negative sign



- energy – P down, above datum, potential energy positive
- virtual work – disp. up, load down, δW_{ext} negative

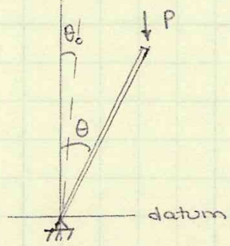
NONLINEAR ANALYSIS

Imperfections



unstrained position, rotated to θ_o
↳ no load in spring
"undeformed" position

Deformed position

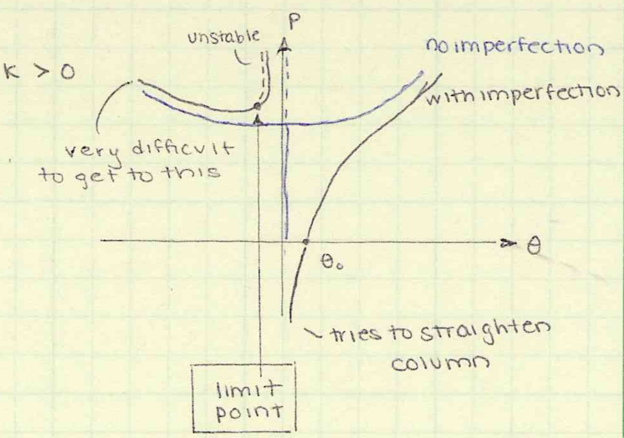


$V = PL\cos\theta$
↑ must be to vertical
if is only additional rotation,
would need to add θ_o

$U = \frac{1}{2}K(\theta - \theta_o)^2$
↑ if defined from original position,
do not need to subtract θ_o

$$\frac{d\Pi}{d\theta} = -PL\sin\theta + K(\theta - \theta_o) = 0 \quad , \quad P(\theta) = \frac{K}{L}\frac{(\theta-\theta_o)}{\sin\theta}$$

$$\frac{d^2\Pi}{d\theta^2} = -PL\cos\theta + K > 0$$



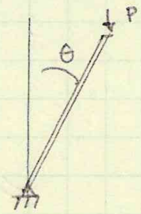
transition between stable and unstable equilibrium on the same equilibrium curve

NONLINEAR ANALYSIS

Imperfections

Example

Deformed position



θ includes θ_o

$V = PL\cos\theta$

~~$U = \frac{1}{2}kL\sin(\theta-\theta_o)\cdot L\sin\theta$~~

$U = \frac{1}{2}k\,(L\sin\theta - L\sin\theta_o)^2$

$\quad = \frac{1}{2}kL^2(\sin\theta - \sin\theta_o)^2$

For equilibrium:

$$\frac{d\Pi}{d\theta} = -PL\sin\theta + kL^2(\sin\theta - \sin\theta_o)\cos\theta = 0$$

$$PL\sin\theta = kL^2(\sin\theta - \sin\theta_o)\cos\theta$$

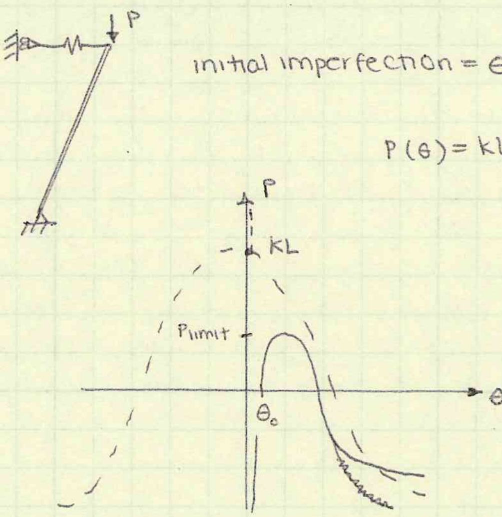
$$P(\theta) = kL\,\frac{\sin\theta - \sin\theta_o}{\tan\theta}$$

if $\theta_o = 0$, $P(\theta) = kL\cos\theta$

same as before

NONLINEAR ANALYSIS

Example, cont'd / revisited



initial imperfection = θ_o

$$P(\theta) = KL \frac{(\sin\theta - \sin\theta_o)}{\tan\theta}$$

For stability:

$$\frac{d^2\Pi}{d\theta} > 0$$

evaluated at $P(\theta)$

only consider at an equilibrium position, not just anywhere

$$\underbrace{KL^2\left(\frac{\sin\theta_o}{\sin\theta} - \sin^2\theta\right)}_{f(\theta)} > 0$$

θ	$f(\theta)$
$\theta < 0$	$f(\theta) < 0$
$0 < \theta < \theta_{cr}$	> 0
$\theta_{cr} < \theta < \pi - \theta_{cr}$	< 0
$\pi - \theta_{cr} < \theta < \pi$	> 0

θ_{cr}:

$$\sin^3\theta_{cr} = \sin\theta_o$$
$$\sin\theta_{cr} = \sin^{1/3}\theta_o$$

Koiter's 2/3 Power Rule:

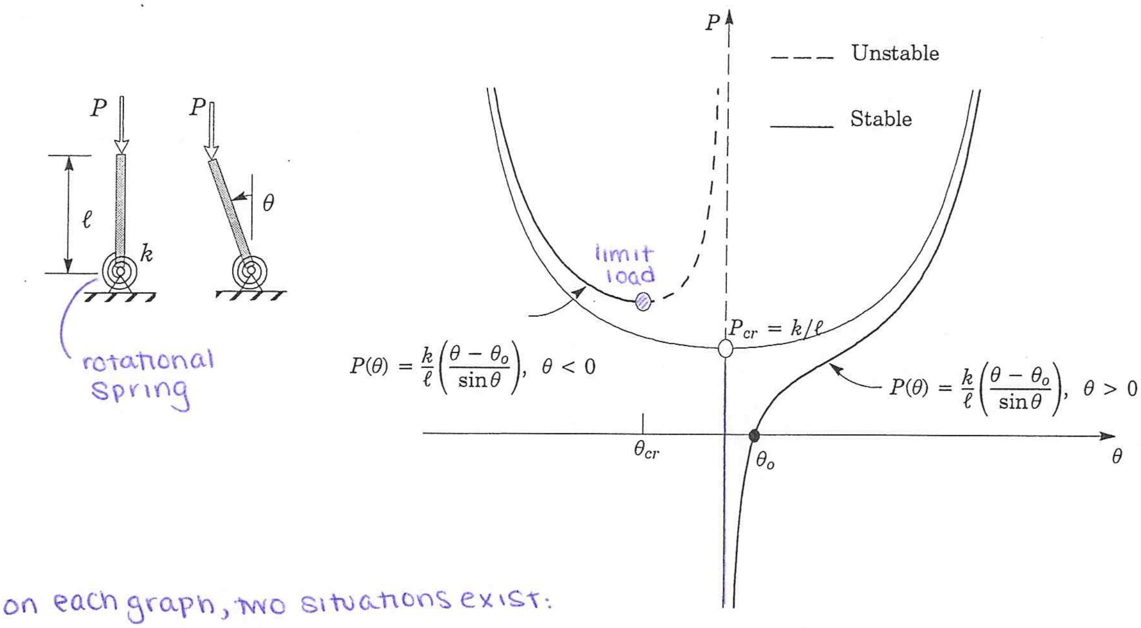
- symmetric bifurcation diagram
- all unstable post-critical behavior

$>$ very sensitive to initial imperfections, related to 2/3s power

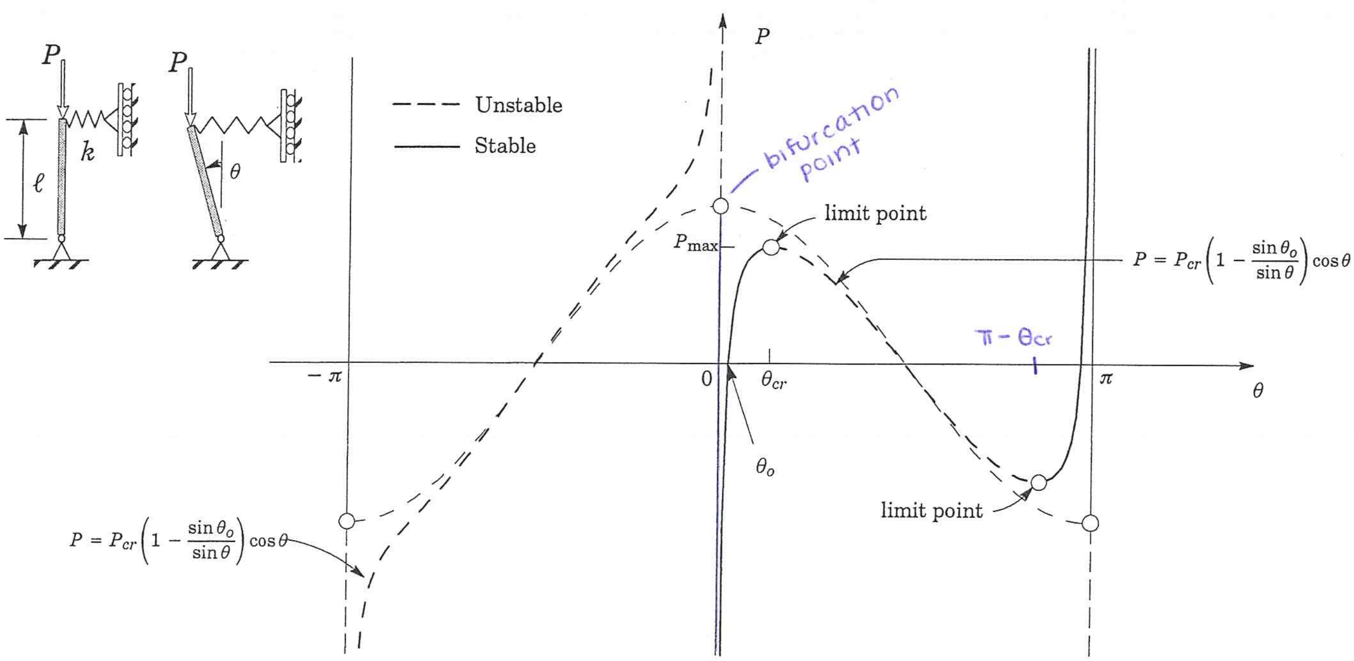
$$P_{max} = KL\left[1 - \sin^{2/3}\theta_o\right]^{3/2}$$

for this particular case

NONLINEAR STABILITY EXAMPLE



on each graph, two situations exist:
- perfect case
- initial imperfection θ_o



$$P_{max} = P(\theta_{cr}) = \underset{\uparrow \text{ KL}}{P_{cr}} \left[1 - \sin^{2/3}\theta_o\right]^{3/2}$$

KOITER'S 2/3 POWER RULE

NONLINEAR ANALYSIS

Stability of multi-DOF systems

Let $\pi = \pi(x_1, x_2 \ldots x_n)$

n degrees of freedom

$\underset{\sim}{x} = \underset{\sim}{a}$ corresponds to an equilibrium position

$\pi(a_1, a_2 \ldots a_n)$: equilibrium!

evaluate:

$$\pi(\underset{\sim}{a} + \Delta \underset{\sim}{x}) = \pi(a_1 + \Delta x_1, a_2 + \Delta x_2 \ldots a_n + \Delta x_n)$$

$$= \pi(\underset{\sim}{a}) + \left(\Delta x_1 \cdot \frac{\partial \pi}{\partial x_1}\bigg|_{\underset{\sim}{x}=\underset{\sim}{a}} + \Delta x_2 \frac{\partial \pi}{\partial x_2}\bigg|_{\underset{\sim}{x}=\underset{\sim}{a}} + \ldots \right)$$

Taylor series expansion:

$$f(x+\Delta x) = f(x) + \Delta x \cdot \frac{df}{dx}\bigg|_x + \frac{\Delta x^2}{2} \frac{d^2 f}{dx^2}\bigg|_x + \ldots$$

$$+ \left(\frac{\Delta x_1^2}{2} \frac{\partial^2 \pi}{\partial x_1^2}\bigg|_{\underset{\sim}{x}=a} + \frac{\Delta x_2^2}{2} \frac{\partial^2 \pi}{\partial x_2^2}\bigg|_{\underset{\sim}{x}=a} + \Delta x_1 \Delta x_2 \frac{\partial^2 \pi}{\partial x \partial y}\bigg|_{\underset{\sim}{x}=\underset{\sim}{a}} \cdots \right)$$

Skipping lots of math we don't seem to know,

For equilibrium,

$\nabla \pi(\underset{\sim}{x}) = 0$

gradient; all of the partial first derivatives

$$\begin{bmatrix} \frac{\partial \pi}{\partial x_1} \\ \frac{\partial \pi}{\partial x_2} \\ \vdots \\ \frac{\partial \pi}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

For stability,

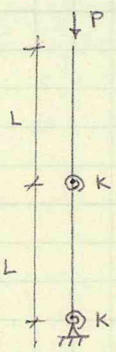
$\underset{\sim}{H}(\pi(\underset{\sim}{x}))$ ~~and~~ is positive definite

Hessian matrix; partial second derivatives

$$\underset{\sim}{H} = \begin{bmatrix} \partial^2/\partial x_1^2 & \partial^2/\partial x_1 \partial x_2 & \cdots \\ \vdots & & \\ \frac{\partial^2}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2}{\partial x_n^2} \end{bmatrix}$$

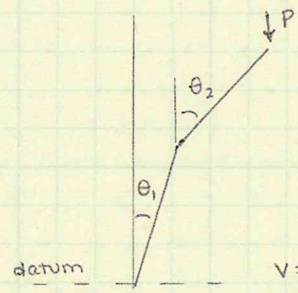
MDOF STABILITY

Example



2 DOFs
rigid beams

Deformed Shape:



$\Pi = V + U$

$V = \left[L\cos\theta_1 + L\cos\theta_2\right]P$

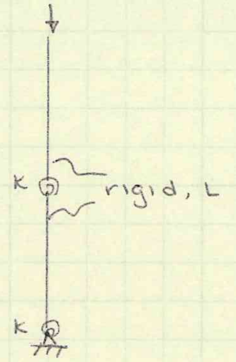
$U = \frac{1}{2}k\theta_1^2 + \frac{1}{2}k(\theta_2 - \theta_1)^2$

further opening of the spring

$\Pi(\theta) = PL(\cos\theta_1 + \cos\theta_2) + \frac{1}{2}k(\theta_1^2 + (\theta_2 - \theta_1)^2)$

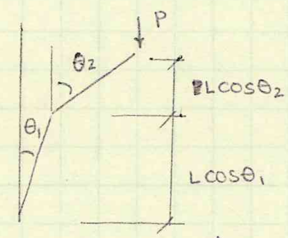
MDOF STABILITY

Example, cont'd



From the other day,

$$\pi(\theta) = PL(\cos\theta_1 + \cos\theta_2) + \tfrac{1}{2}k\left[\theta_1^2 + (\theta_2-\theta_1)^2\right]$$



xP, for potential energy in load

Equilibrium:

$$\frac{\partial\pi}{\partial\theta_1} = -PL\sin\theta_1 + k\theta_1 - k(\theta_2-\theta_1)$$

$$= 2k\theta_1 - k\theta_2 - PL\sin\theta_1$$

$$\frac{\partial\pi}{\partial\theta_2} = -LP\sin\theta_2 + \tfrac{1}{2}(2)k(\theta_2-\theta_1)$$

$$= k(\theta_2-\theta_1) - PL\sin\theta_2$$

$$\nabla(\pi(\underset{\sim}{\theta})) = \begin{bmatrix} \dfrac{\partial\pi}{\partial\theta_1} \\ \dfrac{\partial\pi}{\partial\theta_2} \end{bmatrix}$$

gradient, vector

→ For equilibrium,

$\nabla(\pi(\underset{\sim}{\theta})) = \underset{\sim}{0}$, or, both $\frac{\partial\pi}{\partial\theta_1}$ and $\frac{\partial\pi}{\partial\theta_2}$ must equal zero

two coupled, non-linear equations

$$\begin{bmatrix} (1) & 2k\theta_1 - k\theta_2 - PL\sin\theta_1 = 0 \\ (2) & k(\theta_2-\theta_1) - PL\sin\theta_2 = 0 \end{bmatrix}$$

very hard to solve in closed form — sine makes things odd / difficult

verify equilibrium in initial, undeformed shape:

$$\theta_1 = \theta_2 = 0, \quad \nabla(\pi(\underset{\sim}{\theta})) = \underset{\sim}{0}$$

MDOF STABILITY

Example, cont'd

For stability, find Hessian matrix

$$\frac{\partial^2 \Pi}{\partial \theta_1^2} = 2K - PL\cos\theta_1$$

$$\frac{\partial^2 \Pi}{\partial \theta_1 \partial \theta_2} = -K$$

$$\frac{\partial^2 \Pi}{\partial \theta_2^2} = K - PL\cos\theta$$

$$\underset{\sim}{H} = \begin{bmatrix} 2K - PL\cos\theta_1 & -K \\ -K & K - PL\cos\theta \end{bmatrix}$$

symmetric matrix

must be positive definite for stability

$\underset{\sim}{x}^T \underset{\sim}{H} \underset{\sim}{x} > 0$ for any $\underset{\sim}{x}$

For this to be true,

- all eigenvalues for $\underset{\sim}{H} > 0$
- all principal minors > 0

Principal minors

come from square matrices

$$\underset{\sim}{B} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

Principal minor $1 = |B_{11}| = B_{11}$

$$2 = \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix} = (B_{11}B_{22}) - (B_{12} - B_{21})$$

$$3 = |\underset{\sim}{B}|$$

For each additional row/column, increase the number of involved values to get the next principal minor

Evaluate stability at equilibrium points

(1) $\theta_1 = \theta_2 = 0$

$$\underset{\sim}{H} = \begin{bmatrix} 2K - PL & -K \\ -K & K - PL \end{bmatrix}$$

$2K - PL > 0$ for stability

$\cancel{P/L}$ $P < 2K/L$ principal minor #1

PM #2: $(2K - PL)(K - PL) - K^2 > 0$

$$2K^2 - 3KPL + (PL)^2 - K^2 > 0$$

$$(PL)^2 - 3KPL + K^2 > 0$$

cont'd →

MDOF STABILITY

Example, cont'd

Evaluating stability

PM#1 says $P < 2K/L$

PM#2 says

$$(PL)^2 - 3KPL + K^2 > 0$$

$$PL = \frac{3K \pm \sqrt{9K^2 - 4K^2}}{2} = \frac{3 \pm \sqrt{5}}{2} \cdot K$$

thus, $P < \left(\frac{3 \pm \sqrt{5}}{2}\right) \frac{K}{L}$

$\underline{P < 0.382\, K/L}$, $P < 2.618\, K/L$

smallest, controls the stability of system

much smaller than value for stiff, shorter column (KL)

Find buckling mode shapes

- evaluate equilibrium equations at Pcr

$$\frac{\partial \Pi}{\partial \theta_1} = 2K\theta_1 - K\theta_2 - (0.382\, K/L)\, L \sin\theta_1 = 0$$

$$\frac{\partial \Pi}{\partial \theta_2} = -K\theta_1 + K\theta_2 - (0.382\, K/L)\, L \sin\theta_2 = 0$$

- considering we want to know about onset of buckling (θs are small), linearized approximations will be good

$$\left.\begin{aligned} 2K\theta_1 - K\theta_2 - 0.382K\theta_1 &= 0 \\ -K\theta_1 + K\theta_2 - 0.382K\theta_2 &= 0 \end{aligned}\right\rangle \begin{bmatrix} 2K - 0.382K & -K \\ -K & K - 0.382K \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

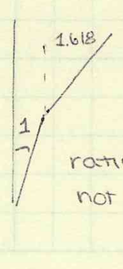
matrix is singular

θs equal zero, or relative to each other

- can determine shape, not magnitude

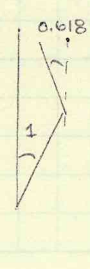
assume $\theta_1 = 1$; then $\theta_2 = 1.618$

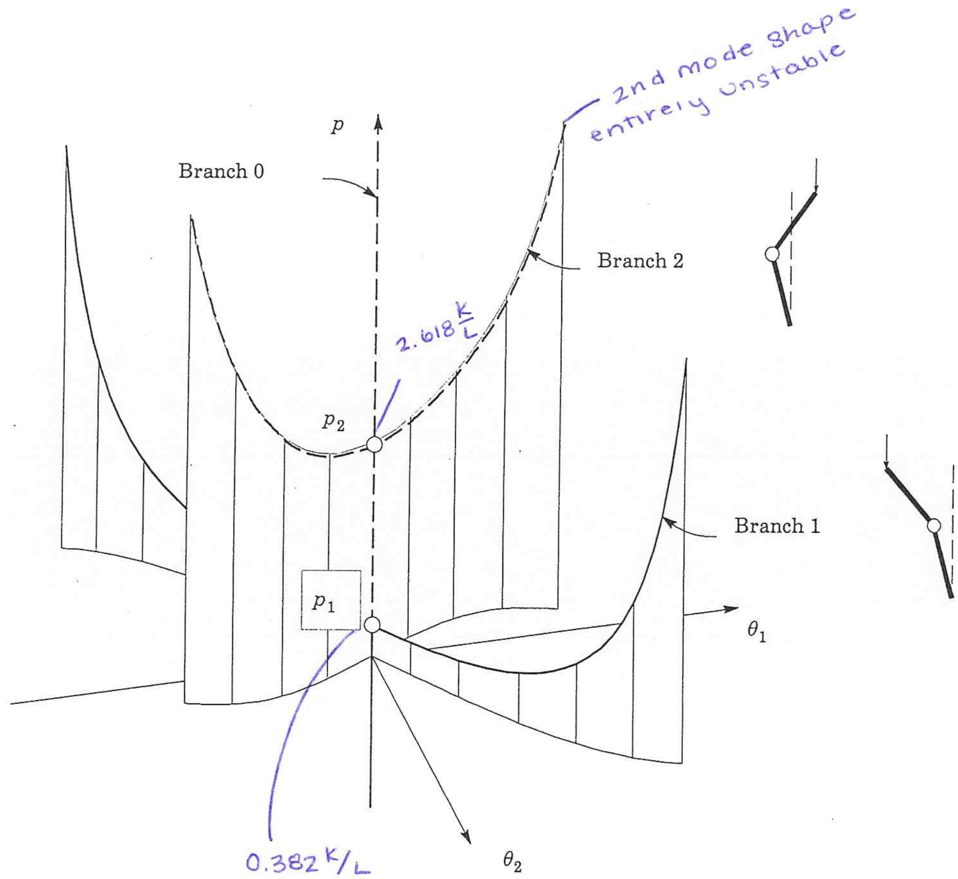
Mode 1



ratio of angles, not values!

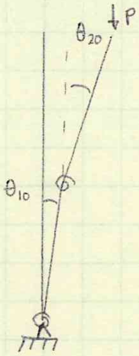
Mode 2 — comes from when $P = 2.618\, K/L$




2nd mode shape
entirely unstable
p
Branch 0
Branch 2
2.618 K/L
p_2
Branch 1
p_1
θ_1
0.382 K/L
θ_2

MDOF ANALYSIS

Initial imperfections



deformed position looks the same as previously

$$U = \tfrac{1}{2}k(\theta_1 - \theta_{10})^2 + \tfrac{1}{2}k\left[(\theta_2 - \theta_{20}) - (\theta_1 - \theta_{10})\right]^2$$

$$V = PL(\cos\theta_1 + \cos\theta_2)$$

For equilibrium,

$$\frac{\partial \pi}{\partial \theta_1} = k\theta_1 - k\theta_{10} + k(\theta_2 - \theta_{20} - \theta_1 + \theta_{10})(-1) - PL\sin\theta_1$$

$$\frac{\partial \pi}{\partial \theta_2} = k(\theta_2 - \theta_{20} - \theta_1 + \theta_{10}) - PL\sin\theta_2$$

again, difficult to solve in closed form

always (almost) tends towards mode 1 shape

CONTINUOUS SYSTEMS

Introduction

Definition:

$$\frac{\text{strain energy}}{\text{unit volume}} = u_o = \int \sigma \, d\varepsilon$$

for a linearly elastic system,

$$\sigma = E\varepsilon$$

$$u_o = \int E\varepsilon \, d\varepsilon = \frac{1}{2}E\varepsilon^2$$

apply to an axial force member

total strain energy = U

$$= \int_{vol} u_o \, dvol$$

Energy must be conserved in this theory. Nonlinear is okay; inelastic is not

For a prismatic member, A, L constant (<u>and</u> ε)

$$U = \int_{vol} \frac{1}{2}E\varepsilon^2 \, dvol$$

don't vary with volume (for prismatic member)

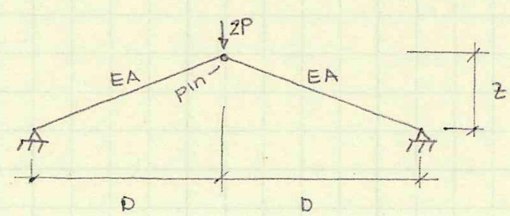
$$U = \frac{1}{2}E\left(\frac{\Delta}{L}\right)^2 \cdot A \cdot L \quad \longleftarrow \text{assumes } \varepsilon = \frac{\Delta}{L}$$

$$= \frac{1}{2}\frac{EA}{L}\Delta^2$$

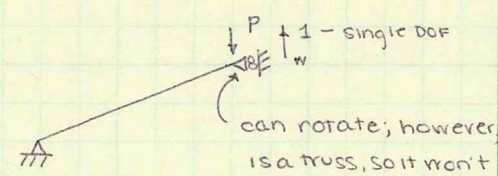
consider a spring: $K = \frac{EA}{L}$, $U = \frac{1}{2}Kx^2$

$$\underline{\underline{U = \frac{1}{2}\left(\frac{EA}{L}\right)\Delta^2}}$$

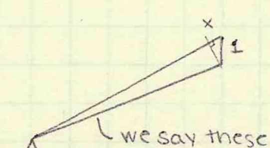
Example



by symmetry, only one DOF



Approach 1: Review of 363



$$x = 1\sin\theta = \sin\theta = \frac{z}{L}$$

$$K_{11} = \frac{EA}{L} \cdot \frac{z}{L} \cdot \frac{z}{L}$$

$$R = -P$$

$$\frac{EA}{L^3} \cdot z^2 \cdot u_1 = -P$$

CONTINUOUS SYSTEMS

Example, con't'd

Now, solve the 381P way

$$U = \frac{1}{2}\frac{EA}{L}\Delta^2$$

energy

$\Delta = L_f - L_o$ ← original length L

final length

$$L = \sqrt{D^2 + z^2}$$

$$L_f = \sqrt{D^2 + (z+u_1)^2}$$

to simplify things, assume:

arch is shallow $(z \ll D)$

using this,

$$L = \sqrt{D^2(1+z/D)^2}$$

$$L_f = \sqrt{D^2\left(1+\frac{z+u_1}{D}\right)^2}$$, but z/D is small

use binomial expansion:

$$\sqrt{1+x^2} \sim 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 \ldots$$

– also needs us to assume u_1 is small compared to D

– not small enough to ignore, but not too large, either

$$L = D\left(1 + \frac{1}{2}(z/D)^2\right)$$

$$L_f = D\left(1 + \frac{1}{2}\left(\frac{z+u_1}{D}\right)^2\right)$$

$$\Delta = D\left[1 + \frac{1}{2}\left(\frac{z+u_1}{D}\right)^2 - 1 - \frac{1}{2}(z/D)^2\right]$$

$$= \frac{D}{2}\left[\frac{1}{D^2}(z^2 + 2u_1 z + u_1^2) - \frac{1}{D^2}z^2\right]$$

$$= \frac{1}{2D}\left[2u_1 z + u_1^2\right] = \underline{\frac{u_1 z}{D} + \frac{u_1^2}{2D}}$$

$$U = \frac{1}{2}\frac{EA}{DL}\left(u_1 z + u_1^2/2\right)^2$$

$$U = \frac{1}{2}\frac{EA}{D^2 L}\left((u_1 z)^2 + u_1^3 z + u_1^4/4\right)$$ – accounts for actual change in length of the member

$$V = P(z + u_1)$$

$$\Pi = U + V = \frac{1}{2}\frac{EA}{D^2L}\left(u_1 z + u_1^2/2\right)^2 + P(z+u_1)$$

$$\frac{d\Pi}{du_1} = \frac{EA}{D^2L}\left(u_1 z + u_1^2/2\right)(z+u_1) + P \quad \text{or} \quad -P = \frac{EA}{L}\left[\underbrace{\frac{z^2}{D^2}u_1}_{\text{linear part}} + \underbrace{\frac{3}{2}\frac{z u_1^2}{D^2} + \frac{u_1^3}{2D^2}}_{\text{nonlinear part}}\right]$$

if $D \sim L$, same as 363 answer

CONTINUOUS NONLINEAR SYSTEMS

strain energy

$$\frac{S.E.}{Vol} = u_0 = \int \sigma d\varepsilon$$

for a spring, $u = \frac{1}{2}\frac{EA}{L}\Delta^2$

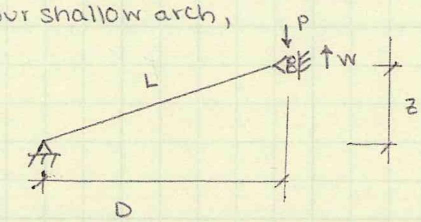
- using engineering strain

$$\varepsilon = \frac{\Delta}{L} = \frac{L_f - L_o}{L_o}$$

not the only type of strain

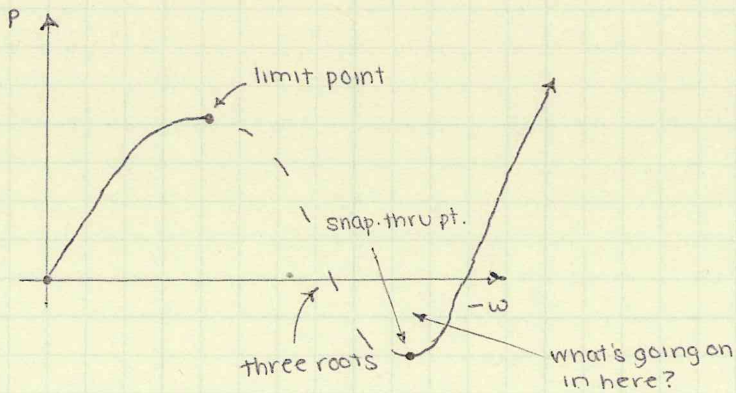
- assuming prismatic member (ε constant)
- constant E

for our shallow arch,



$$P = \frac{EA}{L}(-1)\left[\frac{z^2}{D^2}w + \frac{3}{2}\frac{zw^2}{D^2} + \frac{1}{2}\frac{w^3}{D^2}\right]$$

assuming:
- shallow arch, $z \ll L$
- L approximately equals D
- used binomial expansion $\sqrt{1+x^2} \sim 1 + \frac{x^2}{2} - \frac{x^4}{8} \cdots$



Bifurcation diagram
- needs to indicate which portions are stable vs. unstable
- equilibrium curve shown, too

limit point — one eq. curve, stable to unst.

bifurcation point — two eq. curves intersecting

$\boxed{K_T \equiv \text{tangent stiffness}}$

$$K_T \equiv \frac{dP(w)}{dw}$$

instantaneous stiffness along curve of $P(w)$ [compare to secant stiffness, which links back to the origin]

$= 0$ at limit points

$$K_T = \frac{-EA}{L}\left[z^2/D^2 + 3\,zw/D^2 + \frac{3}{2}\frac{w^2}{D^2}\right] = \frac{-EA}{D^2L}\underbrace{\left[z^2 + 3zw + \frac{3}{2}w^2\right]}$$

at limit load, this is equal to zero

$$w = \frac{-3z \pm \sqrt{9z^2 - 4z^2(3/2)}}{3} = -z \pm \frac{z}{3}\sqrt{3}$$

CONTINUOUS NONLINEAR SYSTEMS

Tangent stiffness and bifurcation diagrams

to get load at limit point, use

$$w = z\left(-1 \pm \sqrt{3}/3\right)$$ and plug into the equation for $P(w)$

use sign that corresponds to smaller negative value (as shown on curve)

Types of strain

- engineering, $\varepsilon = \frac{\Delta}{L} = \frac{L_f - L_o}{L_o}$
- natural, $\varepsilon_N = \frac{L_f - L_o}{L_f}$

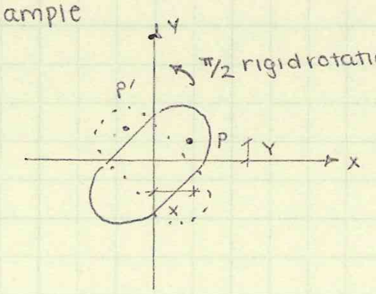
normalized differently
huge difference if Δ is large

- Green's, $\varepsilon_g = \frac{L_f^2 - L_o^2}{2L_o^2}$
- logarithmic, $\varepsilon_L = \int_{L_f}^{L_o} \frac{dL_f}{L_f} = \ln\left(\frac{L_f}{L_o}\right)$
- Almansi, $\varepsilon_A = \frac{L_f^2 - L_o^2}{2L_f^2}$

small displacements → all yield the same strain

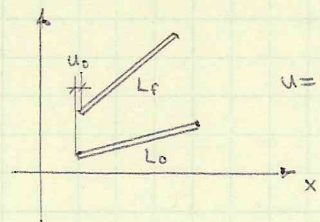
nlgeom: large displacements, varying strain values

Example



new coords: $(-y, x)$

strain measures for planar displacements



u = horizontal disp. varies with x & y $u(x,y)$

v = vertical disp. $v(x,y)$

Green's strain

$$\varepsilon_{Gx} = \frac{\partial u}{\partial x} + \underbrace{\frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]}_{\text{non-linear strain contribution}}$$

Engineering strain:

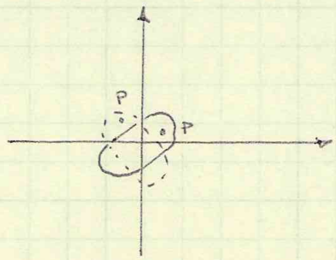
$$\varepsilon_x = \frac{\partial u(x,y)}{\partial x}$$

$$\varepsilon_y = \frac{\partial v(x,y)}{\partial y}$$ (v, not u)

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$ used in plates and solids

CONTINUOUS NONLINEAR SYSTEMS

Applying equations to example



P: (x, y)

P′: (−y, x)

$u(x,y) = -y - x$

$v(x,y) = x - y$

Engineering strain

$$\varepsilon_x = \frac{\partial u}{\partial x} = -1$$

Green's strain

$$\varepsilon_{Gx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$= -1 + \frac{1}{2}\left[(-1)^2 + (1)^2\right]$$

$$= 0$$

not the same value!

Green's strain is more accurate:
rigid body rotation should result in zero strain

- consideration of large displacements is important when they exist
- equations derived from deformed shape, not original (unlike linear calcs, where it's assumed to essentially not move)

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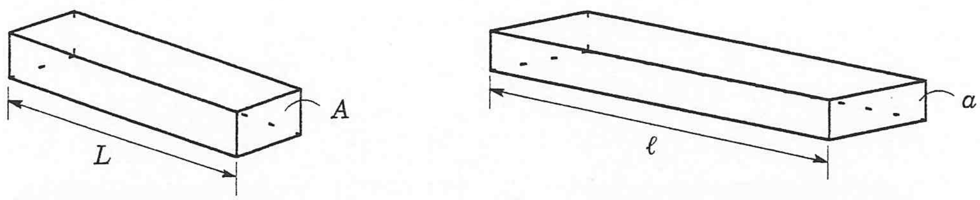
Dept. of Civil, Arch. & Env. Eng.

Instructor: E. B. Williamson
CE 381P Computer Methods in Structural Analysis

Strain of an Axially Loaded Bar

Consider the deformation of an axially loaded bar shown in Fig. 1. Initially, before the applica-



Initial Configuration **Deformed Configuration**

Fig. 1. Deformation of an axially loaded bar

tion of any loads, the bar has length L and cross-sectional area A. After deformation, the bar has length ℓ and cross-sectional area a. One of the most simple ways to measure the strain in the bar is by the well known *engineering strain*. By definition,

$$\epsilon_E = \frac{\ell - L}{L}$$

L = original length
ℓ = final length

The change in length could also be normalized by the final length to give the so-called *natural strain*

$$\epsilon_N = \frac{\ell - L}{\ell}$$

Aside from these measures of strain, it would be possible for us to invent a variety of other quantities that could be used to describe the deformation of the bar. In fact, almost any function of the stretch (defined as $\ell - L$) can be used to give us a suitable definition of strain. Other well known strain measures include the "true" or *logarithmic strain*, *Green's strain*, and the *Almansi strain*. Definitions for each of these terms, respectively, are given by the following formulae:

$$\epsilon_L = \int_L^{\ell} \frac{d\ell}{\ell} = \ln\left(\frac{\ell}{L}\right)$$

$$\epsilon_G = \frac{\ell^2 - L^2}{2L^2}$$

$$\epsilon_A = \frac{\ell^2 - L^2}{2\ell^2}$$

A complete derivation of these definitions can be found in most books on continuum mechanics. For the case where strains are small ($\ell \approx L$), a simple Taylor Series expansion shows that all the definitions of strain above give the "small strain" formula of $\epsilon = \Delta L/L$. In order for us to make use of these definitions for problems involving large deformations, we must develop expressions that give functional relationships between displacements and strains. With such definitions known, we can use our virtual work approach for developing truss and beam stiffness matrices that account for large displacements. Thus, it is useful for us to study how these definitions apply to a 2-D continuum.

Continuum Strain Measures

Under the assumption of "small displacements", the strain of a continuum depends only linearly upon the displacements. Typically, strain terms are grouped together in the mathematical construct of the strain tensor $\boldsymbol{\epsilon}$. While it will be possible for us to proceed entirely from the mathematical basis of tensor calculus, we will find the calculations to be more transparent by maintaining the strain and displacement terms in component form. For those interested in learning more about a rigorous, tensor-based approach, there are several good books on nonlinear continuum mechanics that address this issue. For us, we will simply be referring to the tensor components relative to a global X-Y coordinate system. As such, we can represent these quantities in a simple matrix form. To wit, recall from strength of materials the definition of *engineering* strain, in terms of the displacement components, to be

$$\epsilon_{xx} = \frac{\partial u}{\partial X}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial Y}$$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X}\right)$$

where u and v are the displacement components in the X-direction and the Y-direction, respectively. These equations rely on the assumption that the displacements are small. The applicability of these expressions for problems that involve "large" displacements can be demonstrated through a simple example.

Example: Rigid Body Rotation

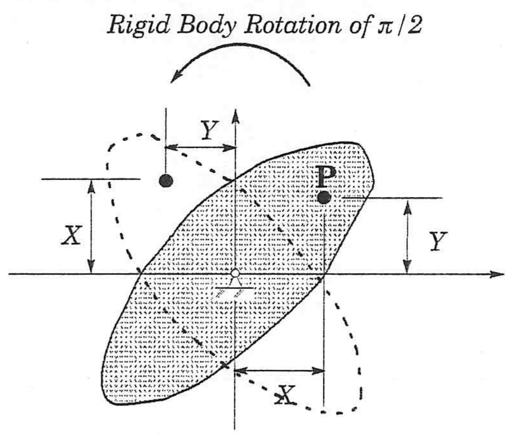


Fig. 2. Large displacement of a structure

Consider the case of a two-dimensional solid undergoing a counterclockwise rotation of $\pi/2$ as shown in Fig. 2. For this movement of the structure, the displacement components are given as

$$u = -Y - X$$

$$v = X - Y$$

According to the definition of engineering strain given above, the resulting values for the strain components are

$$\epsilon_{xx} = \epsilon_{yy} = -1$$
$$\epsilon_{xy} = 0$$

These values do not make sense physically — under rigid body motion, we expect no strain to accrue in the structure. Clearly, it is necessary to re-establish the definition of stain for a continuum so that physically correct results are obtained when a body is subjected to a finite motion or deformation process. Focusing specifically on the 2-D case, we can derive the necessary expressions by analyzing the response of a 2-D solid displacing from some reference position to some other deformed position. In doing so, we will develop expressions for Green's Strain.

Derivation of Green's Strain

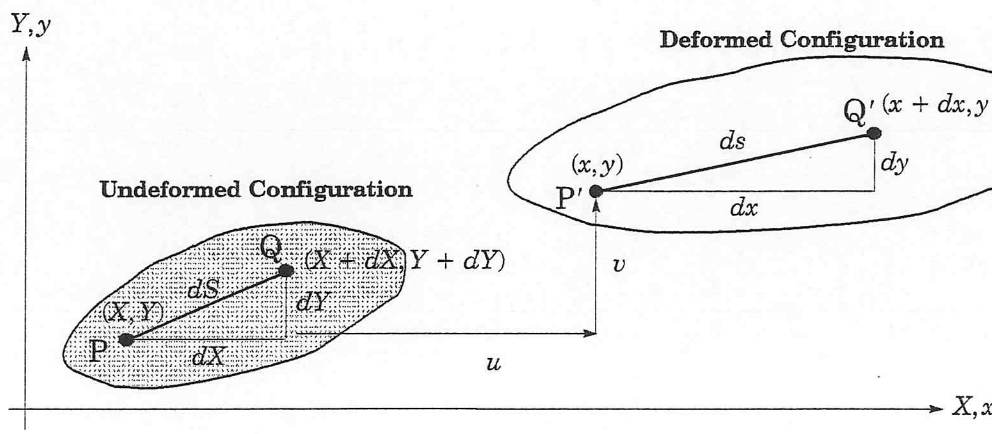


Fig. 3. Displacement from deformed to undeformed configuration

Consider a body initially in equilibrium that is then deformed and displaced to some new position as shown in Fig. 3. We will characterize the nature of the deformation of a typical line segment **PQ**. Before the deformation, point **P** is located at the point (X, Y) and **Q** is located at the point $(X + dX, Y + dY)$. Thus, the initial length of the line connecting these points can be obtained from the expression

$$L^2 = dS^2 = (dX)^2 + (dY)^2$$

After deformation, point **P** moves to point **P**′ with coordinates (x, y) and **Q** moves to point **Q**′ with coordinates $(x + dx, y + dy)$. Note that the line segment **P**′**Q**′ has a different *length* and *orientation* than **PQ**. The length of the line segment in the deformed configuration can be obtained from the relationship

$$\ell^2 = ds^2 = (dx)^2 + (dy)^2$$

In terms of displacement components

$$u = x - X \qquad v = y - Y$$

Considering differential quantities, we have

$$dx = du + dX \qquad dy = dv + dY$$

In terms of displacements, the deformed length of the line segment can be computed as

$$\ell^2 = ds^2 = (dX)^2 + (dY)^2 + 2(dX)du + 2(dY)dv + (du)^2 + (dv)^2$$

and

$$\ell^2 - L^2 = 2(dX)du + 2(dY)dv + (du)^2 + (dv)^2$$

By definition, the *total derivative* of a function of two variables $u = f(X, Y)$ is given by

$$du = \frac{\partial u}{\partial X}dX + \frac{\partial u}{\partial Y}dY$$

$$dv = \frac{\partial v}{\partial X}dX + \frac{\partial v}{\partial Y}dY$$

The rate of change of the displacement components with respect to the coordinates in the undeformed configuration ($\partial u/\partial X$, $\partial u/\partial Y$, $\partial v/\partial X$, $\partial v/\partial Y$) are known as the *displacement gradients*.

In order to help reduce the resulting expressions, we can, with no loss in generality, assume that the line segment **PQ** was originally oriented along the X axis. This assumption is analogous to defining a local coordinate system for a frame or truss member and then determining the global properties by means of a coordinate transformation. Therefore, based on our assumption, $dY = 0$, and the resulting expression for Green's Strain is given as

$$\epsilon_{G_X} \equiv \frac{\ell^2 - L^2}{2L^2} = \frac{\partial u}{\partial X} + \frac{1}{2}\left(\left(\frac{\partial u}{\partial X}\right)^2 + \left(\frac{\partial v}{\partial X}\right)^2\right)$$

Likewise, we could have just as easily assumed the line segment **PQ** to be oriented along the Y axis in order to derive the Green's Strain in the Y-direction as

$$\epsilon_{G_Y} \equiv \frac{\ell^2 - L^2}{2L^2} = \frac{\partial v}{\partial Y} + \frac{1}{2}\left(\left(\frac{\partial u}{\partial Y}\right)^2 + \left(\frac{\partial v}{\partial Y}\right)^2\right)$$

Returning to the example above, we see that when the strain is defined in terms of Green's Strain rather than engineering strain, the results follow our expectations. That is, for the case of a rigid body rotation, $\epsilon_{G_X} = \epsilon_{G_Y} = 0$. As such, for problems that involve large displacements, we expect that the Green's Strain will be a better measure of deformation than engineering strain.

Nonlinear Truss Example

Let's now consider the problem of analyzing the response of a shallow arch consisting of truss members. For this example, we can take advantage of symmetry to produce the analytical model shown in Fig. 4. In analyzing this structure, we will pay special attention to the impact that the definition

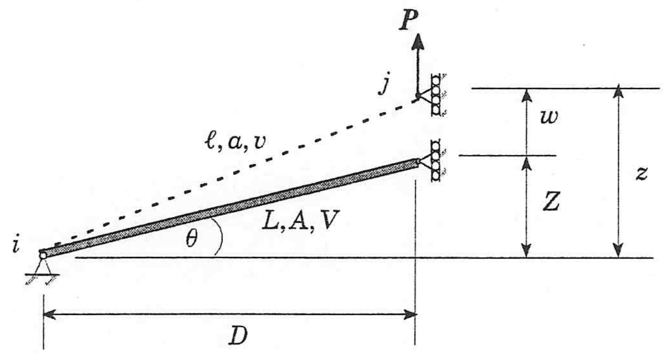


Fig. 4. Response of a shallow arch

of strain has on the computed response. In addition, we will be careful about referring to quantities in the initial, undeformed configuration versus in the deformed configuration. Referring to Fig. 4, the bar in its initial configuration has length L, cross-sectional area A, and volume V. In addition, its initial geometry is such that the loaded end is located at a height of Z above the global X axis. After deformation, the bar has new length ℓ, cross-sectional area a, and volume v. The loaded end is now at a height of z above the global X axis. We will use the variable w to describe the displacement of the loaded end from the original configuration to its final position.

For simplicity, we will assume that the material comprising the truss is *incompressible*. This assumption results in the following simplification:

$$V = v \quad \Rightarrow \quad AL = a\ell$$

It is more common to assume that *mass* is conserved so that

$$\rho v = PV$$

where ρ is the mass density of the material in the deformed configuration, and P is the original mass density. For the purposes of this example, however, it is sufficient to assume incompressibility.

We will study the response of the structure shown in Fig. 4 considering two different measures of strain – Green's strain and logarithmic strain. We will further assume that Young's Modulus, E, is suitable to use as the elastic constant for either strain measure.

Previously, we used an energy-based approach to derive the governing equilibrium equations for the structure. It is also possible to develop the appropriate equations simply by looking at a free-body-diagram of the forces acting at point j of the structure in the deformed position. Thus, equilibrium in the global Y direction requires that

$$R(z) = T(z) - P = 0$$

where $T(z)$ is the vertical component of the internal force in the bar and $R(z)$ is defined as the *residual*. If the structure indeed satisfies equilibrium, the residual will equal zero. The tension force in the truss member, as a function of position, is given by

$$T(z) = \sigma a \sin\theta = \sigma a\left(\frac{z}{\ell}\right)$$

In terms of the two strain measures,

$$T(z) = \frac{Evz}{\ell^2}\left(\frac{\ell^2 - L^2}{2L^2}\right)$$

or

$$T(z) = \frac{Evz}{\ell^2}\ln\left(\frac{\ell}{L}\right)$$

Note that ℓ is also a function of z so that T is highly nonlinear in z. Further insight into the nature of the nonlinearity in the presence of large deformations can be revealed for this example if we consider the rate of change of force with displacement as given by the tangent stiffness matrix K_T. By definition,

$$K_T = \frac{dR(z)}{dz}$$

For the particular case where P is constant,

$$K_T = \frac{dT(z)}{dz} = \frac{d}{dz}\left(\sigma a\frac{z}{\ell}\right)$$

where ℓ, a, and σ all vary as a function of z. Because of our assumption of incompressibility, the resulting expression for the tangent stiffness is somewhat simplified to

$$K_T = \frac{dT(z)}{dz} = \frac{d}{dz}\left(\sigma\frac{V}{\ell^2}z\right)$$

Based on the geometry defined in Fig. 4, $\ell^2 = D^2 + z^2$. Differentiating based upon the product rule, we obtain

$$K_T = \frac{d\sigma}{dz}\left(\frac{Vz}{\ell^2}\right) + \sigma V\frac{d}{dz}\left(\frac{z}{\ell^2}\right)$$

$$= \frac{d\sigma}{dz}\left(\frac{az}{\ell}\right) + \sigma V\left(\frac{\ell^2 - 2z^2}{\ell^4}\right)$$

$$= \frac{d\sigma}{d\ell}\frac{d\ell}{dz}\left(\frac{az}{\ell}\right) + \sigma V\left(\frac{\ell^2 - 2z^2}{\ell^4}\right)$$

$$= \frac{d\sigma}{d\ell}\left(\frac{az^2}{\ell^2}\right) + \sigma\frac{a}{\ell} - 2\sigma\frac{az^2}{\ell^3}$$

$$= a\left(\frac{d\sigma}{d\ell} - \frac{2\sigma}{\ell}\right)\frac{z^2}{\ell^2} + \frac{\sigma a}{\ell}$$

All that remains is to find $d\sigma/d\ell$ for each of the strain definitions being considered. Thus,

$$\left(\frac{d\sigma}{d\ell}\right)_G = \frac{E\ell}{L^2} \quad \text{and} \quad \left(\frac{d\sigma}{d\ell}\right)_L = \frac{E}{\ell}$$

and

$$K_{T_G} = \frac{A}{L}\left(E - 2\sigma\frac{L^2}{\ell^2}\right)\frac{z^2}{\ell^2} + \frac{\sigma a}{\ell}$$

$$K_{T_L} = \frac{a}{\ell}(E - 2\sigma)\frac{z^2}{\ell^2} + \frac{\sigma a}{\ell}$$

Although these expressions are similar, they are clearly not the same. To be consistent with typical continuum mechanics terminology, it is instructive to express K_{T_G} in an alternative form as

$$K_{T_G} = \frac{A}{L}(E - 2\Sigma)\frac{z^2}{\ell^2} + \frac{\Sigma A}{L}$$

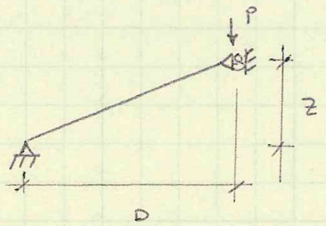
where $\Sigma = \sigma L^2/\ell^2$ is known as the *Second Piola-Kirchhoff Stress*. It gives the force per unit undeformed area but transformed by what is known as the inverse of the deformation gradient $(\ell/L)^{-1}$. Apparent in this formulation is the fact that the Green strain measures deformation relative to the initial, undeformed configuration, but stress (normally referred to as the "true" or Cauchy stress) needed for establishing equilibrium is defined in the deformed configuration. In order for the stress and strain measures to be consistent, they both must be defined with respect to the same state of deformation. Thus, when using the Green strain, it is necessary to use the second Piola-Kirchhoff stress, which relates stress in the deformed configuration back to the original, undeformed configuration.

Allowing for the local-to-global force transformation implied by $(z/\ell)^2$, we see that the stiffness can be expressed in terms of the initial, undeformed configuration or the current configuration. The term $\sigma a/\ell = \Sigma A/L$ is generally known as the *initial stress stiffness* or the *geometric stiffness* since the same term can be derived by considering the change in equilibrating global end forces occurring when an initially stressed rod rotates by a small amount.

The geometric stiffness is the term that, in general, is of interest in stability analyses because a very large negative geometric stiffness can make the overall stiffness matrix singular. It should be noted that the geometric stiffness is unrelated to changes in cross-sectional area and is purely associated with the force changes caused by rigid body rotation.

CONTINUOUS NONLINEAR SYSTEMS

Revisit arch example



DOF 1 at load

$\uparrow w$

$\Pi = U + V$

V: potential energy

$P(z+w)$

negative work, (+) u

U: strain energy

$U = \frac{1}{2} EAL\varepsilon^2$

- constant strain
- AL refer to original volume, not deformed; assumptions still limit response
- elastic response

using Green's strain,

$$L_F^2 = \left[(D)^2 + (z+w)^2 \right]$$

$$L_o^2 = D^2 + z^2$$

$$\varepsilon_G = \frac{L_F^2 - L_o^2}{2L_o^2} = \frac{D^2 + (z+w)^2 - D^2 - z^2}{2(D^2+z^2)}$$

$$= \frac{2zw + w^2}{2(D^2+z^2)}$$

$$U = \frac{1}{2} E(AL)_o \left(\frac{2zw+w^2}{2(D^2+z^2)} \right)^2$$

$$= \frac{EA}{8L_o^3} \left[2zw + w^2 \right]^2$$

$$= \frac{EAw^2}{8L_o^3} \left[4z^2 + 4zw + w^2 \right]$$

$$\Pi = P(z+w) + \frac{EAw^2}{8L_o^3} \left[4z^2 + 4zw + w^2 \right]$$

$$\frac{d\Pi}{dw} = P + \frac{EA}{8L_o^3} \left[8z^2 w + 12zw^2 + 4w^3 \right] = 0$$ for equilibrium

$$\frac{-EA}{L_o} \left[\frac{z^2}{L_o^2} w + \frac{3}{2} \frac{z}{L_o^2} w^2 + \frac{w^3}{2L_o^3} \right] = P$$

first term is the same as linear; w is small. other terms add nonlinear behavior

Use Approximate Length (Binomial Expansion) for "Shallow Arch" Analysis

$$L_1(D,z) := D\cdot\left[1+\frac{1}{2}\cdot\left(\frac{z}{D}\right)^2\right] \qquad L_2(D,z,w) := D\cdot\left[1+\frac{1}{2}\cdot\left(\frac{z+w}{D}\right)^2\right]$$

Engineering strain

$$\varepsilon_E(D,z,w) := \frac{L_2(D,z,w)-L_1(D,z)}{L_1(D,z)} \text{ simplify } \rightarrow w\cdot\frac{2\cdot z+w}{2\cdot D^2+z^2}$$

$$U_{E1}(E,A,D,w,z) := \frac{1}{2}\cdot E\cdot A\cdot L_1(D,z)\cdot\left(\varepsilon_E(D,z,w)\right)^2 \left| \begin{array}{l} \text{simplify} \\ \text{collect}, E \\ \text{expand}, 5 \end{array} \right. \rightarrow \frac{w^2}{D}\cdot E\cdot\frac{A}{2\cdot D^2+z^2}\cdot z^2+\frac{w^3}{D}\cdot E\cdot\frac{A}{2\cdot D^2+z^2}\cdot z+\frac{1}{4}\cdot\frac{w^4}{D}\cdot E\cdot\frac{A}{2\cdot D^2+z^2}$$

$$\frac{d}{dw}U_{E1}(E,A,D,w,z) \text{ simplify } \rightarrow w\cdot E\cdot A\cdot\frac{2\cdot z^2+3\cdot w\cdot z+w^2}{D\cdot\left(2\cdot D^2+z^2\right)}$$

Approximate L_1 = D for total volume of member

$$U_{E2}(E,A,D,w,z) := \frac{1}{2}\cdot E\cdot A\cdot D\cdot\left(\varepsilon_E(D,z,w)\right)^2 \left| \begin{array}{l} \text{simplify} \\ \text{collect}, E \\ \text{expand}, 5 \end{array} \right. \rightarrow 2\cdot E\cdot A\cdot D\cdot\frac{w^2}{\left(2\cdot D^2+z^2\right)^2}\cdot z^2+2\cdot E\cdot A\cdot D\cdot\frac{w^3}{\left(2\cdot D^2+z^2\right)^2}\cdot z+\frac{1}{2}\cdot E\cdot A\cdot D\cdot\frac{w^4}{\left(2\cdot D^2+z^2\right)^2}$$

27 April 07

95a

$$\frac{d}{dw}U_{E2}(E,A,D,w,z) \text{ simplify } \rightarrow 2\cdot E\cdot A\cdot D\cdot w\cdot\frac{2\cdot z^2+3\cdot w\cdot z+w^2}{\left(2\cdot D^2+z^2\right)^2}$$

Engineering strain -- Approximate answer (deformations are small -> D ~ L1)

$$\varepsilon_E(D,z,w) := \frac{L_2(D,z,w)-L_1(D,z)}{D} \text{ simplify } \rightarrow \frac{1}{2\cdot D^2}\cdot w\cdot(2\cdot z+w)$$

$$U_{E3}(E,A,D,w,z) := \frac{1}{2}\cdot E\cdot A\cdot L\cdot\varepsilon_E(L,z,w)^2 \text{ simplify } \rightarrow \frac{1}{8}\cdot E\cdot\frac{A}{L^3}\cdot w^2\cdot(2\cdot z+w)^2$$

$$\frac{d}{dw}U_{E3}(E,A,D,w,z) \text{ simplify } \rightarrow \frac{1}{2}\cdot E\cdot A\cdot w\cdot(2\cdot z+w)\cdot\frac{z+w}{L^3}$$

$$\frac{d^2}{dw^2}U_{E3}(E,A,D,w,z) \text{ simplify } \rightarrow \frac{1}{2}\cdot E\cdot A\cdot\frac{2\cdot z^2+6\cdot w\cdot z+3\cdot w^2}{L^3}$$

$$\frac{d^2}{dw^2}U_{E2}(E,A,D,w,z) = 0 \;\Big|\begin{array}{l}\text{simplify}\\ \text{solve}, w\end{array} \rightarrow \begin{bmatrix}\left(-1+\frac{1}{3}\cdot 3^{\frac{1}{2}}\right)\cdot z\\ \left(-1-\frac{1}{3}\cdot 3^{\frac{1}{2}}\right)\cdot z\end{bmatrix}$$

$$w_1(z) := \left(-1+\frac{1}{3}\cdot\sqrt{3}\right)\cdot z$$

$$P_{max}(E,A,D,z,w) := \frac{1}{2}\cdot E\cdot A\cdot w_1(z)\cdot\left(2\cdot z+w_1(z)\right)\cdot\frac{\left(z+w_1(z)\right)}{L^3} \;\Big|\begin{array}{l}\text{simplify}\\ \text{float}, 6\end{array} \rightarrow -.192450\cdot E\cdot A\cdot\frac{z^3}{L^3}$$

Use Actual Length Formula

$L_1(D,z) := \sqrt{D^2 + z^2}$ $\qquad$ $L_2(D,z,w) := \sqrt{D^2 + (z+w)^2}$

Engineering strain

$$\varepsilon_E(D,z,w) := \frac{L_2(D,z,w) - L_1(D,z)}{L_1(D,z)} \text{ simplify } \rightarrow \frac{-\left[-\left(D^2 + z^2 + 2\cdot w\cdot z + w^2\right)^{\frac{1}{2}} + \left(D^2 + z^2\right)^{\frac{1}{2}}\right]}{\left(D^2 + z^2\right)^{\frac{1}{2}}}$$

$$U_{E4}(E,A,D,w,z) := \left(\frac{1}{2}\cdot E\cdot A\cdot L_1(D,z)\cdot \varepsilon_E(D,z,w)^2\right) \rightarrow \frac{1}{2}\cdot E\cdot \frac{A}{\left(D^2 + z^2\right)^{\frac{1}{2}}}\cdot\left[\left(D^2 + z^2 + 2\cdot w\cdot z + w^2\right)^{\frac{1}{2}} - \left(D^2 + z^2\right)^{\frac{1}{2}}\right]^2$$

No binomial expansion
(shallow arch) approximation

$$\frac{d}{dw}U_{E4}(E,A,D,w,z) \text{ simplify } \rightarrow E\cdot A\cdot\left[\left(D^2 + z^2 + 2\cdot w\cdot z + w^2\right)^{\frac{1}{2}} - \left(D^2 + z^2\right)^{\frac{1}{2}}\right]\cdot\frac{z + w}{\left(D^2 + z^2\right)^{\frac{1}{2}}\cdot\left(D^2 + z^2 + 2\cdot w\cdot z + w^2\right)^{\frac{1}{2}}}$$

Exact length formula using Green Strain

Green strain

$$\varepsilon_G(D,z,w) := \frac{(L_2(D,z,w))^2 - (L_1(D,z))^2}{2\cdot(L_1(D,z))^2} \text{ simplify } \rightarrow \frac{1}{2}\cdot w\cdot\frac{2\cdot z + w}{D^2 + z^2}$$

$$U_{G1}(E,A,D,w,z) := \frac{1}{2}\cdot E\cdot A\cdot L_1(D,z)\cdot\varepsilon_G(D,z,w)^2 \text{ simplify } \rightarrow \frac{1}{8}\cdot E\cdot\frac{A}{\left(D^2 + z^2\right)^{\frac{3}{2}}}\cdot w^2\cdot(2\cdot z + w)^2$$

Approximate $L_1 = D$ for total volume of member

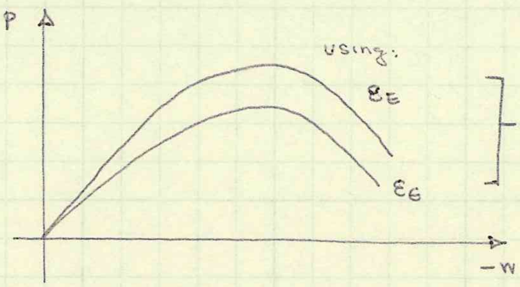
$$U_{G2}(E,A,D,w,z) := \frac{1}{2}\cdot E\cdot A\cdot D\cdot\varepsilon_G(D,z,w)^2 \text{ simplify } \rightarrow \frac{1}{8}\cdot A\cdot E\cdot D\cdot w^2\cdot\frac{(2\cdot z + w)^2}{\left(D^2 + z^2\right)^2}$$

$$\frac{d}{dw}U_{G1}(E,A,D,w,z) \text{ simplify } \rightarrow \frac{1}{2}\cdot E\cdot A\cdot w\cdot(2\cdot z + w)\cdot\frac{z + w}{\left(D^2 + z^2\right)^{\frac{3}{2}}}$$

$$\frac{d}{dw}U_{G2}(E,A,D,w,z) \text{ simplify } \rightarrow \frac{1}{2}\cdot A\cdot E\cdot D\cdot w\cdot(2\cdot z + w)\cdot\frac{z + w}{\left(D^2 + z^2\right)^2}$$

CONTINUOUS NONLINEAR SYSTEMS

Response using different strains



not the same, but E used was
- if E varied for each method, response would be much closer
- especially in data comparison, need to know what is collected, what is calculated

(ABAQUS uses Green's)

Generalized Approach

derivation of a truss stiffness matrix for the general case

for the linear solution (small Δ)

$$\underset{\sim}{K} = \int_0^L \underset{\sim}{B}^T E(x) A(x) \underset{\sim}{B} \, dL$$

strain-displacement vector

prismatic, 2-noded ~~bar~~ element:

$$\underset{\sim}{k} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

If the displacements are large,

use E, A_o, L_o

$\underset{\sim}{B}$ is the only thing that changes

$$\varepsilon_{E_{linear}} = \frac{\partial u}{\partial x}$$

$$\varepsilon_G = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

TRUSS ANALYSIS

Stiffness matrix formulation

Basic principle

$$\delta W_{int} = \int_{vol} \delta \varepsilon \cdot \sigma \, dvol$$

For nonlinear geometry:

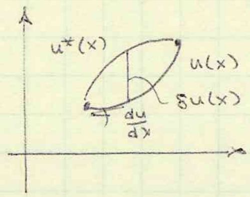
use Green's strain

$$\varepsilon_G = \frac{L_f^2 - L_o^2}{2L_p^2} \quad \text{or,} \quad \varepsilon_G = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

$u(x)$ = displacement in the x·direction

$v(x)$ = displacement in the y·direction

Find virtual Green's strain, $\delta \varepsilon_G$



$\delta u(x)$ not the same as du/dx but, has similar behaviors

$$\delta \varepsilon_G = \frac{\partial}{\partial x}(\delta u) + \cancel{\tfrac{1}{2}}\frac{\partial}{\partial x}(\delta u)\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}(\delta v)\frac{\partial v}{\partial x}$$

Reapplying to δW functions:

$$\delta W_{int} = \underbrace{\int_{vol} \frac{\partial}{\partial x}(\delta u)\cdot \sigma \, dvol}_{(1)} + \underbrace{\int_{vol} \frac{\partial}{\partial x}(\delta u)\frac{\partial u}{\partial x}\cdot \sigma \, dvol}_{(2)} + \underbrace{\int_{vol} \frac{\partial}{\partial x}(\delta v)\frac{\partial v}{\partial x}\, \sigma \, dvol}_{(3)}$$

(1): $\int_{vol} \frac{\partial}{\partial x}(\delta u)\sigma \, dvol$

– Assume linear material behavior ($\sigma = E\varepsilon_E$) ← $\frac{\partial u}{\partial x}$

$$= \int_{vol} \frac{\partial}{\partial x}\delta u \cdot E \cdot \frac{\partial u}{\partial x} \, dvol$$

$$= \int_{L} \frac{\partial}{\partial x}\delta u \cdot E(x)\cdot A(x)\cdot \frac{\partial u}{\partial x} \, dL$$

or, dx

– Assume $u(x) = \underset{\sim}{L}_x \underset{\sim}{u}$

$\underset{\sim}{L}_x = [\,1 - x/L \quad 0 \quad x/L \quad 0\,]$

v_1 v_2 u_1 u_2

$\underset{\sim}{u}^T = [\,u_1 \ v_1 \ u_2 \ v_2\,]$

$v(x) = \underset{\sim}{L}_y \underset{\sim}{u}$

$\underset{\sim}{L}_y = [\,0 \quad 1 - x/L \quad 0 \quad x/L\,]$

Galerkin Method: u, δu of same form

$\delta u(x) = \underset{\sim}{L}_x \, \delta \underset{\sim}{u}$, etc.

NONLINEAR TRUSS ANALYSIS

Stiffness matrix formulation

$$\delta W_{int} = \underbrace{\int_{vol} \frac{\partial}{\partial x}(\delta u)\sigma \, dvol}_{(1)} + \underbrace{\int_{vol} \frac{\partial}{\partial x}(\delta u)\frac{\partial u}{\partial x}\sigma \, dvol}_{(2)} + \underbrace{\int_{vol} \frac{\partial}{\partial x}(\delta v)\frac{\partial v}{\partial x}\sigma \, dvol}_{(3)}$$

(1) continued:

$$\int_0^L \delta \underset{\sim}{u}^T \underset{\sim}{L}_x'^T \cdot E(x) \cdot A(x) \cdot \underset{\sim}{L}_x' \, \underset{\sim}{u} \, dx \quad , \quad \underset{\sim}{B}_x = \underset{\sim}{L}_x'$$

$$\delta \underset{\sim}{u}^T \underbrace{\int_0^L \underset{\sim}{B}_x^T E(x) A(x) \underset{\sim}{B}_x \, dx}_{\underset{\sim}{K}_L \equiv \text{ linear stiffness matrix, same as before}} \cdot \underset{\sim}{u}$$

(2) $\int_{vol} \frac{\partial}{\partial x}(\delta u)\frac{\partial u}{\partial x}\sigma \, dvol$

in a truss bar, $\sigma \cdot A(x)$ is constant along length

$\sigma(x) \cdot A(x) \equiv$ axial force, F

loads are applied as nodal forces, not distributed

$$\int_0^L \frac{\partial}{\partial x}\delta u \cdot \frac{\partial u}{\partial x} \underbrace{\sigma(x) \cdot A(x)}_{F} \, dx$$

$$F \cdot \underbrace{\int_0^L \frac{\partial}{\partial x}\delta u}_{\delta u^T \cdot B_x^T} \cdot \frac{\partial u}{\partial x} \cdot dx$$

$$F \cdot \delta \underset{\sim}{u}^T \int_0^L \underset{\sim}{B}_x^T \cdot \underset{\sim}{B}_x \, dx \cdot \underset{\sim}{u} \longleftarrow \text{no } E(x), A(x) \text{ terms}$$

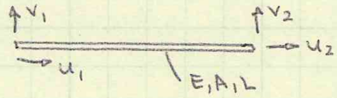
(3) $\int_{vol} \frac{\partial}{\partial x}(\delta v)\frac{\partial v}{\partial x}\sigma \, dvol$

$$F \cdot \delta \underset{\sim}{u}^T \int_0^L \underset{\sim}{B}_y^T \cdot \underset{\sim}{B}_y \, dx \cdot \underset{\sim}{u}$$

Σ (2), (3) give the nonlinear contribution to response

NONLINEAR TRUSS ANALYSIS

Example – constant E, A



$$\underset{\sim}{K}_L = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{K}_G = \frac{F}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \equiv \text{geometric stiffness matrix}$$

no super-obvious nonlinearity

$\underset{\sim}{K} = \underset{\sim}{K}_L + \underset{\sim}{K}_G$ but, F exists on both sides of equation
to solve, proceed iteratively

tension F – more stiff
compression F – less stiff

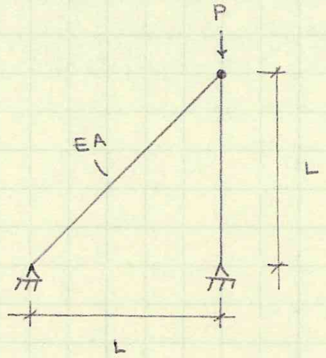
Sign of F is very important!

ASSUMPTIONS:

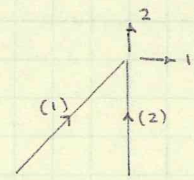
– linear material behavior, $\sigma = E\left(\frac{\partial u}{\partial x}\right)$
– integrated over $vol_{original}$
could make large difference
– small deformations (near-constant volume)

CONCLUDING EXAMPLE

Introduction



- 2 DOFs
- statically determinate
- use direct stiffness method
- joint is a hinge



Consider member (1)

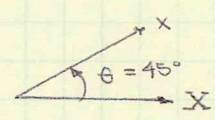
$E, A, \sqrt{2}L$

$$\underset{\sim}{k}_L^1 = \frac{EA}{\sqrt{2}L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\underset{\sim}{k}_G^1 = 0$ — axial force in the member is zero when the structure is undeformed

Transformation matrix

$$\underset{\sim}{T}_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\frac{\sqrt{2}}{2}$$



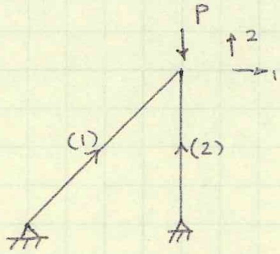
$\underset{\sim}{k}_1 = \underset{\sim}{k}_L^1 + \underset{\sim}{k}_G^1$

$\underset{\sim}{K}_1 = \underset{\sim}{T}_1^T \underset{\sim}{k}_1 \underset{\sim}{T}_1$ stiffness matrix of element 1 in global coordinates (unassembled)

Reminder ~~am~~ of T formulation:

$$\begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

CONCLUDING EXAMPLE



using direct stiffness

Member (2)

EA, L

$$\underset{\sim}{K}_L^2 = \frac{EA}{L}\begin{bmatrix} \cdots \end{bmatrix}$$

$$\underset{\sim}{K}_G^2 = \frac{-P}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$ — load actually detracts from overall stiffness of structure

$$\underset{\sim}{K}_2 = \frac{1}{L}\begin{bmatrix} EA-P & 0 & -EA+P & 0 \\ 0 & -P & 0 & P \\ -EA+P & 0 & EA-P & 0 \\ 0 & P & 0 & -P \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

mapping to global coords, use 3rd and 4th row/columns of both $\underset{\sim}{K}_1$ and $\underset{\sim}{K}_2$

Full structure:

$$\underset{\sim}{K} = \frac{EA}{L}\begin{bmatrix} \sqrt{2}/4 & \sqrt{2}/4 \\ \sqrt{2}/4 & 1+\sqrt{2}/4 \end{bmatrix} - \frac{P}{L}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{F} = \begin{bmatrix} 0 \\ -P \end{bmatrix}$$

Nonlinear problem:

P exists on both sides of equation $\underset{\sim}{K}\cdot\underset{\sim}{u} = \underset{\sim}{F}$

→ must iterate to solve ←

Geometric Stiffness Matrix for a Planar Beam

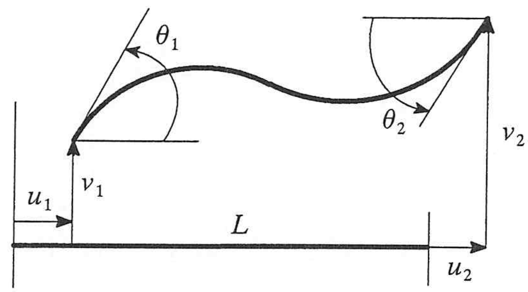


Fig. 1 Deformation of a Planar Beam

Accounting for both flexural and axial response in the expression for strain, the geometric stiffness matrix for a planar beam can be developed using the same approach that was used for the truss element. In the formulation given below, transverse displacements are assumed to vary according to the standard cubic Hermitian functions.

$$k_G = \frac{P}{\ell}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & \frac{6}{5} & \frac{\ell}{10} & 0 & -\frac{6}{5} & \frac{\ell}{10} \\ 0 & \frac{\ell}{10} & \frac{2\ell^2}{15} & 0 & -\frac{\ell}{10} & -\frac{\ell^2}{30} \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{6}{5} & -\frac{\ell}{10} & 0 & \frac{6}{5} & -\frac{\ell}{10} \\ 0 & \frac{\ell}{10} & -\frac{\ell^2}{30} & 0 & -\frac{\ell}{10} & \frac{2\ell^2}{15} \end{bmatrix}$$

CONCLUDING EXAMPLE

Bifurcation buckling / linearized buckling analysis
- finds buckling load
- does not show post-buckling behavior

To solve:

1. Apply a reference level of load $\underset{\sim}{R}_{Ref}$
2. compute axial forces in members with a linearized analysis
3. increase the load by a scalar multiplier, λ
4. Assume axial forces in members scale by the same multiplier, λ — can be bad, depending on pre-buckling deformation
5. Form $\underset{\sim}{k}$ for structure and compute lowest eigenvalue for buckling load

From previous example, find eigenvalues (or, principle minors)

$$\underset{\sim}{k} = \frac{1}{L}\begin{bmatrix} \sqrt{2}/4\,EA - P & \sqrt{2}/4\,EA \\ \sqrt{2}/4\,EA & (1+\sqrt{2}/4)EA - P \end{bmatrix}$$

principle minors: determinants along the diagonal

$$(1)\quad \sqrt{2}/4\,EA \cdot \frac{1}{L} - \frac{P}{L} > 0 \quad \text{for stability}$$

$$P < \frac{\sqrt{2}}{4}EA, \quad \underline{P < 0.35355EA}$$

$$(2)\quad \frac{1}{L}\left[\left(\frac{\sqrt{2}}{4}EA - P\right)\left((1+\sqrt{2}/4)EA - P\right) - \left(\sqrt{2}/4\,EA\right)^2\right] > 0$$

quadratic equation for P
solve numerically

$P_{cr} < 0.2412EA$

$P_{cr} < 1.466EA$

↑ for stability

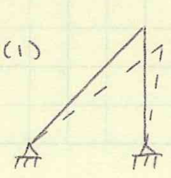
controlling case is $P_{cr} < 0.2412EA$

Solving for buckled shapes
- evaluate $\underset{\sim}{k}$ at critical load value
  matrix will be singular
- assume values for u or v
  if $u = 1$, $v = -0.318$

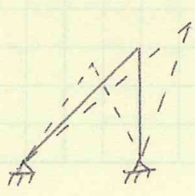
$\underset{\sim}{k} \cdot \underset{\sim}{u} = [0]$

↑ increment in load is zero

(1)



(2)

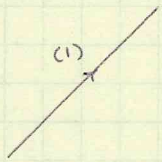


(using P_{cr_2}) $u = 1, v = 3.15$
$u = -1, v = -3.15$

CONCLUDING EXAMPLES

Alternate method

Energy-based analysis



$L_f^2 = (L+u)^2 + (L+v)^2$ to calculate Green's strain

two equilibrium equations

add total energies from (1) and (2)

partial derivatives

$$\frac{\partial \pi}{\partial u} = 0, \quad \frac{\partial \pi}{\partial v} = 0$$

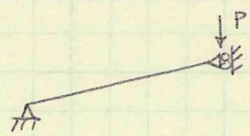
no axial force in non-linear equations

if linearized, resulting equations are the same as we got using direct stiffness

linearized analysis – not always good

initial imperfections

large deformations pre-buckling



REVIEW

Bifurcation analysis – determine load that causes buckling
(modification from pg. 102)

1. Apply small load – $P=1$, or $P = P/\lambda$...
2. compute linear displacements using K_L
3. Calculate axial forces in linear case
4. Use calculated axial forces in K_G formulations

$$K_G = \frac{P}{L}\left[\quad\right]$$, where P is axial force in member

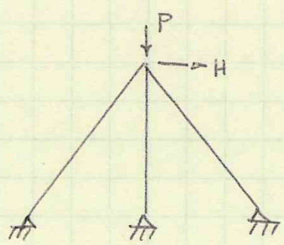
5. increase loads by λ (now $P=\lambda$ or $P=P$)
6. Assume axial forces scale by λ

$$K_G' = K_G \cdot \lambda$$

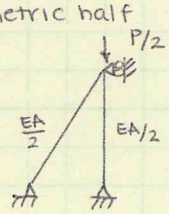
assumes pre-buckling Δ is small
linear response still

7. use eigenvalue or principal minors analysis to solve for λ, or P

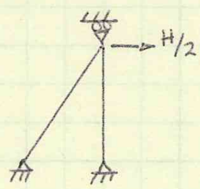
Symmetric example



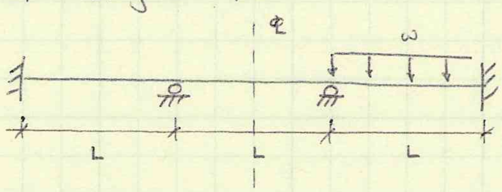
symmetric half



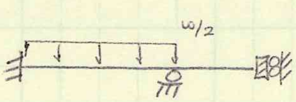
antisymmetric half



Better symmetry example



symmetric:



Anti-symmetric: